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On fuzzy γ_μ-open sets in generalized fuzzy topological spaces

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Abstract

In this paper, we explore the existence of operation approach on open sets in a generalized fuzzy topological space. We introduce the concept of fuzzy γ_μ-open set and study some basic properties of it. We obtain an interesting result that the intersection of two fuzzy γ_μ-open sets may not be a fuzzy γ_μ-open set, but if the operation is regular then the intersection becomes a fuzzy γ_μ-open set. We also initiate the notions of fuzzy minimal γ_μ-open set and fuzzy γ_μ-locally finite space and establish various results related to these.

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1. Introduction

In the year 1979, Kasahara [13] first introduced the concept of operations in general topological space. The very basic definition of operation in a topological space (X, τ) given by him is as follows:
An operation $\gamma : \tau \rightarrow P(X)$ is a function from the topology τ to the power set of X such that $V \subseteq V^{\gamma}$, for each open set $V \in \tau$, where V^{γ} denotes the value of γ at V. He had shown the existence of such kind of various operations, viz. $\gamma(V) = V, \gamma(V) = cl(V)$ etc.
The use of operations became a topic of research since its inception. The investigation on operations were done on various sets in the topological space. Krishnan et. al [14] studied the operation approach on semi-open sets in a topological space and showed various applications of the same. Van An et. al [22] extended the notion of operations on pre-open set in 2008. Tahiliani [18] defined an operation on the class of all β-open sets, whereas Carpintero et. al [3] studied the same on the class of all b-open sets in a topological space. Very recently, Hussain [11] introduced the notion of operations in generalized closed sets with the same approach and studied some of its applications. Ogata [16] called the operation α as γ-operation and he introduced τ_{γ} as the collection of all γ-open sets in a topological space (X, τ). Some more significant works on operation approaches may be seen in [1] [12].
Nakaoka and Oda [15] first initiated the notion of minimal open set in general topological space in the year 2001. After the introduction of fuzzy topology, lot of development on continuity, quasicoincedence etc which lead to mixed topology etc [9, 10, 19, 20, 21] . In literature there are several interesting research done on this concepts in various environments in recent days on generalization of open sets in crisp, as well as in fuzzy environment [2, 5, 6, 7, 8]. In this present treatise, we extend the concept of operations in fuzzy sense on generalized open set in a generalized fuzzy topological space due to Chetty [4] and examine the characteristic of these operations. Through operation approach, we define and characterize fuzzy γ_{μ}-open set in a generalized fuzzy topological space extensively and then introduce the notion of fuzzy minimal γ_{μ}-open set therein.

Some important related definitions are recalled below as ready references of our present work.

2. Preliminaries

Before going to the contribution section, we need some basic and preliminary ideas about the existing definitions and results which will play a major role in this study.

2.1. Definition (Chetty, 2008)

Let X be a non-empty set and μ be any collection of fuzzy sets having the following properties:

(i) $0_X \in \mu$ and

(ii) $\forall \lambda_1, \lambda_2 \in \mu, \lambda_1 \vee \lambda_2 \in \mu$.

Then μ is called a generalized fuzzy topology (in short, GFT) and the structure (X, μ) is called a generalized fuzzy topological space (for short, GFTS).
Each member of μ is said to be a fuzzy μ-open set and the complement of a fuzzy μ-open set is called fuzzy μ-closed set.

2.2. Definition (Chetty, 2008)

Let (X, μ) be a generalized fuzzy topological space. Then the interior of a fuzzy set λ is the supremum of all fuzzy μ-open set contained in λ and it is denoted by $i_\mu(\lambda)$. Also the closure of λ is the infimum of the fuzzy μ-closed sets which contains λ and it is denoted by $c_\mu(\lambda)$.
Thus

$$i_\mu(\lambda) = \bigvee\{\delta \in \mu : \delta \leq \lambda, \delta \text{ is a fuzzy } \mu\text{-open set}\} \text{ and;}$$
$$c_\mu(\lambda) = \bigwedge\{\delta : \lambda \leq \delta, \delta \text{ is a fuzzy } \mu\text{-closed set}\}.$$

2.3. Definition (Roy, 2018)

Let (X, μ) be a generalized topological space. An operation γ_μ on the generalized topology μ is a mapping from μ to $P(X)$ such that $V \subseteq V^{\gamma_\mu}$, for each $V \in \mu$, where V^{γ_μ} denotes the value of γ_μ at V. Such an operation is denoted by $\gamma_\mu : \mu \to P(X)$.

2.4. Definition (Roy, 2018)

Let (X, μ) be a generalized topological space and γ_μ an operation on μ. A subset A of X is called γ_μ-open if for each $x \in A$, there exists a μ-open set U containing x such that $U^{\gamma_\mu} \subseteq A$.
A subset B of a generalized topological space (X, μ) is called γ_μ-closed if $X - B$ is γ_μ-open in (X, μ).

