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Convergence of an iteration scheme in convex metric spaces

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Abstract

In this paper, a new iteration scheme in a uniformly convex metric space is defined and its convergence is obtained. A numerical example is also considered to compare the rate of convergences of the iteration with that of an existing iteration scheme.

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1. Introduction

If T is a self mapping on a metric space (X, d), then $F(T)$ denotes the set of all fixed points of T, that is, $F(T) = \Big\{x \in X : Tx = x\Big\}$. In the study of fixed point theory, there is a natural interest in finding conditions on T and X, as general as possible, and which also guarantee the strong convergence of the sequence of iterates $\{x_n\}$ to a fixed point of T in X.

Moreover, if the sequence of iterates converges to a fixed point of T, it is interesting to evaluate the rate of convergence (or, alternately, the error estimate) of the method, i.e., in obtaining a stopping criterion for the sequence of successive approximation. For a weaker contractive condition, the Picard iterates need not converge to the fixed point of T, and some other iteration schemes must be considered. For $x_0 \in X$, the iteration given by

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2, \ldots$$

is called Picard iteration.

In this regard, many authors have introduced and investigated various iteration schemes to approximate fixed point for different classes of contractive conditions (for instance, refer [1], [3], [6], [9], [10], [11], [7], etc. and the references therein).

In 2007, Agarwal et al. [1] introduced the S-iteration scheme for a hyperbolic metric space. Let K be a nonempty subset of a hyperbolic metric space (X, d). For $x_0 \in K$, define

$$\left.\begin{array}{lll} x_{n+1} & = & \mathcal{W}\left(Tx_n, Ty_n, \alpha_n\right) \\ y_n & = & \mathcal{W}\left(x_n, Tx_n, \beta_n\right) \end{array}\right\} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$.

In 2014, Kadioglu and Yildirim [7] defined Picard normal S-iteration scheme for a convex subset of a normed space. The same iteration may be defined in a nonempty closed and convex subset K of a hyperbolic metric space as follows. For $x_0 \in K$,

$$\left.\begin{array}{lll} x_{n+1} & = & Ty_n \\ y_n & = & \mathcal{W}\left(z_n, Tz_n, \alpha_n\right) \\ z_n & = & \mathcal{W}\left(x_n, Tx_n, \beta_n\right) \end{array}\right\} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$.

They showed that (1.2) converges faster than that of Picard, Mann, Ishikawa and (1.1).

In 2018, Ullah and Arshad [16] introduced a three-step iteration process called 'M-iteration' as follows.

For x_0 in K, a nonempty convex subset of a Banach space, the M-iteration process is defined by

$$\left.\begin{array}{lcl} x_{n+1} & = & Ty_n, \\ y_n & = & Tz_n, \\ z_n & = & \mathcal{W}(x_n, Tx_n, \alpha_n) \end{array}\right\} \tag{1.3}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

They showed that the M-iteration converges (to the fixed point) faster than that of S-iteration and Picard S-iteration processes, using numerical examples.

In this paper, we introduce a new iteration scheme (*N-iteration scheme*) and prove the convergence of the sequence generated by it to the fixed point of a fundamentally nonexpansive mapping T, when $F(T) \neq \emptyset$.

For x_0 in K, a nonempty convex subset of a uniformly convex metric space space,

$$\left.\begin{array}{lcl} x_{n+1} & = & Ty_n, \\ y_n & = & \mathcal{W}(Tz_n, z_n, \alpha_n), \\ z_n & = & Tx_n \end{array}\right\} \tag{1.4}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

The convergence rate of the sequence generated by (1.4) is also compared to that of the sequence generated by (1.3), taking a numerical example.

2. Preliminaries

In this section, some preliminary definitions and results required for our subsequent discussion are presented, starting with the notion of convex metric space introduced by Takahashi [15] in 1970.

