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# Powers of cycle graph which are $k$-self complementary and $k$-co-self complementary

*K. Arathi Bhat*
*Manipal Institute of Technology, India*
*and*
*G. Sudhakara*
*Manipal Institute of Technology, India*
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**Abstract**

*E. Sampath Kumar and L. Pushpalatha [4] introduced a generalized version of complement of a graph with respect to a given partition of its vertex set. Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V$ of order $k \geq 1$. The $k$-complement $G_k^P$ of $G$ with respect to $P$ is defined as follows: For all $V_i$ and $V_j$ in $P$, $i \neq j$, remove the edges between $V_i$ and $V_j$, and add the edges which are not in $G$. Analogues to self complementary graphs, a graph $G$ is $k$-self complementary ($k$-s.c.) if $G_k^P \cong G$ and is $k$-co-self complementary ($k$-co.s.c.) if $G_k^P \cong \overline{G}$ with respect to a partition $P$ of $V(G)$. The $m^{th}$ power of an undirected graph $G$, denoted by $G^m$ is another graph that has the same set of vertices as that of $G$, but in which two vertices are adjacent when their distance in $G$ is at most $m$.*

*In this article, we study powers of cycle graphs which are $k$-self complementary and $k$-co-self complementary with respect to a partition $P$ of its vertex set and derive some interesting results. Also, we characterize $k$-self complementary $C_n^2$ and the respective partition $P$ of $V(C_n^2)$. Finally, we prove that none of the $C_n^2$ is $k$-co-self complementary for any partition $P$ of $V(C_n^2)$.*

**Keywords:** *$k$-complement, $k(i)$-complement, $k$-self complementary, $k$-co-self complementary, powers of cycle graph.*

**2010 Mathematics subject classification:** *05C50, 15A24.*

## 1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_q\}$. Two vertices $v_i$ and $v_j, i \neq j$, are said to be adjacent to each other if there is an edge between them. An adjacency between the vertices $v_i$ and $v_j$ in $G$ is denoted by $v_i \sim v_j$, and non adjacency between them is denoted by $v_i \not\sim v_j$.
For the notations and terminology we refer [3].
For $2 \leq k \leq n$, characterization of all $k$-s.c. trees, forests and unicyclic graphs are obtained in [4]. Authors of [4] have also derived some results giving bounds for the maximum degree and the number of edges as given in Theorem 1.1.

**Theorem 1.1.** *If a graph $G$ with $n$ vertices and $q$ edges is $k$-s.c., then*

1. *$G$ has a vertex of degree* $\dfrac{n(k-1)}{2k}$
2. $\dfrac{(k-1)(2n-k)}{4} \leq q \leq \dfrac{2n(n-k)+k(k-1)}{4}$.

Later, E. Sampath Kumar et al. [5], defined an invariant of $k-$ complement, known as $k(i)-$complement given as follows;
Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V$ of order $k \geq 1$. For each set $V_r$ in the partition $P$, remove the edges of $G$ inside $V_r$, and add the edges of $\overline{G}$ (the complement of $G$) joining the vertices of $V_r$. The graph $G^P_{k(i)}$ thus obtained is called the $k(i)$-complement of G with respect to the partition $P$ of $V(G)$. The graph $G$ is $k(i)$- self complementary ($k(i)$-s.c.) if $G^P_{k(i)} \cong G$ for some partition $P$ of order $k$.
A graph $G$ is $k(i)$-co-self complementary ($k(i)$-co-s.c.) if $G^P_{k(i)} \cong \overline{G}$.
Authors of [5] have obtained all $k(i)$-self complementary trees for $k = 2, 3$ and $2(i)$- self complementary unicyclic graphs. They also obtained some necessary conditions for a tree/ unicyclic graph to be $k(i)$-self complementary. It is noted that The graph $G$ is $k$-s.c. if and only if $G$ is $k(i)$-co.s.c. and $G$ is $k(i)$-s.c. if and only if it is $k$-co-s.c. with respect to a partition $P$ of $V(G)$. The $k$-self complementary and $k$-co-self complementary graphs with respect to a partition $P$ of $V(G)$ among various classes of graphs have

been studied.
Few other facts concerned with $k$- complements and $k(i)$- complements are available in the literature [1], [2], [4], [5] and [6].
In this article we answer the question whether there is any $C_n^2$ which is $k$-s.c. or $k$-co.s.c. for some partition $P$ of $V(C_n^2)$. Throughout this article, we considered the graph $C_n$ with $v_1, v_2, \ldots, v_n$ as its $n$ vertices and $v_i \sim v_{i+1}$ for $1 \le i \le n-1$ and $v_1 \sim v_n$ while discussing the graph $C_n^m$.