2.5. Definition (Roy, 2018)

A fuzzy open set λ of a fuzzy space (X, τ) is called minimal fuzzy open set if λ is non-zero and there is no non-zero proper fuzzy open subset of λ.

3. Fuzzy γ_μ-Open Sets and their Properties

In this section we introduce the concept of γ_μ-operation and fuzzy γ_μ-open set in a generalized fuzzy topological space. We study the basic properties of the fuzzy γ_μ-open set and find an interesting result which explains that the collection of all fuzzy γ_μ-open sets forms a generalized fuzzy topology therein. Then we initiated the notion of regular γ_μ-operation and show that in case of a regular γ_μ-operation, the collection of all fuzzy γ_μ-open sets forms a fuzzy topology.

3.1. Definition

Let (X, μ) be a GFTS. We define an operation γ_μ on the GFT μ which is a function from μ to I^X such that for each $\lambda \in \mu$, $\lambda \leq \lambda^{\gamma_\mu}$, where λ^{γ_μ} is the value of γ_μ at λ.
In a GFTS, the fuzzy μ-closure operator is an example of a γ_μ-operation.

3.2. Definition

Let us consider a GFTS (X, μ) and a γ_μ-operation on μ. Any fuzzy set λ of X is said to be a fuzzy γ_μ-open set in μ if $\forall$ fuzzy point $x_p \in \lambda$, $\exists$ a μ-open set δ containing x_p such that $\delta^{\gamma_\mu} \leq \lambda$. A fuzzy set α is said to be a fuzzy γ_μ-closed set if its complement that is, $1_X - \alpha$ is a fuzzy γ_μ-open set in μ. The collection of all fuzzy γ_μ-open sets and fuzzy γ_μ-closed sets are denoted by $F\gamma_\mu$-$O(X)$ and $F\gamma_\mu$-$C(X)$ respectively.
The γ_μ-closure and the γ_μ-interior of any fuzzy set λ are denoted by $cl_{\gamma_\mu}(\lambda)$ and $int_{\gamma_\mu}(\lambda)$, which are defined as follows:

$$cl_{\gamma_\mu}(\lambda) = \bigwedge\{\alpha : \lambda \leq \alpha, \alpha \text{ is a fuzzy } \gamma_\mu\text{-closed set}\}$$

and

$$int_{\gamma_\mu}(\lambda) = \bigvee\{\alpha : \alpha \leq \lambda, \alpha \text{ is a fuzzy } \gamma_\mu\text{-open set}\}.$$

3.3. Theorem

In a GFTS (X, μ), every fuzzy γ_μ-open set is a fuzzy μ-open set, that is, Fγ_μ-$O(X) \subseteq \mu$.

Proof: We consider a fuzzy set $\lambda \in$Fγ_μ-$O(X)$ and a fuzzy point $x_p \in \lambda$. Then there exists a μ-open set δ containing x_p such that $\delta^{\gamma_\mu} \leq \lambda$. Consequently, we have $x_p \in \delta \subseteq \lambda$ and so x_p belongs to the μ-interior of λ, which proves that λ is a fuzzy μ-open set, that is, $\lambda \in \mu$. Therefore, Fγ_μ-$O(X) \subseteq \mu$.

3.4. Remark

The converse of the above theorem is not true in general. It is demonstrated in the following example.

3.5. Example

We take a non-empty set $X = \{a, b\}$ and a GFT $\mu = \{\phi, X, \{(a, 0.1), (b, 0.6)\}, \{(a, 0.7), (b, 0.3)\}, \{(a, 0.7), (b, 0.6)\}\}$ defined on X. We define a γ_μ-operation from μ to I^X as follows:

$$\gamma_\mu(\lambda) = c_\mu(\lambda)$$

Here the fuzzy set $\lambda = \{(x, 0.1), (y, 0.6)\}$ is a fuzzy μ-open set but it is not a fuzzy γ_μ-open set.

3.6. Theorem

Arbitrary union of fuzzy γ_μ-open sets in a GFTS (X, μ) is a fuzzy γ_μ-open set therein.

Proof: We take the collection $\{\lambda_n : n \in \Gamma\}$ of all fuzzy γ_μ-open sets of X. Let the fuzzy point $x_p \in \bigvee\{\lambda_n : n \in \Gamma\}$, then $\exists\ \lambda_k \in \lambda_n$, for some $k \in \Gamma$. Now $x_p \in \lambda_k$ implies $\exists$ a fuzzy μ open set δ containing x_p such that $\delta^{\gamma_\mu} \leq \lambda_k \leq \bigvee\{\lambda_n : n \in I\}$. Thus, we conclude arbitrary union of fuzzy γ_μ-open sets in a GFTS (X, μ) is a fuzzy γ_μ-open set.