Definition 2.1. *[15] A convex metric space $(X, d, \mathcal{W})$ is a metric space with a convex structure $\mathcal{W} : X \times X \times [0, 1] \longrightarrow X$ satisfying*

$$d\Big(z, \mathcal{W}(x, y, t)\Big) \leq td(z, x) + (1 - t)d(z, y)$$

for all x, y and z in X, and $t \in [0,1]$.

Definition 2.2. *[13] A convex metric space $(X, d, \mathcal{W})$ is said to be uniformly convex if for any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that, for all $r > 0$ and $x, y, z \in X$ with $d(z,x) \leq r$, $d(z,y) \leq r$ and $d(x,y) \geq r\varepsilon$,*

$$d\Big(z, \mathcal{W}(x, y, 1/2)\Big) \leq r(1-\alpha) < r.$$

In 2012, Khan et al. [8] proved the following result for uniformly convex hyperbolic spaces which is also valid for uniformly convex metric spaces.

Lemma 2.1. *[8] Let $(X, d, \mathcal{W})$ be a uniformly convex hyperbolic metric space with monotone modulus of uniform convexity η. Let $z \in X$ and $\{t_n\}$ be a sequence in $[a,b]$ for some $a, b \in (0,1)$. If $\{x_n\}$, $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, z) \leq c$, $\limsup_{n\to\infty} d(y_n, z) \leq c$ and $\limsup_{n\to\infty} d\Big(\mathcal{W}(x_n, y_n, t_n), z\Big) = c$ for some $c \geq 0$, then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

In 2014, Ghoncheh and Razani [2] introduced a class of mappings called fundamentally nonexpansive mappings in a metric space (X, d) which generalizes Suzuki mappings, where a mapping $T : X \longrightarrow X$ is said to be fundamentally nonexpansive if

$$d(T^2x, Ty) \leq d(Tx, y)$$

for all x and y in X. They showed that every mapping which satisfies condition C is fundamentally nonexpan- sive, but the converse is not true.

A mapping $T : X \longrightarrow X$ is said to be a fundamental contraction [5] if there exists a positive number $k < 1$ such that

$$d(T^2x, Ty) \leq kd(Tx, y)$$

for all x and y in X.

Senter and Doston [12] defined the following condition to obtain a convergence result for nonexpansive mappings in metric spaces.

Definition 2.3. *[12] Let K be a nonempty subset of a metric spaces (X, d). A mapping $T : K \longrightarrow K$ with $F(T) \neq \emptyset$ is said to satisfy Condition (I) if* *there exists a non-decreasing function* $f : [0, \infty) \longrightarrow [0, \infty)$ *with* $f(0) = 0$, $f(t) > 0$ *for all* $t \in (0, \infty)$ *such that*

$$f\left(d(x, F(T))\right) \leq d(x, Tx) \quad \text{for all} \;\; x \in K,$$

where $d(x, F(T)) = \inf \Big\{ d(x, p) : p \in F(T) \Big\}$.

The following is the definition of T-stability of iteration schemes given by Harder and Hicks [4].

Definition 2.4. *[4] Let* $T : X \longrightarrow X$ *and* w *be a fixed point of* T*. For any* $x_0 \in X$*, let the sequence* $\{x_n\}$ *generated by the iteration scheme* $x_{n+1} = \mu(T, x_n)$*,* $n = 0, 1, 2, \ldots$ *converges to* w*. Let* $\{u_n\}$ *be an arbitrary sequence, and set* $\epsilon_n = d\left(u_{n+1}, x_{n+1})\right)$*,* $n = 0, 1, 2, \ldots$*. Then the iterative scheme* $\mu(T, x_n)$ *is called* T*-stable if and only if* $\lim_{n\to\infty} \epsilon_n = 0$ *implies* $\lim_{n\to\infty} u_n = w$*.*

3. Convergence of N-iteration scheme

In this section, we obtain convergence results for the N-iteration scheme for fundamentally nonexpansive mappings T with $F(T) \neq \emptyset$, where $F(T) = \{x \in X : Tx = x\}$.