## 2. $k$-self complementary $C_n^m$

Let $C_n^m$ be the $m^{th}$ power of a cycle graph $C_n$ on $n$ vertices. Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V(C_n^m)$. We consider the case when $k < n$, as $(C_n^m)_n^P = (\overline{C_n^m})$. In this section, we obtain bounds on number of vertices in terms of the parameter $m$.
Further, if $m \ge \lfloor \frac{n}{2} \rfloor$, then $C_n^m$ is complete. So, we consider $C_n^m$ with $m < \lfloor \frac{n}{2} \rfloor$. In $C_n^m$, degree of all the vertices is $2m$ and the number of edges is $mn$. From Theorem 1.1, $C_n^m$ is $k$-s.c. if there exists at least one vertex of degree at least $\dfrac{(k-1)n}{2k}$, i.e., if $n \le \dfrac{4km}{k-1}$. Therefore,

$$C_n^m \text{ is } k\text{-s.c. with respect to } P \text{ if } 2m+1 \le n \le \frac{4km}{k-1}. \tag{2.1}$$

With every vertex $v \in V_j, j = 1, 2, \ldots, k$, with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$, we introduce two new terminologies.

**Definition 2.1.** *The $i$-degree of $v$ is the degree of $v$ in the induced graph $V_j$, i.e., $\langle V_j \rangle$, and $o$-degree of $v$ = (degree of $v$ in $G$) $-$ ($i$-degree of $v$).*

**Lemma 2.2.** *Let $G = C_n^m$ be $k$-s.c. with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$. Then all the vertices in $V_j$ are of same $i$-degree and same $o$-degree for $1 \le j \le k$.*

**Proof.** Let $v, w \in V_j,\ 1 \le j \le k$. Let the $i$-degrees of the vertices $v$ and $w$ be $r$ and $s$, $(0 \le r, s \le 2m-1)$ respectively. As degree of $v$ in $C_n^m$ and $(C_n^m)_k^P$ is $2m$, outside $V_j$, the vertex $v$ is adjacent to $(2m-r)$ vertices and non adjacent to $(2m-r)$ vertices. Thus, outside $V_j$, there are $(4m-2r)$ vertices. Similarly, outside $V_j$ there are $(4m-2s)$ vertices. This is possible only if $r = s$, in other words, $i$-degree of all the vertices in $V_j$ are same and so is the $o$-degree. □

**Lemma 2.3.** *Let $G = C_n^m$ be $k$-s.c. with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$. If $k$ is even, then $n$ is also even and if $n$ is odd then $k$ is also odd.*

**Proof.** Let $v \in V_1$, and $i$-degree of $v$ be $r$. Then, as outside $V_1$ there are $(4m - 2r)$ vertices, we have, $|V_2| + |V_3| + \ldots + |V_k| = (4m - 2r)$, an even number.
Similarly, $|V_1| + |V_3| + \ldots + |V_k|$, $\cdots$, $|V_1| + |V_2| + \ldots + |V_{k-1}|$ are even numbers. On adding all these $k$ combinations we get,
$(k-1)(|V_1| + |V_2| + \ldots + |V_k|)$ an even number. Thus, $(k-1)n$ is even. Hence the result follows. □

**Theorem 2.4.** *Let $G = C_n^m$ be $k$-s.c. with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ for some even integer $k$. Then $|V_j|,\ 1 \le j \le k$ is even.*

**Proof.** From Lemma 2.3, when $k$ is even, $n$ is also even. As $|V_2| + |V_3| + \ldots + |V_k|$ is even, implies $|V_1|$ is also even. Hence, $|V_j|,\ 1 \le j \le k$ is even. □

Let $G = C_n^m$ be $k$-s.c. with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$. From the above results, we note the following remarks.