3.7. Remark

Even the intersection of two fuzzy γ_μ-open sets in a GFTS (X, μ) may not be a fuzzy γ_μ-open set. In the following example this claim is verified.

3.8. Example

Let us consider a GFTS (X, μ) with $X = \{a, b, c\}$ and $\mu = \{0_X, 1_X, \{(a, 1), (b, 1), (c, 0)\}, \{(a, 1), (b, 0), (c, 1)\}\}$ and a γ_μ-operation on μ in such a way that

$$\gamma_\mu(\lambda) = \begin{cases} \lambda, & \text{if } \lambda \neq \{(a,1),(b,0),(c,0)\} \\ \{(a,1),(b,1),(c,0)\}, & \text{otherwise} \end{cases}$$

It can be easily verified that $\lambda_1 = \{(a, 1), (b, 1), (c, 0)\}$ and $\lambda_2 = \{(a, 1), (b, 0), (c, 1)\}$ are both fuzzy γ_μ-open sets. But their intersection $\lambda_1 \wedge \lambda_2 = \{(a, 1), (b, 0), (c, 0)\}$ is not a fuzzy γ_μ-open set therein.

3.9. Remark

The empty set 0_X is obviously a fuzzy γ_μ-open set. Thus considering the Theorem 3.6 and the Remark 3.7, we establish that Fγ_μ-$O(X)$ forms a generalized fuzzy topology in X.

3.10. Definition

Let (X, μ) be a GFTS. The γ_μ-operation is said be regular if for two fuzzy μ-open sets λ_1 and λ_2 containing the fuzzy point x_p, there exists another fuzzy μ-open set δ containing x_p such that $\delta^{\gamma_\mu} \leq \lambda_1^{\gamma_\mu} \wedge \lambda_2^{\gamma_\mu}$.

3.11. Theorem

Let $\gamma_\mu : \mu \to I^X$ be a regular operation. Then the intersection of any two fuzzy γ_μ-open set is again a γ_μ-open set.

Proof: We consider two fuzzy γ_μ-open sets λ_1 and λ_2 in a GFTS X. We take an arbitrary fuzzy point $x_p \in \lambda_1 \wedge \lambda_2$. Then $x_p \in \lambda_1$ and $x_p \in \lambda_2$. Thus there exists two fuzzy μ-open sets δ_1 and δ_2 such that $\delta_1^{\gamma_\mu} \leq \lambda_1$ and $\delta_2^{\gamma_\mu} \leq \lambda_2$. Now, since γ_μ is a regular operation, then there is another fuzzy μ-open set α in X such that $\alpha^{\gamma_\mu} \leq \delta_1^{\gamma_\mu} \wedge \delta_2^{\gamma_\mu}$ and so $\alpha^{\gamma_\mu} \leq \lambda_1 \wedge \lambda_2$. Hence, $\lambda_1 \wedge \lambda_2$ is also a fuzzy γ_μ-open set.

3.12. Theorem

If the operation $\gamma_\mu : \mu \to I^X$ is regular, then the collection of all fuzzy γ_μ-open sets in X forms a fuzzy topology therein.

Proof: Obviously, 1_X is a fuzzy γ_μ-open set. From the Remark 3.9 and the Theorem 3.11, we establish that the collection of all fuzzy γ_μ-open sets forms a fuzzy topology in X.

4. Fuzzy Minimal γ_μ-Open Sets

In this particular section, we extend the concept of minimal open set in fuzzy environment. we define fuzzy minimal γ_μ-open set in a generalized fuzzy topological space and establish several results based on this concept. We also define and discuss fuzzy γ_μ-pre-open set, fuzzy γ_μ-γ-open set and fuzzy γ_μ-locally finite space therein.

4.1. Definition

Let (X, μ) be a GFTS and γ_μ be any operation defined on μ. A non-empty fuzzy γ_μ-open set λ is said to be fuzzy minimal γ_μ-open set in X if there does not exists any fuzzy γ_μ-open subset of λ other than the fuzzy set 0_X.

4.2. Remark

A fuzzy minimal γ_μ-open set may not be a subset of any other fuzzy γ_μ-open set. The following example will support this claim.

4.3. Example

Let us consider a GFTS (X, μ) with $X = \{a, b\}$ and $\mu = \{0_X, 1_X,$ $\{(a, 1), (b, 0)\}, \{(a, 0), (b, 1)\}\}$. We define a γ_μ operation on μ such that

$$\gamma_\mu(\lambda) = \begin{cases} \lambda, & \text{if } a_1 \in \lambda \\ c_\mu(\lambda), & \text{otherwise} \end{cases}$$

Here the only fuzzy γ_μ-open sets in X are $0_X, \{(a, 1), (b, 0)\}, \{(a, 0), (b, 1)\}$ and 1_X. Obviously both the sets $\{(a, 1), (b, 0)\}$ and $\{(a, 0), (b, 1)\}$ are fuzzy minimal γ_μ-open sets but neither of them is the subset of the other.