Lemma 3.1. *Let* K *be a nonempty closed convex subset of a complete convex metric space* $(X, d, \mathcal{W})$ *and* $T : X \longrightarrow X$ *be a fundamentally nonexpansive mapping with* $F(T) \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated by (1.4). Then the sequence* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$*.*

Proof. For $w \in F(T)$, using (1.4) and the condition $d(T^2x, Ty) \leq d(Tx, y)$, one can easily show that

$$d(x_{n+1}) \leq d(x_n, w), \qquad n = 0, 1, 2, \ldots$$

Since the sequence of positive real numbers $\{d(x_n, w)\}$ is monotonically decreasing, it must be convergent, say to $\mu \geq 0$, and therefore,

$$d(x_m, x_n) \leq d(x_m, w) + d(x_n, w) = 2\mu,$$

from which we conclude that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, w)$ exists.

Let $\lim_{n\to\infty} d(x_n, w) = \mu \geq 0$. If $\mu = 0$, then $d(x_n, Tx_n) \leq 2d(x_n, w)$ and taking the limit as $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

If $\mu > 0$, since $d(Tx_n, w) \leq d(x_n, w)$, taking $\limsup$ as $n \to \infty$, we get

$$\limsup_{n\to\infty} d(Tx_n, w) \leq \mu.$$

Now, since $y_n = \mathcal{W}(Tz_n, z_n, \alpha_n)$ and $z_n = Tx_n$, we have

$$d(y_n, w) \leq (1-\alpha_n)d(Tz_n, w) + \alpha_n d(z_n, w) \leq d(z_n, w) \leq d(x_n, w)$$

and as in the above, we get

$$\limsup_{n\to\infty} d(y_n, w) \leq \mu.$$

Since $d(x_{n+1}, w) = d(Ty_n, w) \leq d(y_n, w)$, taking $\liminf$ as $n \to \infty$, we get

$$\mu = \liminf_{n\to\infty} d(x_{n+1}, w) \leq \liminf_{n\to\infty} d(y_n, w) \leq \mu,$$

i.e.,

$$\lim_{n\to\infty} d(y_n, w) = \mu.$$

This implies that

$$\begin{aligned} \mu &= \limsup_{n\to\infty} d(y_n, w) = \limsup_{n\to\infty} d\Big(\mathcal{W}(Tz_n, z_n, \alpha_n), w\Big) \\ &\leq \limsup_{n\to\infty} \Big\{(1-\alpha_n)d(Tz_n, w) + \alpha_n d(z_n, w)\Big\} \\ &\leq \limsup_{n\to\infty} \Big\{(1-\alpha_n)d(Tx_n, w) + \alpha_n d(x_n, w)\Big\} \\ &= \limsup_{n\to\infty} d\Big(\mathcal{W}(Tx_n, x_n, \alpha_n), w\Big) \\ &\leq \limsup_{n\to\infty} d\Big(x_n, w) = \mu, \end{aligned}$$

i.e.,

$$\limsup_{n\to\infty} \mathcal{W}(Tx_n, x_n, \alpha_n) = \mu.$$

It then follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a uniformly convex metric space $(X, d, \mathcal{W})$ and $T : X \longrightarrow X$ be a fundamentally nonexpansive mapping with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (1.4) converges strongly to an element of $F(T)$ if and only if $\liminf_{n\to\infty} d\Big(x_n, F(T)\Big) = 0$, where, $d\Big(x_n, F(T)\Big) = \inf_{w\in F(T)} d(x_n, w)$.*

Proof. If $\{x_n\}$ defined by (1.4) strongly converges to a fixed point of T, then obviously $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

To show the sufficiency part, we first note that $F(T)$ is closed. For, if $\{w_k\}$ is a sequence in $F(T)$ which converges to some $w \in K$, then T is fundamentally nonexpansive, $d(w_n, Tw) = d(T^2 w_n, Tw) \leq d(Tw_n, w) = d(w_n, w)$, and thus,

$$0 = \lim_{n\to\infty} d(w_n, w) \geq \lim_{n\to\infty} d(w_n, Tw) = d\left(\lim_{n\to\infty} w_n, Tw\right) = d(w, Tw),$$

showing that $w \in F(T)$, and hence $F(T)$ is closed.

