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Stability problem in a set of Lebesgue measure zero of bi-additive functional equation

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Abstract

Let X be a vector space and Y be a Banach space. Our aim in this paper is to investigate the Hyers-Ulam stability problem of the following bi-additive functional equation

$$f(x+y,s-t)+f(x-y,s+t)=2f(x,s)-2f(y,t), \quad x,y,s,t \in X,$$

where $f : X \times X \to Y$. As a consequence, we discuss the stability of the considered functional equation in a restricted domain and in the set of Lebesgue measure zero.

Keywords: *Bi-additive functional equation; Hyers-Ulam stability; functional equation; Baire category theorem; First category; Lebesgue measure.*

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1. Introduction

Let X be a vector space and Y be a Banach space. Throughout this paper, we denote by $\mathbf{N}$ the set of natural numbers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and by $\mathbf{R}$ the set of real numbers.

In 1989, Aczél and Dhombres [1] proved that a mapping $g : X \to Y$ satisfies the following quadratic functional equation

$$g(x+y) + g(x-y) = 2g(x) + 2g(y)$$

if and only if there exists a symmetric bi-additive mapping $B : X \times X \to Y$ such that $g(x) = B(x, x)$ where

$$B(x,y) = \frac{1}{4}\Big(g(x+y) - g(x-y)\Big), \quad x, y \in X.$$

We recall that a function $f : X \times X \to Y$ is bi-additive provided

$$f(x+y,s) = f(x,s) + f(y,s) \qquad \text{and} \qquad f(x,s+t) = f(x,s) + f(x,t)$$

for all $x, t, s, t \in X$. Consider the functional equation

$$(1.1) \quad f(x+y,s-t) + f(x-y,s+t) = 2f(x,s) - 2f(y,t), \quad x,y,s,t \in X.$$

It is easy to show that $f : X \times X \to Y$ is bi-additive, if and only if, it fulfils Eq. (1.1) for every $x, y, s, t \in X$. Therefore, we can say that Eq. (1.1) is a bi-additive functional equation.
W. G. Park and J. H. Báe [23] have solved and have investigated the stability of Eq. (1.1) in Banach modules over an unital C^*-algebra. In 2017, J. Berzdęk et al. [12] have proved the stability and hyperstability of Eq. (1.1) by using fixed point theorem as a basic tool under some weak assumptions. Let us mention that the concept of stability problem has been a very popular subject of investigation for the last eighty years. The study of such problem was motivated by the following question of S.M. Ulam [28] in 1940.

Ulam's Problem:*Let $(G_1, *_1)$ be a group and let $(G_2, *_2)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality*

$$d\Big(h(x *_1 y), h(x) *_2 h(y)\Big) < \delta$$

for all $x, y \in G_1$*, then there exists a homomorphism* $H : G_1 \to G_2$ *with*

$$d\Big(h(x), H(x)\Big) < \varepsilon$$

for all $x \in G_1$*?*
The affirmative answer of this question is the equation of homomorphism $h(x *_1 y) = h(x) *_2 H(y)$ is stable.
In 1941, D. H. Hyers [17] published the first answer to Ulam's problem in the case of Banach spaces as follows

Theorem 1.1. *[17] Let* E_1 *and* E_2 *be two Banach spaces and* $f : E_1 \to E_2$ *be a function such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ *and for all* $x, y \in E_1$*. Then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$*, and* $A : E_1 \to E_2$ *is the unique additive function such that*

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in E_1$*.*

In 1950, T. Aoki [2], D. G. Bourgin [4] considered the stability problem with unbounded Cauchy differences. In 1978, Th. M. Rassias [24] provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded.