4.4. Theorem

Let us consider a regular operation $\gamma_\mu : \mu \to I^X$ on μ in the GFTS (X, μ). If λ is a fuzzy minimal γ_μ-open set and δ is a fuzzy γ_μ-open set, then either $\lambda \wedge \delta = 0_X$ or $\lambda \leq \delta$.

Proof: If $\lambda \wedge \delta = 0_X$, then there remains nothing to prove. Let $\lambda \wedge \delta \neq 0_X$. Here λ and δ are both fuzzy γ_μ-open sets in X. Then, by the Theorem 3.11, we have $\lambda \wedge \delta$ is also a fuzzy γ_μ-open set. By the minimality condition, we have $\lambda \leq \lambda \wedge \delta$. As a consequence $\lambda \leq \delta$ and hence our claim.

4.5. Theorem

For a regular operation $\gamma_\mu : \mu \to I^X$ in a GFTS (X, μ), there exists only one fuzzy minimal γ_μ-open set.

Proof: If possible let there are two different fuzzy minimal γ_μ-open sets λ and δ in a given GFTS (X, μ). Then both of them are fuzzy γ_μ open sets and so is their intersection. Thus from the Theorem 4.4, we have $\lambda \leq \delta$, by considering the fact that λ is a fuzzy minimal γ_μ-open set and $\delta \leq \lambda$ as δ is also a fuzzy minimal γ_μ-open set. Combining both only we have $\lambda = \delta$.

4.6. Theorem

Let $\gamma_\mu : \mu \to I^X$ be a regular γ_μ-operation in a GFTS (X, μ). If λ is a fuzzy minimal γ_μ-open set and $x_p \in \lambda$, then $\lambda \leq \delta$, forall fuzzy γ_μ-open set δ containing the fuzzy point x_p.

Proof: Here both λ and δ are both fuzzy γ_μ-open sets such that the fuzzy point $x_p \in \lambda, \delta$. If possible let, λ, which is not a fuzzy subset of δ. Now, since γ_μ is a regular operation, so $\lambda \wedge \delta$ is also a fuzzy γ_μ-open set. Now, $\lambda \wedge \mu$, which is not a fuzzy subset of λ and $\lambda \wedge \delta \neq \phi$ (as $x_p \in \lambda, \delta$). Consequently we have, $\lambda \wedge \delta$ is a fuzzy minimal γ_μ-open set, which is a contradiction to our assumption. Therefore, $\lambda \leq \delta$.

From the above Theorem 4.6, we can directly find a result, which is as follows.

4.7. Remark

Let $\gamma_\mu : \mu \to I^X$ be a regular operation defined in a GFTS (X, μ) and λ be a fuzzy minimal γ_μ-open set in X. Then for any fuzzy point $x_p \in \lambda$, $\lambda = \bigwedge\{\delta : \delta$ is a fuzzy γ_μ-open set containing $x_p\}$.

4.8. Theorem

Suppose γ_μ be a regular operation defined in a GFTS (X, μ). Also let λ be a fuzzy minimal γ_μ-open set with $x_p \in 1_X - \lambda$ and $\chi_x = \bigwedge\{\delta \in$F$\gamma_\mu$-$O(X) : x_p \in \delta\}$. Then either $\chi_x \wedge \lambda = 0_X$ or $\lambda \leq \chi_x$.

Proof: The proof is done for the following two cases:
Case I: For $\lambda \leq \delta$ with $x_p \in \delta$
Then $\lambda \leq \bigwedge\{\delta \in$F$\gamma_\mu$-$O(X) : x_p \in \delta\}$ that is, $\lambda \leq \chi_x$.

Case II: For $\delta < \lambda$ with $x_p \in \delta$
In this case there exists a fuzzy γ_μ-open set δ containing x_p such that $\lambda \wedge \delta = 0_X$ and thus $\lambda \wedge \chi_x = 0_X$.

4.9. Theorem

Let $\gamma_\mu : \mu \to I^X$ be a regular operation in the GFTS (X, μ). Then λ is a fuzzy minimal γ_μ-open set iff for any non-empty fuzzy subset β of λ, $\lambda \leq cl_{\gamma_\mu}(\beta)$ and $cl_{\gamma_\mu}(\lambda) = cl_{\gamma_\mu}(\beta)$.