From the proof of Lemma 3.1 that $\lim_{n\to\infty} d(x_n, w)$ exists for all w in $F(T)$ so that $d\Big(x_{n+1}, F(T)\Big) \leq d\Big(x_n, F(T)\Big)$, which implies the sequence $\Big\{d\Big(x_n, F(T)\Big)\Big\}$ is non-increasing and bounded below, and so, $\liminf_{n\to\infty} d\Big(x_n, F(T)\Big)$ exists.

Since $\liminf_{n\to\infty} d\Big(x_n, F(T)\Big) = 0$, it follows that $\lim_{n\to\infty} d\Big(x_n, F(T)\Big) = 0$. Consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, w_k) < \frac{1}{2^k}$ for all $k \geq 1$ and $\{w_k\} \subseteq F(T)$. Then $d(x_{n_{k+1}}, w_k) \leq d(x_{n_k}, w_k), w_k) < \frac{1}{2^k}$ which implies

$$d(w_{k+1}, w_k) \leq d(w_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, w_k) < \frac{1}{2^{k-1}},$$

showing that $\{w_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{w_k\}$ converges in $F(T)$. Let $\lim_{k\to\infty} w_k = w$. Then as $k \to \infty$,

$$d(x_{n_k}, w) \leq d(x_{n_k}, w_k) + d(w_k, w) \longrightarrow 0,$$

showing that $\lim_{k\to\infty} d(x_{n_k}, w) = 0$. Now, since $\lim_{n\to\infty} d(x_n, w)$ exists, we must have

$$\lim_{n\to\infty} d(x_n, w) = 0,$$

as required. □

Next, we prove a strong convergence result using the definition of condition (I) given by Senter and Doston [12] for metric spaces.

Theorem 3.3. *Let K be a nonempty closed convex subset of a uniformly convex metric space $(X, d, \mathcal{W})$ and $T : K \longrightarrow K$ be a fundamentally nonexpansive mapping with $F(T) \neq \emptyset$. If T satisfies Condition (I), then the sequence defined by (1.4) converges strongly to some fixed point of T.*

Proof. As in the proof of Theorem 3.2, $F(T)$ is closed. We observe that by Lemma 3.1, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T satisfies Condition (I), we have

$$\lim_{n\to\infty} f\left(d(x_n, F(T))\right) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since f is a non-decreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$,

$$\lim_{n\to\infty} d\left((x_n, F(T)\right) = 0.$$

The conclusion of the proof follows as in the proof of Theorem 3.2. □

Next, we prove a stability result for the iterative scheme (1.4).

Theorem 3.4. *Let K be a nonempty closed convex subset of a uniformly convex metric space $(X, d, \mathcal{W})$ and $T : X \longrightarrow X$ be a fundamental contraction mapping with $F(T) \neq \emptyset$. For $x_0 \in K$, let $\{x_n\}$ be the sequence generated by the iterative scheme $x_{n+1} = \mu(T, x_n)$, $n \geq 0$ as defined in (1.4). Then the iteration scheme is T-stable if $\liminf_{n\to\infty} d\Big(x_n, F(T)\Big) = 0$ or T satisfies condition (I).*

Proof. Let $\{u_n\}$ be an arbitrary sequence in K and $\varepsilon_n = d(u_n, x_{n+1})$, $n \geq 0$, where $x_{n+1} = \mu(T, x_n)$. Then, for $w \in F(T)$, we have

$$d(u_{n+1}, w) \leq d(u_{n+1}, x_{n+1}) + d(x_{n+1}, w) = \varepsilon_n + d(x_{n+1}, w).$$

By Theorem 3.2 and Theorem 3.3, $\{x_n\}$ converges to a fixed point of T, i.e., $\lim_{n\to\infty} x_n = w$ and hence the result. □

4. Numerical examples

In this section, some examples are considered.