Theorem 1.2. *Let* $f : X \to Y$ *be a mapping satisfying the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in X\ \{0\}$*, where* θ *and* p *constants with* $\theta > 0$ *and* $p \neq 1$*. Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x+y) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p,$$

for all $x \in X\ \{0\}$*.*

Theorem 1.2 is due to T. Aoki [2] for $0 < p < 1$, Z. Gajda [14] for $p > 1$, D. H. Hyers[17] for $p = 0$ and Th. M. Rassias [25] for $p < 0$. Subsequently, several authors have studied different functional equations in various spaces (see, for example, [5, 7, 11, 15, 24]). The stability problem of functional equations on a restricted domain have been extensively investigated by a number of authors (see, for example, [6, 8, 9, 13, 16, 19, 20, 27]). S. M. Jung [18] and J. M. Rassias [26] proved the Hyers-Ulam stability of the quadratic functional equation in a restricted domain.
It's very natural to ask if the restricted domain $D = \{(x, y), (s, t) \in X^2 \times X^2 : \|(x, y)\| + \|(s, t)\| \geq d\}$ can be replaced by much smaller $\Gamma \subset D$ (i.e. a subset of measure zero) in a measurable space X.
In 2013, J. Chung [9] found the answer to this question by considering the stability of the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

in a set $\Gamma \subset \{(x, y) \in \mathbf{R}^2 : |x| + |y| \geq d\}$ where $m(\Gamma) = 0$ and $f : \mathbf{R} \to \mathbf{R}$.
In 2014, J. Chung and J.M. Rassias [10] proved the stability of the quadratic functional equation in a set of measure zero.
Our goal, in this paper, is to investigate the Hyers-Ulam stability on a set $\Gamma \subset X^4$ of measure zero for the bi-additive functional equation (1.1). In addition, we apply these results to the asymptotic behavior of Eq. (1.1).

2. Measure zero stability

First, we first study the Hyers-Ulam stability of Eq. (1.1) on X by the use of the direct method and then we deduce the measure zero stability for this equation.

Theorem 2.1. *Let $\varepsilon \geq 0$ be fixed, X be a vector space and Y a Banach space. If a function $f : X \times X \longrightarrow Y$ such that $f(x, -s) = -f(x, s)$, $x, s \in X$ satisfies the inequality*

$$\|f(x + y, s - t) + f(x - y, s + t) - 2f(x, s) + 2f(y, t)\| \leq \varepsilon, \tag{2.1}$$

for all $x, y, s, t \in X$, then there exists a unique bi-additive mapping $B : X \times X \longrightarrow Y$ such that

$$\Big\| f(x, s) - B(x, s) \Big\| \leq \frac{1}{3}\varepsilon$$

for all $x, s \in X$.

Proof. Letting $y = x$ and $s = -t$ in the inequality (2.1), we have

$$\|f(2x, 2s) - 4f(x, s)\| \leq \varepsilon, \quad x, s \in X. \tag{2.2}$$

Let $k \in \mathbf{N}$, replacing x by $2^{k-1}x$ and s by $2^{k-1}s$, where $k \in \mathbf{N}$, in (2.2), we obtain

$$\|f(2^k x, 2^k s) - 4f(2^{k-1}x, 2^{k-1}s)\| \leq \varepsilon, \tag{2.3}$$

for all $x, s \in X$ and $k = 1, 2, ..., n$.
Multiplying both sides of the above inequality by $\frac{1}{4^k}$ and adding the resulting n equalities, we get

$$\sum_{k=1}^{n} \frac{1}{4^k} \left\| f(2^k x, 2^k s) - 4f(2^{k-1}x, 2^{k-1}s) \right\| \leq \sum_{k=1}^{n} \frac{\varepsilon}{4^k}, \ \ x, s \in X \tag{2.4}$$

which yields

$$\sum_{k=1}^{n} \frac{1}{4^k} \left\| f(2^k x, 2^k s) - 4f(2^{k-1}x, 2^{k-1}s) \right\| \leq \frac{\varepsilon}{3} \left(1 - \frac{1}{4^n} \right), \ \ x, s \in X. \tag{2.5}$$

Using the triangle inequality, we obtain

$$\left\| \frac{1}{4^n} f(2^n x, 2^n s) - f(x, s) \right\| \leq \frac{\varepsilon}{3} \left(1 - \frac{1}{4^n} \right), \ \ x, s \in X. \tag{2.6}$$