**Remark 2.5.** *If $n$ is odd, then $|V_i|$ is odd for every $i,\ 1 \le i \le k$. Hence, $i$-degree of all the vertices is even.*

**Remark 2.6.** *The total number of vertices in any $(k-1)$ partite sets is at most $4m$.*

**Remark 2.7.** *For all $V_i$ in the partition $P$, $1 \le |V_i| \le 4m - (k-2)$.*

**Remark 2.8.** *As number of vertices in any $(k-1)$ partite sets is even, $|V_j|, j = 1, 2, \ldots, k$, is either all even or all odd.*

Further, we study 2-s.c. and 3-s.c. m-powers of cycle graph on n vertices with respect to a partition $P$ of its vertex set and obtain some results. $k$-s.c. with respect to a partition $P$ of $V(C_n^2)$.

### 2.0.1. 2-self complementary $C_n^m$

From the Equation (2.1), $C_n^m$ is 2-s.c. if $2m+1 \le n \le 8m$.
As $k$ is even, $n$ is also even, and both the partite sets contain even number of vertices. Therefore $n$ is an even number such that, $2m+2 \le n \le 8m$. We now derive a result, using which we improve the upper bound $8m$ for $n$ in the above equation.

**Lemma 2.9.** *Consider the graph $C_n^m$ where $2m+2 \le n \le 8m$. Let $\alpha(C_n^m)$ denotes the vertex independence number of $C_n^m$. Then,*

$$\alpha(C_n^m) \le \begin{cases} 5, & m = 2 \\ 6, & 3 \le m \le 6 \\ 7, & m \ge 7.. \end{cases}$$

**Proof.** Let $v_1, v_2, \ldots, v_n$ be vertices of $C_n$ and $v_i \sim v_{i+1}$ for $1 \le i \le n-1$ and $v_1 \sim v_n$. Then, in $C_n^m$, the vertex $v_1$ is adjacent to $v_2, v_3, \ldots, v_{m+1}$, $v_n, v_{n-1}, \ldots, v_{n-m}$. Therefore the vertices $v_1, v_{m+2}, v_{2m+3}, v_{3m+4}, v_{4m+5}$, $v_{5m+6}, v_{6m+7}$ are independent.
As $n \le 8m$, if $6m+7 \le 7m$, i.e., for $m \ge 7$, $\alpha(C_n^m) \le 7$.
If $6m+7 > 7m$, i.e., for $m < 7$, then $\alpha(C_n^m) \le 6$.
If $5m+6 > 7m$, i.e., for $m < 3$, then $\alpha(C_n^m) \le 5$.
If $5m+6 \le 7m$, i.e., for $m \ge 3$, then $\alpha(C_n^m) \le 6$.
If $4m+5 > 7m$, i.e., for $m \le 1$, then $\alpha(C_n^m) \le 4$.
If $4m+5 \le 7m$, i.e., for $m \ge 2$, then $\alpha(C_n^m) \le 5$.
Hence the result. □

**Remark 2.10.** *In general,* $\alpha(C_n^m) = \lfloor \frac{n}{m+1} \rfloor$.

**Remark 2.11.** *Let $C_n^m$ be 2-s.c. with respect to the partition $P = \{V_1, V_2\}$ and $|V_1| = 4m$. Then any vertex $v \in V_2$ is adjacent to $2m$ vertices of $V_1$ and non adjacent to $2m$ vertices of $V_1$. Hence $\langle V_2 \rangle$ is independent.*

**Theorem 2.12.** *Let $C_n^m$ be 2-s.c. with respect to the partition $P = \{V_1, V_2\}$. Then $n$ is an even number such that, $2m+2 \le n \le 8m-4$.*

**Proof.** From Remark 2.7, the maximum number of vertices in any partite set is at most $4m$. Let $|V_1| = 4m$, then $|V_2| \leq 4m$. From Remark 2.11, $\langle V_2 \rangle$ is independent. Also from Lemma 2.9, $C_n^m$ has at most 7 independent vertices. If $|V_2| = 4m$, then $4m \leq 7$.
Only when $m = 1$, we have $|V_1| = |V_2| = 4$. Therefore $|V_2| \leq 4m - 2$. If $|V_2| = 4m - 2$, then $(4m - 2) \leq 7 \implies 4m - 2 = 2$ or $4m - 2 = 6$ which implies $m = 1$ or $m = 2$. But for $m = 2$, $C_n^m$ has at most 5 independent vertices. Therefore $n \leq 8m - 4$. □

### 2.0.2. 3-self complementary $C_n^m$

In this section, we prove that there is no 3-self complementary $C_n^m$ on odd number of vertices which gives slight improvement to the inequality, $2m \leq n \leq 6m$, which follows from the Equation (2.1).