Example 4.1. *Consider the uniformly convex metric space $(X, d, \mathcal{W})$, where $X = \mathbf{R}$, $\mathcal{W} : X \times X \times [0,1] :\longrightarrow X$ is defined by $\mathcal{W}(x, y, t) = tx + (1-t)y$ and the metric d is given by*

$$d(x,y) = \begin{cases} x + y, & x \neq y \\ 0, & x = y \end{cases}$$

Then $K = [0, 1]$ *is a closed convex subset of* X. *Consider the mapping* $T : K \longrightarrow K$ *defined by* $Tx = x^2$ *for all* x *in* K. *One can easily check that* T *is fundamentally nonexpansive and* $F(T) \neq \emptyset$.

Consider $f(t) = t$, *for all* $t \geq 0$, *then the mapping* T *satisfies condition (I) as*

$$f\Big(d(x, F(T))\Big) = \inf\Big\{d(x, w) : w \in F(T)\Big\} = x + 0 \leq x + x^2 = d(x, Tx)$$

for all x *in* K. *Thus all the conditions of Theorem 3.3 are satisfied and hence the convergence of the sequence generated by (1.4).*

Now, the iteration (1.4) reduces to

$$\left.\begin{array}{ll} x_{n+1} & = Ty_n, \\ y_n & = (1-\alpha_n)Tz_n + \alpha_n z_n, \\ z_n & = Tx_n \end{array}\right\}$$

where $\{\alpha_n\}$ *is a real sequence in* $(0, 1)$, *which can be written as*

$$x_{n+1} = T\Big((1-\alpha_n)T^2 x_n + \alpha_n T x_n\Big), \quad n = 0, 1, 2, \ldots$$

Taking $\alpha_n = \frac{n+1}{3n+2}$ *we generate and plot the graph of the sequence generated by (1.4) for the initial points* $x_0 = 0.95$, 0.65 *and* 0.35.

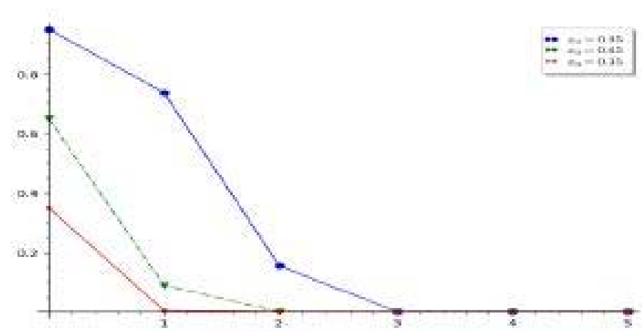

Figure 4.1: N-iteration with different initial points

n	$x_0 = 0.95$	$x_0 = 0.65$	$x_0 = 0.35$
1	0.73702761	0.09030212	0.00472699
2	0.15549611	0.00001090	7.98893E-11
3	0.00008897	1.98586E-21	5.72818E-42
4	8.28630E-18	2.05652E-84	1.42364E-166
5	6.01349E-70	2.28148E-336	5.23957E-665
6	1.62896E-278	3.37503E-1344	9.3883E-2659
7	8.62556E-1113	1.58945E-5375	9.51667E-10634
8	6.696895E-4450	7.7217E-21501	9.9235E-42534
9	2.41008E-17798	4.2598E-86002	1.1619E-170133
10	4.0117E-71192	3.9153E-344007	2.1677E-680533

From Table 1 and Fig. 4.1, it is seen that for any x_0 in K, the sequence $\{x_n\}$ generated by (1.4) converges to the fixed point 0 of T.