Now, if $n > m > 0$, then $n - m$ is a natural number and we can replace n by $n - m$ in (2.6) to obtain

$$\left\| \frac{f(2^{n-m}x, 2^{n-m}s)}{4^{n-m}} - f(x, s) \right\| \leq \frac{\varepsilon}{3} \left(1 - \frac{1}{4^{n-m}} \right), \tag{2.7}$$

for all $x, s \in X$. Multiplying both sides by $\frac{1}{4^m}$ and simplifying, we get

$$\left\| \frac{f(2^{n-m}x, 2^{n-m}s)}{4^n} - \frac{f(x, s)}{4^m} \right\| \leq \frac{\varepsilon}{3} \left(\frac{1}{4^m} - \frac{1}{4^n} \right), \tag{2.8}$$

for all $x, s \in X$. Replacing x by $2^m x$ and s by $2^m s$ in (2.8), we conclude that

$$\left\| \frac{f(2^n x, 2^n s)}{4^n} - \frac{f(2^m x, 2^m s)}{4^m} \right\| \leq \frac{\varepsilon}{3} \left(\frac{1}{4^m} - \frac{1}{4^n} \right), \ \ x, s \in X. \tag{2.9}$$

If $m \to \infty$ in (2.9), then $\frac{1}{4^m} - \frac{1}{4^n} \to 0$ and we have

$$\lim_{m\to\infty} \left\| \frac{f(2^n x, 2^n s)}{4^n} - \frac{f(2^m x, 2^m s)}{4^m} \right\| = 0,$$

for all $x, s \in X$. Hence, $\left\{ \frac{f(2^n x, 2^n s)}{4^n} \right\}_{n=1}^{\infty}$ is a Cauchy sequence in Y and the limit of this sequence exists.
Define $B : X \times X \to Y$ by

$$B(x,s) := \lim_{n\to\infty} \frac{f(2^n x, 2^n s)}{4^n}, \quad x, s \in X.$$

We show that $B : X \times X \to Y$ is a bi-additive function. For this goal, we consider

$$\begin{aligned}
&\|B(x+y, s-t) + B(x-y, s+t) - 2B(x,s) + 2B(y,t)\| \\
&= \left\| \lim_{n\to\infty} \left\{ \frac{f(2^n(x+y), 2^n(s-t))}{4^n} + \frac{f(2^n(x-y), 2^n(s+t))}{4^n} - 2\frac{f(2^n x, 2^n s)}{4^n} \right.\right. \\
&\left.\left. +2\frac{f(2^n y, 2^n t)}{4^n} \right\} \right\| \\
&= \lim_{n\to\infty} \frac{1}{4^n} \Big\| f(2^n x + 2^n y, 2^n s - 2^n t) + f(2^n x - 2^n y, 2^n s + 2^n t) \\
&-2f(2^n x, 2^n s) + 2f(2^n y, 2^n t) \Big\| \\
&\leq \lim_{n\to\infty} \frac{\varepsilon}{4^n} \\
&= 0,
\end{aligned}$$

for all $x, y, s, t \in X$. Therefore,

$$B(x+y, s-t) + B(x-y, s+t) = 2B(x,s) - 2B(y,t),$$

for all $x, y, s, t \in X$.
The next goal is to show, for each $x, s \in X$, that

$$\|B(x,s) - f(x,s)\| \leq \frac{\varepsilon}{3}.$$

Indeed, for every $x, s \in X$, we have

$$\begin{aligned}
\|B(x,s) - f(x,s)\| &= \left\| \lim_{n\to\infty} \frac{f(2^n x, 2^n s)}{4^n} - f(x,s) \right\| \\
&= \lim_{n\to\infty} \left\| \frac{f(2^n x, 2^n s)}{4^n} - f(x,s) \right\| \\
&\leq \lim_{n\to\infty} \frac{\varepsilon}{3} \left(1 - \frac{1}{4^n}\right) = \frac{\varepsilon}{3}.
\end{aligned}$$

Thus, we deduce that

$$\|B(x,s) - f(x,s)\| \leq \frac{\varepsilon}{3},$$

for all $x, s \in X$. Finally, we prove the uniqueness of B. We assume that there exists an other bi-additive mapping $C : X \times X \to Y$ such that

$$\|C(x,s) - f(x,s)\| \le \frac{\varepsilon}{3}$$

for all $x, s \in X$. Therefore, for every $x, s \in X$, we have

$$\begin{aligned} \|B(x,s) - C(x,s)\| &\le \|B(x,s) - f(x,s)\| + \|C(x,s) - f(x,s)\| \\ &\le \tfrac{2\varepsilon}{3}. \end{aligned}$$

