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Near-Zumkeller numbers

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Abstract

A positive integer n is called a Zumkeller number if the set of all the positive divisors of n can be partitioned into two disjoint subsets, each summing to $\frac{\sigma(n)}{2}$. In this paper, Generalizing further, near-Zumkeller numbers and k-near-Zumkeller numbers are defined and also some results concerning these numbers are established. Relations of these numbers with practical numbers are also studied in this paper.

Key-Words: *Perfect Numbers; Zumkeller Numbers; Practical Numbers; Fermat Primes.*

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1. Introduction

A positive integer n is said to be a perfect number if $\sum_{d|n, d\in \mathbf{N}} d = 2n$. The study of perfect numbers and its generalizations are one of the most oldest and beautiful area of research in number theory. The study of perfect numbers was originated most probably in the early Greek times and its generalizations were developed in the seventeenth century.

Zumkeller numbers are the recent development in the study of generalized perfect numbers. In 2003, R. H. Zumkeller generalized the concept of perfect numbers by defining a new type of number called Zumkeller number, by partitioning the set of all the positive divisors of an integer n into two disjoint subsets of equal sum. In 2008 Clark et al. announced [4] several results and conjectures related to Zumkeller numbers. In [5], Y. Peng and K. P. S. Bhaskara Rao established several results on Zumkeller numbers. In [3] the authors have studied Zumkeller numbers with two distinct prime factors and the relation of these numbers with practical numbers and harmonic mean numbers are also studied.

In section 2 of this paper, Zumkeller numbers are generalized to near-Zumkeller numbers and various results on these numbers are studied. Several results on practical numbers and Fermat primes in connection with near-Zumkeller numbers are also derived in this section. In section 3, near-Zumkeller numbers are further generalized to k-near-Zumkeller numbers and several results on these numbers are also established.

2. Near-Zumkeller Numbers

Definition 2.1. *A positive integer n is a near-Zumkeller number if we can partition the set of all the positive divisors of n into two disjoint subsets of equal sum, except for one of the divisor d where $1 < d < n$. The divisor d is called the redundant divisor. A near-Zumkeller partition for a positive integer n is a partition $\{P, Q\}$ of the set of positive divisors of n such that sum of all the elements in both P and Q is equal to $\frac{\sigma(n)-d}{2}$.*

The numbers $12, 18, 20, 24, 30, 36$ are the first few near-Zumkeller numbers. 18 is a near-Zumkeller number with redundant divisor 3.

Proposition 2.1. *[2] Let the prime factorization of a positive integer n be*

$\prod_{i=1}^{s} p_i^{m_i}$, *then the sum of all the positive divisors of* n *is given by the sigma function* $\sigma(n)$ *and*

$$\sigma(n) = \prod_{i=1}^{s} \frac{p_i^{m_i+1} - 1}{p_i - 1}.$$

Proposition 2.2. *If* n *is a near-Zumkeller number with redundant divisor* d *then*

(a) $\sigma(n)$ *is even (odd) if and only if* d *is even (odd).*

(b) $\sigma(n) \geq 2n + d$.

(c) *The prime factorization of* n *must include atleast one odd prime to an odd (even) power if* d *is even (odd).*

Proof. The proof follows as in [5].

(a) Let $\sigma(n)$ be even (odd). Since, n is a near-Zumkeller number with redundant divisor d, so $\sigma(n) - d$ must be even. Therefore d must be even (odd). Converse part can be shown easily.

(b) Let, n be a near-Zumkeller number with redundant divisor d with near-Zumkeller partition $\{P, Q\}$. Without loss of generality we can assume that $n \in P$. Therefore the sum in P is atleast n. Hence we can conclude that $\sigma(n) \geq 2n + d$.

(c) Let, d be even. Therefore $\sigma(n)$ must be even. Hence, the number of odd positive divisors of n must be even. Let the prime factorization of n be $2^l p_1^{l_1} p_2^{l_2} ... p_m^{l_m}$. So total number of odd positive divisors of n is $(l_1 + 1)(l_2 + 1) ... (l_m + 1)$. To make the total number of odd positive divisors of n is even, atleast one of l_i must be odd.

Similarly, we can show that if d is odd then one of the l_i must be even.