Next, we consider a numerical example and compare the convergence rate of N-iteration scheme (1.4) against that of M-iteration scheme (1.3).

Example 4.2. *Consider the uniformly convex metric space $(X, d, \mathcal{W})$, where $X = \mathbf{R}$ and $\mathcal{W} : X \times X \times [0,1] :\longrightarrow X$ is defined by $\mathcal{W}(x,y,t) = tx+(1-t)y$ with the usual metric.*

Let K be the closed convex subset $[0,1]$ of X and $T : K \longrightarrow K$ be defined by $Tx = \frac{2}{3}x$ for all x in K. Then it is easily seen that T is fundamentally non-expansive and $F(T) \neq \emptyset$. Moreover, since $F(T) = \{0\}$ and T satisfies condition (I) for $f(t) = t$, the sequences $\{u_n\}$ and $\{x_n\}$ generated respectively by (1.3) and (1.4) both converges to 0, the fixed point of T.

Now, the iteration (1.3) reduces to

$$\left.\begin{array}{lcl} x_{n+1} & = & Ty_n, \\ y_n & = & Tz_n, \\ z_n & = & (1-\alpha_n)x_n + \alpha_n T x_n \end{array}\right\}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$, which may be written as

$$x_{n+1} = T^2\Big((1-\alpha_n)x_n + \alpha_n T x_n\Big), \quad n = 0, 1, 2, \ldots$$

Taking $\alpha_n = \frac{n+1}{3n+2}$ *we generate and plot the graph of the sequences generated by (1.3) and (1.4) for the initial point* $x_0 = 0.75$.

n	N-iteration	M-iteration
2	0.0987654320987654	0.106995884773663
3	0.0347508001828989	0.040695016003658
4	0.0121686303670757	0.015253895455084
5	0.0042493629853280	0.005645941597439
6	0.0014812594284367	0.002068047307533
7	0.0005156977269372	0.000751047977717
8	0.0001793731224129	0.000270842657831
9	0.0000623462134882	0.000097103656711
10	0.0000216579438937	0.000034645377789

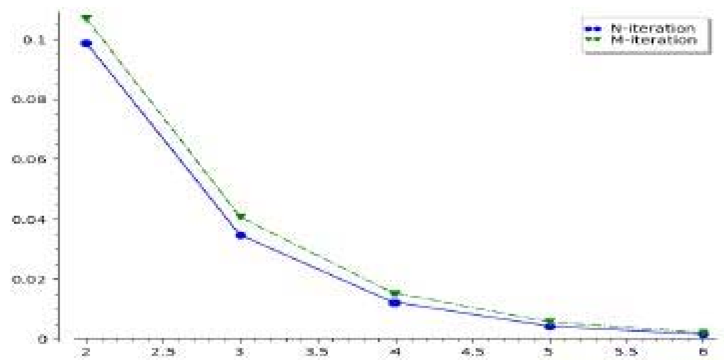

Figure 4.2: Rate of convergence

From Table 2 and Figure 4.2, we can see that the considered iteration converges to the fixed point of T *faster than that of M-iteration.*

Remark 1. *It is interesting to note the following. Considering the following iteration scheme in the settings of Example 4.2. For* x_0 *in* K,

$$\left.\begin{array}{lll} x_{n+1} & = & (1-\alpha_n)Ty_n + \alpha_n y_n, \\ y_n & = & Tz_n, \\ z_n & = & Tx_n \end{array}\right\} \quad (4.1)$$

where $\{\alpha_n\}$ *is a real sequence in* $(0, 1)$.

Then with initial points $x_0 = 0.45$, $x_0 = 0.65$ *and* $x_0 = 0.75$, *the sequences generated by (1.4) and (4.1) are identical.*

The equivalence of the two iterations is however not obtained when taking $x_0 = 0.65$ *with* $Tx = x^2$.

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