Proof: Let $x_p \in \lambda$ and δ be a fuzzy γ_μ-open set containing x_p. Then, $\lambda \leq \delta$ and $\beta = \lambda \wedge \beta \leq \delta \wedge \beta$. Thus $\delta \wedge \beta \neq 0_X$ and evidently $x_p \in cl_{\gamma_\mu}(\beta)$. Therefore, $\lambda \leq cl_{\gamma_\mu}(\beta)$. $\lambda \leq cl_{\gamma_\mu}(\beta) \Rightarrow cl_{\gamma_\mu}(\lambda) \leq cl_{\gamma_\mu}(\beta)$. Again, for any non-empty fuzzy subset β of λ, we have $cl_{\gamma_\mu}(\beta) \leq cl_{\gamma_\mu}(\lambda)$. Consequently, $cl_{\gamma_\mu}(\lambda) = cl_{\gamma_\mu}(\beta)$.
Conversely, if possible let us consider that λ is not a fuzzy minimal γ_μ-open set. Then, there exists a non-empty fuzzy γ_μ-open set β which is not a fuzzy subset of λ in (X, μ). Thus, there exist a fuzzy point $x_p \in \lambda$ such that $x_p \notin \beta$ and so $cl_{\gamma_\mu}(x_p) \subseteq 1_X - \beta$, that means, $cl_{\gamma_\mu}(x_p) \neq cl_{\gamma_\mu}(\lambda)$, which is a contradiction. Hence, λ is a fuzzy minimal γ_μ-open set.

4.10. Definition

Let (X, μ) be a GFTS and $\gamma_\mu : \mu \to I^X$ be an operation defined on (X, μ). A fuzzy subset λ is said to be a fuzzy γ_μ-pre-open set if $\lambda \leq int_{\gamma_\mu}(cl_{\gamma_\mu}(\lambda))$.

4.11. Theorem

Every subset of a fuzzy minimal γ_μ-open set in a GFTS (X, μ) is fuzzy γ_μ-pre-open if $\gamma_\mu : \mu \to I^X$ is a regular operation on μ.

Proof: Let δ be any subset of the fuzzy minimal γ_μ-open set λ. Then by the Theorem 4.8, we have, $\lambda \leq cl_{\gamma_\mu}(\delta) \Rightarrow int_{\gamma_\mu}(\lambda) \leq int_{\gamma_\mu}(cl_{\gamma_\mu}(\delta))$. But as λ is a fuzzy γ_μ-open set, so $\lambda = int_{\gamma_\mu}(\lambda)$ and thus $\delta \leq \lambda = int_{\gamma_\mu}(\lambda) \leq int_{\gamma_\mu}(cl_{\gamma_\mu}(\delta))$. Hence, δ is a fuzzy γ_μ pre-open set.

4.12. Theorem

Let γ_μ be an operation defined on a GFTS (X, μ) and λ be a proper fuzzy γ_μ-open set. Then there exists a fuzzy minimal γ_μ-open set δ such that $\delta \leq \lambda$.

Proof: Let λ be a proper fuzzy γ_μ-open subset in the GFTS (X, μ).

Case-I: λ itself is a fuzzy minimal γ_μ-open set
In this case, the proof is very much obvious by setting $\lambda = \delta$.

Case-II: λ is not a fuzzy minimal γ_μ-open set.
If λ is not a fuzzy minimal γ_μ-open set, then there exists a proper fuzzy γ_μ-open subset λ_1 of λ. If λ_1 is a fuzzy minimal γ_μ-open set then by setting $\delta = \lambda_1$, we get $\delta \leq \lambda$. Again if λ_1 is not a fuzzy minimal γ_μ-open set then we continue this process until we get a fuzzy minimal γ_μ-open set. And since this process will terminate after a finite number of steps (say, n), we will get fuzzy minimal γ_μ-open set $\delta = \lambda_n$ such that $\delta \leq \lambda$.

4.13. Definition

Let $\gamma_\mu : \mu \to I^X$ be an operation defined on μ. Any fuzzy set λ is said to be a fuzzy γ_μ-γ-open set if its intersection with every fuzzy γ_μ-pre-open sets gives a fuzzy γ_μ-pre-open set.

4.14. Theorem

If the GFTS X is singleton, then every fuzzy γ_μ-pre-open set is a fuzzy γ_μ-γ-open set therein.

Proof: Let λ be a fuzzy γ_μ-pre-open set in a GFTS (X, μ) and δ be any other fuzzy γ_μ-pre-open set. Now since X is singleton, then either $\lambda \leq \delta$ or $\delta \leq \lambda$, which implies either $\lambda \wedge \delta = \lambda$ or δ. In both the cases, the intersection is giving a fuzzy γ_μ-pre-open set. Hence λ is a fuzzy γ_μ-γ-open set in X.

4.15. Theorem

Every fuzzy γ_μ-γ-open set is necessarily a fuzzy γ_μ-pre-open set.

Proof: From the definition of fuzzy γ_μ-γ-open set, the proof is obvious and so it is omitted.

4.16. Remark

The converse of the above theorem is not true, that is a fuzzy γ_μ-pre-open set may not be a fuzzy γ_μ-γ-open set.