Since B is bi-additive, for each $n \in \mathbf{N}_0$, we obtain

$$\begin{aligned} \|B(x,s) - C(x,s)\| &= \left\| \tfrac{B(2^n x, 2^n s)}{4^n} - \tfrac{C(2^n x, 2^n s)}{4^n} \right\| \\ &= \tfrac{1}{4^n} \|B(2^n x, 2^n s) - C(2^n x, 2^n s)\| \\ &\le \tfrac{2\varepsilon}{3 \times 4^n}, \end{aligned}$$

for all $x, s \in X$. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} \|B(x,s) - C(x,s)\| \le \lim_{n\to\infty} \frac{2\varepsilon}{3 \times 4^n} = 0$$

for all $x, s \in X$ which means that $B(x,s) = C(x,s)$ for all $x, s \in X$. □

For given $x, y, s, t \in X$, we define

$$P_{x,y,s,t,a,b} :=$$

$$\Big\{(x+y, s-t, a, b); (x-y, s+t, a, b); (x, s, y+a, t+b); (x, s, y-a, t-b); (y, t, a, -b)\Big\}$$

In this section, we assume that a set $\Gamma \subset X \times X \times X \times X$ satisfies the following condition:
For given $x, y, s, t \in X$, there exists $a, b \in X$ such that

$$(C) \qquad P_{x,y,s,t,a,b} \subset \Gamma$$

In the following theorem, we prove the Hyers-Ulam stability for the bi-additive functional equation (1.1) on Γ.

Theorem 2.2. *Let $\varepsilon \ge 0$ be fixed. Suppose that the function $f : X \times X \longrightarrow Y$ such that $f(x,-s) = -f(x,s),\ \ x, s \in X$ satisfies the functional inequality*

$$\|f(x+y, s-t) + f(x-y, s+t) - 2f(x,s) + 2f(y,t)\| \le \varepsilon, \tag{2.10}$$

for all $(x, y, s, t) \in \Gamma$. Then there exists a unique bi-additive mapping $B : X \times X \longrightarrow Y$ such that

$$\|f(x,s) - B(x,s)\| \le \varepsilon$$

for all $x, s \in X$.

Proof. Suppose that $f : X \times X \to Y$ is a mapping satisfying (2.10) for all $(x, y, s, t) \in \Gamma$. Define $D_f : X \times X \times X \times X \to Y$ by

$$D_f(x, y, s, t) := f(x+y, s-t)+f(x-y, s+t)-2f(x, s)+2f(y, t), \quad (x, y, s, t) \in \Gamma.$$

Since Γ satisfies (C), for all $x, y, s, t \in X$, there exists $(a, b) \in X \times X$ such that

$$\|D_f(x+y, a, s, t-b)\| \le \varepsilon, \ \ \|D_f(x-y, a, s, b-t)\| \le \varepsilon, \ \ \|D_f(x, y+a, s, t+b)\| \le \varepsilon,$$

$$\|D_f(x, y-a, s, t-b)\| \le \varepsilon \ \text{ and } \ \|D_f(y, a, t, -b)\| \le \varepsilon$$

In view of the triangle inequality, we get

$$\begin{aligned} \|D_f(x, y, s, t)\| &= \Big\| \tfrac{1}{2} D_f(x+y, a, s, t-b) + \tfrac{1}{2} D_f(x-y, a, s, b-t) \\ &\quad + \tfrac{1}{2} D_f(x, y+a, s, t+b) + \tfrac{1}{2} D_f(x, y-a, s, t-b) \\ &\quad + D_f(y, a, t, -b) \Big\| \\ &\le \tfrac{1}{2}\|D_f(x+y, a, s, t-b)\| + \tfrac{1}{2}\|D_f(x-y, a, s, b-t)\| \\ &\quad + \tfrac{1}{2}\|D_f(x, y+a, s, t+b)\| + \tfrac{1}{2}\|D_f(x, y-a, s, t-b)\| \\ &\quad + \|D_f(y, a, t, -b)\| \\ &= 3\,\varepsilon, \end{aligned}$$

for all $x, y, s, t \in X$. According to Theorem 2.1, there exists a unique bi-additive mapping $B : X \times X \longrightarrow Y$ such that
$\|f(x, s) - B(x, s)\| \le \varepsilon$, for all $x, s \in X$. This completes the proof. □

The following corollary is a particular case of Theorem 2.2, when $\varepsilon = 0$.