□

Proposition 2.3. *[5] A positive integer* n *is a near-Zumkeller number with redundant divisor* d *if and only if* $\frac{\sigma(n)-d}{2} - n$ *is a sum of distinct proper positive divisors of* n*.*

Proof. Let, n be a near-Zumkeller integer with near-Zumkeller partition $\{P, Q\}$. Without loss of generality we can assume that $n \in P$ then $\frac{\sigma(n)-d}{2} - n$ is the sum of the remaining elements of P.

Conversely, let there exist a set of positive divisors of n summing to $\frac{\sigma(n)-d}{2} - n$. If we add n to this set then we have a set of positive divisors of n summing to $\frac{\sigma(n)-d}{2}$. The complementary set of positive divisors of n excluding d sums to the same value $\frac{\sigma(n)-d}{2}$. Hence, these two sets form a near-Zumkeller partition for the integer n. □

Proposition 2.4. *There is no near-Zumkeller number of the form 2^l where $l \geq 1$.*

Proof. Let, 2^l be a near-Zumkeller number with redundant divisor 2^m, where $0 < m < l$, then

$$\sigma\left(2^l\right) \geq 2.2^l + 2^m \Rightarrow 2^{l+1} - 1 \geq 2^{l+1} + 2^m$$

this leads to $2^m \leq -1$, a contradiction.

Hence, there is no near-Zumkeller number of the form 2^l where $l \geq 1$. □

Alternatively, we can prove the above theorem by the following way,

Let, 2^l be a near-Zumkeller number with redundant divisor 2^m, where $0 < m < l$.

$\sigma\left(2^l\right)$ is odd so the redundant divisor must also be odd. Which contradicts the fact that the redundant divisor is of the form 2^m.

Proposition 2.5. *Let p be an odd prime and k be a positive integer then the number $2^k p$ is a near-Zumkeller number with redundant divisor d if $2^{k+1} - 1 \geq p + d$.*

Proposition 2.6. *If $n = 2^2 F_m$ is a near-Zumkeller number, then $n = 12$ or $n = 20$ where $F_m = 2^m + 1$ is a Fermat prime.*

Proof. Let, $n = 2^2 F_m$ be a near-Zumkeller number with redundant divisor d, then

$$\sigma(n) \geq 2.2^2 F_m + d \Rightarrow 7(F_m + 1) \geq 8F_m + d \Rightarrow d \leq 7 - F_m$$

which gives $F_m = 3, 5$. □

Proposition 2.7. *If $n = 2^k F_m p$ is a near-Zumkeller number with redundant divisor $2^k F_m$, then $p \leq \frac{(2^k-1)F_m+(2^{k+1}-1)}{F_m-(2^{k+1}-1)}$, where $F_m > 2^{k+1}$ is a Fermat prime and p is an odd prime such that $(p, F_m) = 1$.*

Proof. Since, $n = 2^k F_m p$ is a near-Zumkeller number with redundant divisor $2^k F_m$

$$\sigma(n) \geq 2n + 2^k F_m \Rightarrow \left(2^{k+1} - 1\right)(F_m + 1)(p + 1) \geq 2^{k+1} F_m p + 2^k F_m$$

$$\Rightarrow -F_m p + \left(2^k - 1\right) F_m + \left(2^{k+1} - 1\right) p + \left(2^{k+1} - 1\right) \geq 0$$
$$\Rightarrow p\{F_m - \left(2^{k+1} - 1\right)\} \leq \left(2^k - 1\right) F_m + \left(2^{k+1} - 1\right)$$
$$\Rightarrow p \leq \frac{(2^k-1)F_m+(2^{k+1}-1)}{F_m-(2^{k+1}-1)}.$$

Here, we must have

$$F_m - \left(2^{k+1} - 1\right) > 0 \Rightarrow F_m \geq 2^{k+1}.$$

But F_m is Fermat prime so $F_m > 2^{k+1}$. □

Example 2.1. *For $F_m = 17$ we have $n = 2^3 F_m p$ is a near-Zumkeller number with redundant divisor 136 for all $p \leq 67$, $(p, F_m) = 1$. In particular taking $p = 59$ we have $n = 8024$ is a near-Zumkeller number with redundant divisor 136 with near-Zumkeller partition $\{8, 8024\}$ and* $\{1, 2, 4, 17, 34, 59, 68, 118, 236, 472, 1003, 2006, 4012\}$.

Proposition 2.8. *There is no near-Zumkeller number of the form $n = p_1 p_2$ where p_1 and p_2 are distinct odd primes.*

Proof. Let, if possible $n = p_1p_2$ be a near-Zumkeller number with redundant divisor d then either $d = p_1$ or $d = p_2$.