**Theorem 2.13.** *Let $C_n^m$ be 3-s.c. with respect to the partition $P = \{V_1, V_2, V_3\}$ of $V(C_n^m)$. Then $n$ is even.*

**Proof.** Assume $n$ is odd. As $k = 3$ is odd, all partite sets contain odd number of vertices. Thus, $i$-degree of all the vertices is even. Let $i$-degree of all the vertices of $V_1$, $V_2$ and $V_3$ be $2r$, $2s$ and $2t$ where $r, s, t = 0, 1, \ldots, m-1$, respectively. Outside $V_1$, any vertex of $V_1$ is adjacent to $(2m - 2r)$ vertices and non adjacent to $(2m - 2r)$ vertices.
Therefore $|V_2| + |V_3| = 4(m - r)$.
Similarly, $|V_1| + |V_3| = |V_1| + |V_2| = 4(m - r)$, which implies $2n = 4(3m - r - s - t)$ i.e., $n = 2(3m - r - s - t)$ a contradiction. Therefore $n$ is even.
□

**Remark 2.14.** *Let $C_n^m$ be 3-s.c. with respect to the partition $P = \{V_1, V_2, V_3\}$ of $V(C_n^m)$. Then $n$ is an even number such that $2m + 2 \leq n \leq 6m$.*

### 2.0.3. $k$-self complementary $C_n^2$

We obtain all possible values of $k$ and $n$ for which $C_n^2$ is $k$-s.c. with respect to a partition $P$ of $V(C_n^2)$.

**Lemma 2.15.** *If $k \geq 8$, then $C_n^2$ is not $k$-s.c. with respect to any partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(C_n^2)$.*

**Proof.** From Equation (2.1), $C_n^2$ is $k$-s.c. if $4 \leq n \leq \dfrac{8k}{k-1}$.
But for $k \geq 10, n \leq 8$, is not possible. When $k = 9$, then $n \leq 9$, i.e., $k = n$. When $k = 8$ then, $n \leq 9$. From Lemma 2.3, it follows that $k = n$. Therefore $C_n^2$ is not $k$-s.c. with respect to any partition $P$ of $V(C_n^2)$ if $k \geq 8$. □

**Theorem 2.16.** *There is no k-s.c. $C_n^2$ with respect to any partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(C_n^2)$, if $k \geq 6$.*

**Proof.** From Lemma 2.15, $C_n^2$ is $k$-s.c. if $k \leq 7$.
If $k = 7$, $n$ can be 8 or 9.
For $k = 7$ and $n = 8$, it is not possible to partition the vertex set such that cardinality of all the partite sets are either all even or all odd. Thus, $C_8^2$ is not 7.s.c.
For $k = 7$ and $n = 9$, the only possible partition of the partite set is $|V_1| = |V_2| = \ldots = |V_6| = 1$ and $|V_7| = 3$. Degree of all the vertices in $C_9^2$ is 4 and $i$-degree of all the vertices of $V_7$ is either 0 or 2. Suppose $i$-degree of all the vertices of $V_7$ is 0, then number of vertices lying outside $V_7$ is 8, which is not true. Suppose $i$-degree of all the vertices of $V_7$ is 2, then number of vertices lying outside $V_7$ is 4, which is not true.
Therefore $C_n^2$ is not 7-s.c.
If $k = 6$, then $n = 8$. But from Theorem 2.4, all the partite sets contain even number of vertices, which is not possible. Thus, $C_n^2$ is not 6-s.c. with respect to any partition $P$ of $V(C_n^2)$. □

Next, we obtain all $C_n^2$ which are $k$-s.c. and the respective partition $P$ of $V(C_n^2)$ of order $k \leq 5$. Let $v_1, v_2, \ldots, v_n$ be vertices of $C_n$ and $v_i \sim v_{i+1}$ for $1 \leq i \leq n-1$ and $v_1 \sim v_n$.