Corollary 2.3. *Suppose that $f : X \times X \longrightarrow Y$ satisfies the functional equation*

$$f(x+y, s-t) + f(x-y, s+t) = 2f(x, s) - 2f(y, t) \tag{2.11}$$

for all $(x, y) \in \Gamma$. Then eq. (2.11) holds for all $x, y, s, t \in X$.

3. Applications

In this section, we construct a set Γ of measure zero satisfying the condition (C) when $X = \mathbf{R}$. The following lemma is a crucial key of our construction.

Lemma 3.1. *[22] The set $\mathbf{R}$ of real numbers can be partitioned as $\mathbf{R} = F \cup K$ where F is of first Baire category, i.e. F is a countable union of nowhere dense subsets of $\mathbf{R}$, and K is of Lebesgue measure zero.*

The following lemma was proved by J. Chung and J. M. Rassias in [9] and [10].

Lemma 3.2. *[9], [10] Let K be a subset of $\mathbf{R}$ of measure 0 such that $K^c := \mathbf{R} \setminus K$ is of first Baire category. Then, for any countable subsets $U \subset \mathbf{R}$, $V \subset \mathbf{R} \setminus \{0\}$ and $M > 0$, there exists $\lambda \geq M$ such that*

$$U + \lambda V = \{u + \lambda v : u \in U, v \in V\} \subset K. \tag{3.1}$$

In the following theorem, we give the construction of a set $\Gamma \subset \mathbf{R}^4$ of Lebesgue measure zero satisfying the condition (C).

Theorem 3.3. *Let K be the set defined as in Lemma 3.2, R be a rotation given by*

$$R = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \tag{3.2}$$

and $\Gamma = R^{-1}(K \times K \times K \times K)$. Then Γ satisfies the condition (C) and has four-dimensional Lebesgue measure 0.

Proof. Let $x, y, s, t, a, b \in \mathbf{R}$ and define

$$\begin{aligned} P_{x,y,s,t,a,b} \quad &:= \Big\{(x+y, s-t, a, b); (x-y, s+t, a, b); (x, s, y+a, t+b); \\ &(x, s, y-a, t-b); (y, t, a, -b)\Big\}. \end{aligned}$$

Then Γ satisfies the condition (C), if and only if, for every $x, y, s, t \in \mathbf{R}$, there exists $a, b \in \mathbf{R}$ such that

$$R(P_{x,y,s,t,a,b}) \subset K \times K \times K \times K. \tag{3.3}$$

The above inclusion relation (3.3) is equivalent to

$$\begin{aligned} S_{x,y,s,t,a,b} \quad &:= \Big\{ \tfrac{\sqrt{2}}{2}p_1 - \tfrac{\sqrt{2}}{2}p_3,\ \tfrac{\sqrt{2}}{2}p_2 - \tfrac{\sqrt{2}}{2}p_4,\ \tfrac{\sqrt{2}}{2}p_1 + \tfrac{\sqrt{2}}{2}p_3,\ \tfrac{\sqrt{2}}{2}p_2 + \tfrac{\sqrt{2}}{2}p_4 \\ &: (p_1, p_2, p_3, p_4) \in P_{x,y,s,t,a,b} \Big\} \subset K. \end{aligned}$$

If we choose $\alpha \in \mathbf{R}$ such that $b = \alpha a$, then we can easily check that

$$S_{x,y,s,t,a,\alpha a} = U + aV$$

where

$$U := \left\{ \frac{\sqrt{2}}{2}(x+y),\ \frac{\sqrt{2}}{2}(x-y),\ \frac{\sqrt{2}}{2}y,\ \frac{\sqrt{2}}{2}(s+t),\ \frac{\sqrt{2}}{2}(s-t),\ \frac{\sqrt{2}}{2}t \right\}$$

and

$$V := \left\{ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\alpha,\ \frac{\sqrt{2}}{2}\alpha \right\}$$