Without loss of generality let $d = p_1$, then

$$\sigma(p_1p_2) \geq 2p_1p_2 + p_1 \Rightarrow (p_1 + 1)(p_2 + 1) \geq 2p_1p_2 + p_1$$

which leads to $p_2(p_1 - 1) \leq 1$, a contradiction. □

Alternatively, we can prove the above theorem by the following way,

Let if possible $n = p_1p_2$ be a near-Zumkeller number with redundant divisor d then either $d = p_1$ or $d = p_2 \Rightarrow d$ is an odd number.

$\sigma(p_1p_2) = (1 + p_1)(1 + p_2)$ is an even number, so the redundant divisor d must be even, a contradiction.

Hence $n = p_1p_2$ is not a near-Zumkeller number.

Proposition 2.9. *There is no near-Zumkeller number of the form $p_1^{\alpha}p_2^{\beta}$ where p_1 and p_2 are distinct odd primes and α and β are natural numbers.*

Proof. The proof is identical to proof of the Lemma 2.1 of [3]. □

Proposition 2.10. *[1] If 2^{k-3} is a prime, then $n = 2^{k-1}\left(2^k - 3\right)$ is a near-Zumkeller number with redundant divisor 2.*

Proposition 2.11. *[1] If $2^k - 5$ is a prime, then $n = 2^{k-1}\left(2^k - 5\right)$ is a near-Zumkeller number with redundant divisor 2 or 4.*

Example 2.2. *Taking $k = 4$ we get $n = 88$ is a near-Zumkeller number with redundant divisor 4 and taking $k = 6$ we get $n = 1888$ is a near-Zumkeller number with redundant divisor 2.*

Proposition 2.12. *[1] For $m < k$, if $2^k - F_m$ is a prime, then $n = 2^{k-1}\left(2^k - F_m\right)$ is a near-Zumkeller number with redundant divisor less than equal to 2^m, where $F_m = 2^m + 1$ is a Fermat prime.*

Definition 2.2. *A positive integer n is called a practical number if every positive integer less than n can be represented by as a sum of distinct proper positive divisors of n.*

Proposition 2.13. *[5] A practical number n is a near-Zumkeller number with redundant divisor d if and only if $\sigma(n) - d$ is even and $\sigma(n) \geq 2n + d$.*

Proposition 2.14. *If n is an odd near-Zumkeller number with odd redundant divisor d then n must be a perfect square.*

Proof. Since n is a near-Zumkeller number with redundant divisor d. Therefore $\sigma(n) - d$ must be even. Again since d is odd, so $\sigma(n)$ must be odd. Since n is odd, hence n must be a perfect square. □

3. k-Near-Zumkeller Numbers

Definition 3.1. *A positive integer n is called a k-near-Zumkeller number if we can partition the set of all the positive divisors of n into two disjoint subsets of equal sum, except for k number of positive divisors of n say $d_1, d_2, ..., d_k$, where $1 < d_i < n, \forall i = 1, 2, ..., k$. The divisors $d_1, d_2, ..., d_k$ are called the redundant divisors. A k-near-Zumkeller partition for a positive integer n is a partition $\{P, Q\}$ of the set of positive divisors of n such that sum of all the elements in both P and Q is equal to $\dfrac{\sigma(n) - \sum_{i=1}^{k} d_i}{2}$.*

Near-Zumkeller numbers are 1-near-Zumkeller numbers. 24 is a 2-near-Zumkeller number with redundant divisors $4, 6$ or $4, 8$. Similarly, 30 is also a 2-near-Zumkeller number with redundant divisors $3, 5$. 24 is the first 3-near-Zumkeller number with redundant divisors $2, 4, 6$. 945 is the first odd 2-near-Zumkeller number with redundant divisor $5, 7$.

Proposition 3.1. *If n is a k-near-Zumkeller number with redundant divisors $d_1, d_2, ..., d_k$ then*

(a) $\sigma(n)$ is even (odd) if and only if $\sum_{i=1}^{k} d_i$ is even (odd).

(b) $\sigma(n) \geq 2n + \sum_{i=1}^{k} d_i$.

(c) If $\sum_{i=1}^{k} d_i$ is even (odd) then the prime factorization of n must include atleast one odd prime to an odd (even) power.