4.17. Example

We consider a GFTS (X, μ) with $X = \{a, b\}, \mu = \{0_X, 1_X, \{(a, 0.5), (b, 0)\}, \{(a, 1), (b, 0.2)\}\}$. Also, we define a fuzzy γ_μ-operation from μ to I^X as follows:

$$\gamma_\mu(\lambda) = \begin{cases} \lambda, & \text{if } a_1 \in \lambda \\ cl_\mu(\lambda), & \text{otherwise,} \end{cases}$$

where $a_1 = \{(a, 1), (b, \beta) : \beta[0, 1]\}$.
Then, we have Fγ_μ-$O(X) = \{0_X, 1_X, \{(a, 0.5), (b, 1)\}, \{(a, 1), (b, \alpha)\} : \alpha \in [0, 1]\}$ and thus $F\gamma_\mu$-$C(X) = \{0_X, 1_X, \{(a, 0.5), (b, 0)\}, \{(a, 0), (b, \alpha)\} : \alpha \in [0, 1]\}$. Calculation for the collection all pre-open sets gives that both the fuzzy sets $\lambda_1 = \{(a, 0.5), (b, 1)\}$ and $\lambda_2 = \{(a, 1), (b, 0)\}$ are fuzzy γ_μ-pre-open sets but their intersection $\lambda_1 \wedge \lambda_2 = \{(a, 0.5), (b, 0)\}$ is not a fuzzy γ_μ-pre-open set. Hence, none of these is a fuzzy γ_μ-γ-open set therein.

4.18. Definition

Let γ_μ be an operation on a GFTS (X, μ). X is said to be a fuzzy γ_μ-locally finite space if for every fuzzy point $x_p \in X$ there exists a fuzzy γ_μ-open set $\lambda \neq 1_X$ in X such that $x_p \leq \lambda$.

4.19. Theorem

Let γ_μ be a regular operation defined on a GFTS (X, μ) which is fuzzy γ_μ-locally finite. If δ is a non-empty fuzzy γ_μ-open set, then there exists a fuzzy minimal γ_μ-open set λ such that $\delta \leq \lambda$.

Proof: Suppose a fuzzy point $x_p \in \delta$. Then there exists a fuzzy γ_μ-open set λ such that $x_p \leq U_x$. Now, since $U_x \wedge \delta \neq 0_X$ is a fuzzy γ_μ-open set, then there exists a fuzzy minimal γ_μ-open set λ such that $\lambda \leq U_x \wedge \delta$ and hence $\lambda \leq \delta$.

5. Applications of Fuzzy γ_μ-Open Sets and Fuzzy Minimal γ_μ-Open Sets

In this section we discuss the applications of the newly defined sets fuzzy γ_μ-open sets and fuzzy minimal γ_μ-open sets via fuzzy γ_μ-pre-open sets and fuzzy γ_μ-locally finite space.

5.1. Property (m)

From the Theorem 4.4 and the Theorem 4.19, we see that if γ_μ is a regular operation then for any fuzzy γ_μ-open set λ, we can find a finite collection of fuzzy minimal γ_μ-open sets, say, $\delta_1, \delta_2, ..., \delta_n$ such that $\delta_i \wedge \delta_j = 0_X, \forall i, j = 1, 2, ..., n, i \neq j$. Moreover, if δ is any proper fuzzy minimal γ_μ-open set in λ, then $\delta = \delta_k$, for some $k = 1, 2, ..., n$.

We now define a property on a GFTS (X, μ) namely Property (m) as follows:
A GFTS (X, μ) is said to said satisfy the property (m) if

$$int_{\gamma_\mu}(\bigwedge \lambda_i) = 0_X \Rightarrow \bigwedge \lambda_i = 0_X$$

5.2. Theorem

Let γ_μ be a regular operation defined on a GFTS (X, μ) which satisfies property (m) and λ be a proper fuzzy γ_μ-open subset not necessarily minimal. If $\delta_1, \delta_2, ..., \delta_n$ be the collection of all fuzzy minimal γ_μ-open sets in λ, $y_p \in \lambda - \bigvee \delta_i$ and $\lambda_y = \bigwedge\{\chi \in \mathrm{F}\gamma_\mu\text{-}O(X) : y_p \in \chi\}$, then $\delta_k \leq \lambda_y$, for some $k = 1, 2, ..., n$.

Proof: Suppose there does not exist any fuzzy minimal γ_μ-open set $\delta_k \leq \lambda_y$. Now, it is clear that for any fuzzy minimal γ_μ-open set δ_i in λ, $\delta_i \wedge \lambda = 0_X$ and obviously $\lambda_y \leq \lambda$. Then, $\lambda_y = \lambda \wedge \lambda_y$ is a proper fuzzy set and so is $int_{\gamma_\mu}(\lambda_y)$. Thus there exists a fuzzy minimal γ_μ-open set α such that $\alpha \leq int_{\gamma_\mu}(\lambda_y)$. Consequently $\delta_k \wedge \alpha \leq \delta \wedge \lambda_y = 0_X$ and so $\alpha \neq \delta_k$, $\forall k \in \{1, 2, ..., n\}$. This is a contradiction to our assumption, which complete the proof.