**2-s.c.** $C_n^2$

From Theorem 2.12, $C_n^2$ is 2-s.c., if $n$ is an even number between 6 and 12. Also, $|V_i|$ is an even number such that $2 \leq |V_i| \leq 8$. The first possible case is $|V_1| = 2$ and $|V_2| = 4$. As $|V_2| = 4$, all the vertices of $V_1$ are adjacent with 2 vertices and non adjacent with 2 vertices of $V_2$. Which implies all the vertices in $V_1$ must be of $i$-degree 2. Similarly, since $|V_1| = 2$, all the vertices of $V_2$ are of $i$-degree 3. We list all the possible cases in the following table.

| $\lvert V_1\rvert$ | $\lvert V_2\rvert$ | $i$-degree of $V_1$ | $i$-degree of $V_2$ | $n$ |
|---|---|---|---|---|
| 2 | 4 | 2 | 3 | 6 |
| 2 | 6 | 1 | 3 | 8 |
| 2 | 8 | 0 | 3 | 10 |
| 4 | 4 | 2 | 2 | 8 |
| 4 | 6 | 1 | 2 | 10 |
| 4 | 8 | 0 | 2 | 12 |
| 6 | 6 | 1 | 1 | 12 |

When $|V_1| = 2, |V_2| = 4$, the $i$-degree of all the vertices of $V_1$ being equal to 2 is not possible. When $|V_1| = 2$ and $|V_2| = 6$, the $i$-degree of vertices of $V_1$ and $V_2$ are 1 and 3 respectively. If $V_1 = \{v_1, v_2\}$, then the other partite set $V_2$ has two vertices with $i$-degree 4. Also if $V_1 = \{v_1, v_3\}$, then the other partite set $V_2$ has one vertex of $i$-degree 4. Therefore these cases are not possible.
When $|V_1| = |V_2| = 6$, then the only possibility is $V_1 = \{v_1, v_2, v_5, v_6, v_9, v_{10}\}$ in which case the graphs $C_{12}^2$ and $(C_{12}^2)_2^P$ are not isomorphic.

Thus, the above three cases are discarded and the remaining 4 cases are possible and the four $C_n^2$ as shown in Figures 1, 2, 3 and 4 are the only $C_n^2$ which are 2-s.c. with respect to the partition $P$ of $V(C_n^2)$.

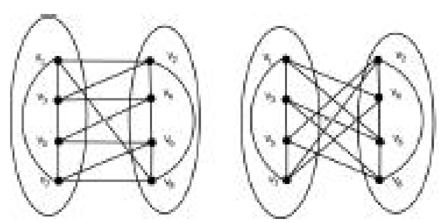

Figure 1: $C_8^2$ and its 2-complement with $C_8^2 \cong (C_8^2)_2^P$ .

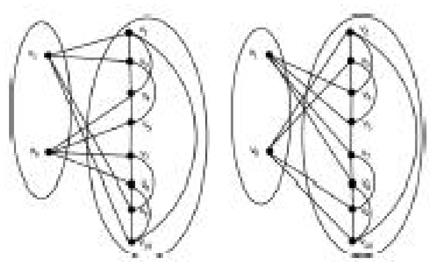

Figure 2: $C_{10}^2$ and $(C_{10}^2)_2^P$ with $C_{10}^2 \cong (C_{10}^2)_2^P$ .

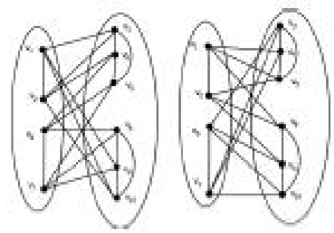

Figure 3: $C_{10}^2$ and $(C_{10}^2)_2^P$ with $C_{10}^2 \cong (C_{10}^2)_2^P$ .