Proposition 3.2. *Any positive integer n is a k-near-Zumkeller number with redundant divisors $d_1, d_2..., d_k$ if and only if $\frac{\sigma(n) - \sum_{i=1}^{k} d_i}{2} - n$ is a sum of distinct proper positive divisors of n.*

Proposition 3.3. *There is no k-near-Zumkeller number of the form 2^l where $l \geq 1$,$k < l$.*

Proof. Let 2^l be a k-near-Zumkeller number with redundant divisors $2, 2^2, ..., 2^k$, $0 < k < l$ then

$$\sigma\left(2^l\right) \geq 2.2^l + \sum_{m=1}^{k} 2^m \Rightarrow 2^{l+1} - 1 \geq 2^{l+1} + 2\left(2^k - 1\right)$$

this leads to $2^{k+1} \leq 1$, a contradiction.

Hence, there is no k-near-Zumkeller number of the form 2^l where $l \geq 1$, $k < l$. □

Proposition 3.4. *If n is a near-Zumkeller number with redundant divisor d and p is a prime such that $(n, p) = 1$, then np^l is a $(l + 1)$-near-Zumkeller number with redundant divisors $d, dp, dp^2, ..., dp^l$.*

Proof. Let $\{P, Q\}$ be the near-Zumkeller partition of the integer n with redundant divisor d. Then clearly, $\{P \cup (pP) \cup (p^2P) \cup ... \cup \left(p^lP\right), Q \cup (pQ) \cup (p^2Q) ... \cup \left(p^lQ\right)\}$ is a near-Zumkeller partition of np^l with redundant divisors $d, dp, dp^2, ..., dp^l$. □

Example 3.1. *We know $n = 12$ is a near-Zumkeller number with redundant divisor 4 and if we take $p = 5$ then $12 \times 5 = 60$ is a 2-near-Zumkeller number with redundant divisors $4, 20$. Similarly, 12×5^2 is a 3-near-Zumkeller number with redundant divisors $4, 20, 100$.*

Corollary 3.5. *If n is a near-Zumkeller number with redundant divisor d and m is relatively prime to n, then mn is a 2-near-Zumkeller number with redundant divisors d and dm.*

Proposition 3.6. *Let $p_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$ be the prime factorization of a positive integer n and also let n be a near-Zumkeller number with redundant divisor d then $p_1^{\alpha_1+\beta(\alpha_1+1)} p_2^{\alpha_2} ... p_m^{\alpha_m}$ is a $(\beta + 1)$-near-Zumkeller number with redundant divisors $d, dp_1^{(\alpha_1+1)}, dp_1^{2(\alpha_1+1)}, ..., dp_1^{\beta(\alpha_1+1)}$ for any positive integer β.*

Proof. Let, $\{P, Q\}$ be the near-Zumkeller partition of the integer n with redundant divisor d. Now if M is the set of all the positive divisors of n except d then the set of positive divisors of $p_1^{\alpha_1+\beta(\alpha_1+1)}p_2^{\alpha_2}...p_m^{\alpha_m}$ is $M\cup\left(p_1^{(\alpha_1+1)}M\right)\cup\left(p_1^{2(\alpha_1+1)}M\right)\cup...\cup\left(p_1^{\beta(\alpha_1+1)}M\right)\cup d\cup dp_1^{(\alpha_1+1)}\cup dp_1^{2(\alpha_1+1)}\cup ...\cup dp_1^{\beta(\alpha_1+1)}$.

Now, the two sets $P\cup\left(p_1^{(\alpha_1+1)}P\right)\cup\left(p_1^{2(\alpha_1+1)}P\right)\cup...\cup\left(p_1^{\beta(\alpha_1+1)}P\right)$ and $Q\cup\left(p_1^{(\alpha_1+1)}Q\right)\cup\left(p_1^{2(\alpha_1+1)}Q\right)\cup...\cup\left(p_1^{\beta(\alpha_1+1)}Q\right)$ serve as the near-Zumkeller partition sets for the integer $p_1^{\alpha_1+\beta(\alpha_1+1)}p_2^{\alpha_2}...p_m^{\alpha_m}$ with redundant divisors $d, dp_1^{(\alpha_1+1)}, dp_1^{2(\alpha_1+1)}, ..., dp_1^{\beta(\alpha_1+1)}$. □

Example 3.2. *We have* $12 = 2^2 \times 3$ *is a near-Zumkeller number with redundant divisor* 4*. Taking* $\beta = 2$ *we have* $2^{2+2(2+1)} \times 3$=*768 is a* 3*-near-Zumkeller number with redundant divisors* $4, 32, 256$.