5.3. Remark

Let γ_μ be a regular operation defined on a GFTS (X, μ) which satisfies the property (m) and λ be a proper fuzzy γ_μ-open set. If $\delta_1, \delta_2, ..., \delta_n$ be the collection of all fuzzy minimal γ_μ-open sets and the fuzzy point $y_p \in \lambda - \bigvee \delta_i$, then

(a) for any fuzzy γ_μ-open set χ_y containing the point y_p, $\delta_k \leq \chi_y$, for some $k = 1, 2, ..., n$,

(b) $y_p \in cl_{\gamma_\mu}(\delta_k)$, for some $k = 1, 2, ..., n$.

5.4. Definition

A GFTS (X, μ) is said to be a fuzzy γ_μ-pre T_2 space if for any two distinct fuzzy points x_p and $y_q \in X$, there exists fuzzy γ_μ-pre-open sets λ and δ such that $x_p \in \lambda, y_q \in \delta, \lambda \wedge \delta = 0_X$.

5.5. Theorem

Let γ_μ be regular operation defined on a fuzzy γ_μ-locally finite GFTS (X, μ) satisfying property (m) and any fuzzy minimal γ_μ-open set have points with non-zero membership values. Then (X, μ) is a fuzzy γ_μ-pre T_2 space.

Proof: Since X is a fuzzy γ_μ-locally finite space, then for any two distinct fuzzy points $x_p, y_q \in X$, there exist two proper fuzzy γ_μ-open sets λ and δ

containing x_p and y_q respectively. Consider two collections of fuzzy minimal γ_μ-open sets $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and $\{\delta_1, \delta_2, ..., \delta_m\}$ with respect to λ and δ respectively.

Case I: $x_p \in \lambda_i$ and $y_q \in \delta_j$, for some $i = 1, 2, ..., n, j = 1, 2, ..., m$
From theorem 4.11, we have fuzzy γ_μ-pre-open sets α which have membership value p for x and 0 for others and β which have membership value q for y and 0 for others such that $x_p \in \alpha, y_q \in \beta$ and $\alpha \wedge \beta = 0_X$. Thus (X, μ) is a fuzzy γ_μ-pre T_2 space.

Case II: $x_p \in \lambda_i$, for some $i = 1, 2, ..., n$ and $y_q \notin \delta_j, \forall j \in \{1, 2, ..., m\}$.
In this case choose fuzzy points $y_{q_j} \in \delta_j$. Then we have fuzzy γ_μ-pre-open sets α which have membership value p for x and 0 for others and β which have membership value 0 of x for sure, q of y and some certain values of others depending upon the selection of the decision maker. In this case also $x_p \in \alpha, y_q \in \beta$ and $\alpha \wedge \beta = 0_X$.

Case III: $y_q \in \delta_j$, for some $j = 1, 2, ..., m$ and $x_p \notin \lambda_i, \forall i \in \{1, 2, ..., n\}$.
Following the previous case, one can easily obtain the desired result.

Case IV: $x_p \notin \lambda_i, y_q \notin \delta_j, \forall i \in \{1, 2, ..., n\}$ and $\forall j \in \{1, 2, ..., m\}$
Choosing $x_{p_k} \in \lambda_k$ and $y_{q_j} \in \delta_j$, we can find two fuzzy γ_μ-pre-open sets α containing x_p and β containing y_q such that $\alpha \wedge \beta = 0_X$.
Hence (X, μ) is a fuzzy γ_μ-pre T_2 space.

6. Conclusion

In this paper operation approaches on open sets studied for the first time in a fuzzy environment. The notion of fuzzy γ_μ-open set have been introduced in a GFTS and it is found that the collection of all fuzzy γ_μ-open sets forms a GFT therein. It has been established that fuzzy γ_μ-open sets forms a fuzzy topology when the γ_μ-operation is regular. Moreover, the concept of fuzzy minimal γ_μ-open set has been initiated and it is proved that a fuzzy minimal γ_μ-open set need not be a subset of all fuzzy γ_μ-open set, that is in a GFTS there may be multiple numbers of fuzzy minimal γ_μ-open sets. It also been proved that in a singleton GFTS every fuzzy γ_μ-pre-open set becomes a fuzzy γ_μ-γ-open set but a fuzzy γ_μ-γ-open set may not be a fuzzy γ_μ-pre-open set. Finally the applications of fuzzy γ_μ-open set has been discussed via fuzzy γ_μ-pre T_2 space.