According to (3.1) in Lemma 3.2, for every $x, y, s, t \in \mathbf{R}$ and $M > 0$, there exists $a \geq M$ such that

$$S_{x,y,s,t,a,\alpha a} \subset U + aV \subset K.$$

Thus, Γ satisfies the condition (C). This completes the proof. □

Corollary 3.4. *Let $\varepsilon \geq 0$ be fixed. Suppose that the function $f : \mathbf{R}^2 \longrightarrow Y$ such that $f(x, -s) = -f(x, s)$, $x, s \in \mathbf{R}$ satisfying the functional inequality*

$$\|f(x+y, s-t) + f(x-y, s+t) - 2f(x,s) + 2f(y,t)\| \leq \varepsilon, \tag{3.4}$$

for all $(x, y, s, t) \in \Gamma$. Then there exists a unique bi-additive mapping $B : \mathbf{R}^2 \longrightarrow Y$ such that

$$\|f(x,s) - B(x,s)\| \leq \varepsilon$$

for all $(x, s) \in \mathbf{R}^2$.

Corollary 3.5. *Suppose that $f : \mathbf{R}^2 \to Y$ such that $f(x, -s) = -f(x, s)$, $x, s \in \mathbf{R}$ satisfies*

$$\|f(x+y, s-t) + f(x-y, s+t) - 2f(x,s) + 2f(y,t)\| \to 0 \tag{3.5}$$

as $(x, y, s, t) \in \Gamma$ and $|x| + |y| + |s| + |t| \to \infty$. Then f is bi-additive.

Proof. The condition (3.5) implies that, for each $n \in \mathbf{N}$, there exists $d_n > 0$ such that

$$\|f(x+y, s-t) + f(x-y, s+t) - 2f(x,s) + 2f(y,t)\| \leq \tfrac{1}{n},$$

for all $(x,y,s,t) \in \Gamma_{d_n} := \Big\{(x,y,s,t) \in \Gamma : |x|+|y|+|s|+|t| \geq d_n\Big\}$. Let $n \in \mathbf{N}$ be fixed. In view of the proof of Theorem 2.1 and the inclusion (3.3), we conclude that, for every $x,y,s,t \in \mathbf{R}$ and $M > 0$, there exist $a \in \mathbf{R}$ such that $a \geq M$ and

$$S_{x,y,s,t,a,\alpha a} \subset \Gamma. \tag{3.6}$$

For given $x,y,s,t \in \mathbf{R}$, if we take $M = d_n + |y|$ and if $|a_n| \geq M$, then we get

$$S_{x,y,s,t,a_n,\alpha a_n} \subset \Big\{(p_1,p_2,p_3,p_4) : |p_1|+|p_2|+|p_3|+|p_4| \geq d_n\Big\}. \tag{3.7}$$

It follows from (3.6) and (3.7) that, for each $x,y,s,t \in \mathbf{R}$, there exist $a_n \in \mathbf{R}$ such that

$$S_{x,y,s,t,a_n,\alpha a_n} \subset \Gamma_{d_n}. \tag{3.8}$$

So, Γ_{d_n} satisfies the condition (C). Thus, by Theorem 2.2, there exists a unique mapping $B_n : \mathbf{R}^2 \to Y$ such that B_n is a solution of (1.1) and

$$\|f(x,s) - B_n(x,s)\| \leq \frac{1}{n} \tag{3.9}$$

for all $(x,s) \in \mathbf{R}^2$. Now, replacing $n \in \mathbf{N}$ by $m \in \mathbf{N}$ in (3.9) and using the triangle inequality, we get

$$\|B_m(x,s) - B_n(x,s)\| \leq \|B_m(x,s) - f(x,s) + f(x,s) - B_n(x,s)\| \leq \frac{1}{m} + \frac{1}{n} \leq 2, \tag{3.10}$$

for all $m,n \in \mathbf{N}$ and all $(x,s) \in \mathbf{R}^2$. Hence, $B_m - B_n$ is bounded. So, we conclude that $B_m = B_n$ for all $m,n \in \mathbf{N}$. Finally, letting $n \to \infty$ in (3.9), we get the desired result. □

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