Proposition 3.7. *Let* $n = 2^l M$ *(where* l *is a non-negative integer and* M *is an odd integer) be a near-Zumkeller number with redundant divisor* d*. Also let* $m_1, m_2...m_r$ *are the positive divisors of* M*. Then*

(a) $2n$ *is a near-Zumkeller number with redundant divisor* d *if* $d \neq 2^l m_i$ *for all* $i = 1, 2, ..., r$.

(b) $2n$ *is a* 2*-near-Zumkeller number with redundant divisors* $d, 2d$ *if* $d = 2^l m_i$ *for some* $i = 1, 2, ..., r$.

Proof. Let, $\{P, Q\}$ be the near-Zumkeller partition for n. Now every positive divisor of $2n$ which are different from the positive divisor of n is of the form $2^{l+1}m_i$, $i = 1, 2, ..., r$.

(a) Let, $d \neq 2^l m_i$ for all $i = 1, 2, ..., r$. Then we have $2^l m_i$ is either in P or in Q for all $i = 1, 2, ..., r$. Without loss of generality we can assume that $2^l m_i \in P$ for some i. In this case, we move $2^l m_i$ to Q and add $2^{l+1}m_i$ to P. Continuing this process to all the remaining positive divisors of $2n$ which are not the positive divisors of n, we will get an equal-summed partition of all the positive divisors of $2n$. Hence, $2n$ is a near-Zumkeller number with redundant divisor d.

(b) Let $d = 2^l m_i$ for some $i = 1, 2, ..., r$ then we take $2d = 2^{l+1}m_i$ as another redundant divisor and for all the remaining i for which $d \neq 2^l m_i$ we have either $2^l m_i \in P$ or $2^l m_i \in Q$. If we assume that (without loss of generality) $2^l m_i$ is in P for some i then we move $2^l m_i$ to Q and add $2^{l+1}m_i$ to Q. Continuing this process for all the i for which $d \neq 2^l m_i$ will yield an equal summed partition of the positive divisors of $2n$ with redundant divisors $d, 2d$. Hence $2n$ is a 2-near-Zumkeller number with redundant divisors $d, 2d$ if $d = 2^l m_i$ for some $i = 1, 2, ..., r$.

□

Example 3.3. *1. $n = 24 = 2^3 \times 3$ is a near-Zumkeller number with redundant divisor 4. Thus by the above proposition $2n = 48$ is a near-Zumkeller number with redundant divisor 4.*

2. $n = 30 = 2 \times 15$ is a near-Zumkeller number with redundant divisor 6. Thus by the above proposition $2n = 60$ is a 2-near-Zumkeller number with redundant divisors 6, 12.

Proposition 3.8. *Let n be a practical number then n is a k-near-Zumkeller number with redundant divisors $d_1, d_2, ..., d_k$ if and only if $\sigma(n) - \sum_{i=1}^{k} d_i$ is even and $\sigma(n) \geq 2n + \sum_{i=1}^{k} d_i$.*

Proposition 3.9. *Let n be a practical number and p be a prime such that $(n, p) = 1$. Then for the positive integers l and d, (where $d|n, 1 < d < n$) if $\sigma(n) - d$ is even and $\sigma(n) \geq 2n + d$ then np^l is a $(l+1)$-near-Zumkeller number with redundant divisors $d, dp, dp^2, ..., dp^l$.*

Proof. Since $\sigma(n) - d$ is even and $\sigma(n) \geq 2n + d$. n is a near-Zumkeller number with redundant divisor d by Proposition 2.13. $\Rightarrow np^l$ is a $(l+1)$-near-Zumkeller number with redundant divisors $d, dp, dp^2, ..., dp^l$; by Proposition 3.4. □

Example 3.4. *Taking $n = 18$, $d = 3$, $p = 5$ and $l = 3$ by the above Proposition we have $18 \times 5^3 = 2250$ is a 4-near-Zumkeller number with redundant divisors 3, 15, 75, 375.*

4. Conclusion

In this paper we have studied near-Zumkeller numbers and k-near-Zumkeller numbers. We may attempt to study

1. the odd near-Zumkeller numbers.

2. the graphical and algebraic structures of near and k-near-Zumkeller numbers.

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