References

[1] B. A. Asaad and N. Ahmed, “Operation on Semi Generalized Open Sets with Its Separation Axioms”, *International Journal of Pure and Applied Mathematics,* vol. 118, no. 3, pp. 701-711, 2018.

[2] C. Carpintero, E. Roses, M. S. Brown, and J. Sanabiro, “Minimal Open Sets on Generalized Topological Spaces”, *Proyecciones (Antofagasta),* vol. 36, no. 4, pp. 739-751, 2017. doi: 10.4067/S0716-09172017000400739

[3] C. Carpintero, N. Rajesh, and E. Roses, “Operation Approaches on b-open Sets and Applications”, *Boletim da Sociedade Paranaense de Matematica,* vol. 20, no. 1, pp. 21-33, 2012.

[4] G. P. Chetty, “Generalized Fuzzy Topology”, *Italian Journal of Pure and Applied Mathematics,* vol. 24, pp. 91-96, 2008.

[5] B. Das, B. Bhattacharya, and A. K. Saha, “Some remarks on fuzzy infi topological spaces”, *Proyecciones (Antofagasta),* vol. 40, no. 2, pp. 399-415, 2021. doi: 10.22199/issn.0717-6279-2021-02-0024

[6] B. Das, B. Bhattacharya, J. Chakaraborty, S. A. Ganapathiraju, and A. Paul, “A new type of generalized closed set via γ-open set in a fuzzy bitopological space”, *Proyecciones (Antofagasta),* vol. 38, no. 3, pp. 511-536, 2019. https://doi.org/10.22199/issn.0717-6279-2019-03-0033

[7] B. Das, J. Chakraborty, G. Paul, and B. Bhattacharya, “A new approach for some applications of generalized fuzzy closed sets”, *Computational and Applied Mathematics,* vol. 40, no. 2, pp. 1-14, 2021.

[8] B. Das, B. Bhattacharya, J. Chakraborty, and B. C. Tripathy, "Generalized fuzzy closed sets in a fuzzy bitopological space via γ-open sets", *Afrika Mathematika,* vol. 32, no. 2, pp. 1-13, 2021.

[9] B. Das, B. C. Tripathy, P. Debnath, and B. Bhattacharya, "Study of Matrix Transformation of Uniformly Almost Surely Convergent Complex Uncertain Sequences", *Filomat*, vol. 34, no. 14, pp. 4907-4922, 2021.

[10] B. Das, B. C. Tripathy, P. Debnath, and B. Bhattacharya, "Almost convergence of complex uncertain double sequences", *Filomat,* vol. 35, no. 1, pp. 61-78, 2021.

[11] S. Hussain, "On Generalized Open Sets", *Hacettepe Journal of Mathematics and Statistics,* vol. 47, no. 6, pp. 1438-1446, 2018.

[12] Y. B. Jun, S. W. Jeong, H. J. Lee, and J. W. Lee, "Application of Preopen Sets", *Applied General Topology,* vol. 9, no. 2, pp. 213-228, 2008.

[13] S. Kasahara, "Operation-Compact Spaces", *Mathematica Japonica*, vol. 24, pp. 97-105, 2007.

[14] G. S. S. Krishnan, M. Ganster, and K. Balachandran, "Operation Approcahes on Semiopen Sets and Its Applications", *Kochi Journal of Mathematics,* vol. 2, pp. 21-33, 2007.

[15] E. Nakaoda and N. Oda, "Some Applications of Minimal Open Sets", *International Journal of Mathematics and Mathematical Sciences,* vol. 27, no. 8, pp. 471-476, 2001.

[16] H. Ogata, "Operations on Topological Space and Associated Topologies", *Mathematica Japonica,* vol. 36, pp. 175-184, 1991.

[17] B. Roy, "Applications of Operations on Minimal Generalized Open Sets", *Afrika Mathematika,* vol. 29, pp. 1097-1104, 2018.

[18] S. Tahiliani, "Operation Approach to β-Open Sets and Applications", *Mathematical Communication,* vol. 16, pp. 577-591, 2011.

[19] B. C. Tripathy and S. Debnath, "γ-open sets and γ-continuous mappings in fuzzy bitopological spaces", *Journal of Intelligent and Fuzzy Systems,* vol. 24, no. 3, pp. 631-635, 2013.

[20] B. C. Tripathy and G. C. Ray, "Mixed fuzzy ideal topological spaces", *Applied Mathematics and Computations,* vol. 220, pp. 602-607, 2013.

[21] B. C. Tripathy and G. C. Ray, "Weakly continuous functions on mixed fuzzy topological spaces", *Acta Scientiarum. Technology*, vol. 36, no. 2, pp. 331-335, 2014.

[22] T. Van An, D. X. Cuong, and H. Maki, "On Operation Preopen Sets in Topological Space", *Scientiae Mathematicae Japonicae*, vol. 21, pp. 241-260, 2008.

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