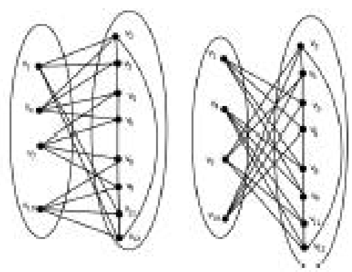

Figure 4: $C_{12}^2$ and $(C_{12}^2)_2^P$ with $C_{12}^2 \cong (C_{12}^2)_2^P$ .

**Note 2.17.** We know that $G$ is $k$-s.c. if and only if $G$ is $k(i)$-co.s.c [5]. Therefore all the 4 graphs as shown in Figure 2.0.3, 2.0.3, 2.0.3 and 2.0.3 which are 2-s.c are also $2(i)$-co.s.c. with respect to the same partition $P$ of $V(C_n^2)$.

**3-self complementary $C_n^2$:**

From Theorem 2.12, $n$ is an even number between 6 and 12. Number of edges in $C_n^2$ is $2n$. From (ii) of Theorem 1.1, we have
$2n \leq \dfrac{2n(n-3)+6}{4}$, which implies $n \geq 7$. Thus $n = 8,\ 10$ or $12$. Further, $|V_i|$ is an even number and number of vertices in any 2 partite sets is at most 8. Thus $2 \leq |V_i| \leq 6$. Hence we have the following cases.

| $\|V_1\|$ | $\|V_2\|$ | $\|V_3\|$ | i-deg of $V_1$ | i-deg of $V_2$ | i-deg of $V_3$ | n |
|---|---|---|---|---|---|---|
| 2 | 2 | 4 | 1 | 1 | 2 | 8 |
| 2 | 2 | 6 | 0 | 0 | 2 | 10 |
| 2 | 4 | 4 | 0 | 1 | 1 | 10 |
| 4 | 4 | 4 | 0 | 0 | 0 | 12 |

For the first and the last case, respectively, the graphs $(C_8^2)_3^P$ and $V_1 = \{v_1, v_4, v_7, v_{10}\}$, $V_2 = \{v_2, v_5, v_8, v_{11}\}$ and $V_3 = \{v_3, v_6, v_9, v_{12}\}$. For this partition, $(C_{12}^2)_3^P$ are of regularity 4, but not 3-s.c. with respect to any partition $P$ of $V(C_n^2)$.
For the second and third case we get $C_n^2$ which are 3-s.c. with respect to the partition $P$ of $V(C_n^2)$ as shown in Figures 5 and 6.

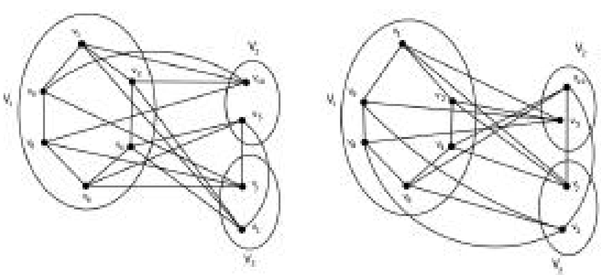

Figure 5: $C_{10}^2$ and $(C_{10}^2)_3^P$ with $C_{10}^2 \cong (C_{10}^2)_3^P$

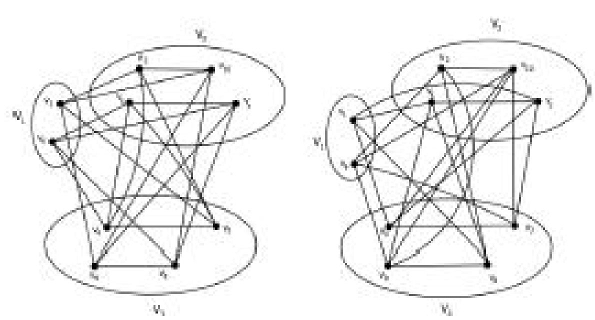

Figure 6: $C_{10}^2$ and $(C_{10}^2)_3^P$ with $C_{10}^2 \cong (C_{10}^2)_3^P$

**4-self complementary $C_n^2$**

From the Equation (2.1) and Lemma 2.3, $n$ can take the values 6, 8 or 10. But from (ii) of Theorem 1.1, $n \geq 8$ and $|V_i|$ is an even number such that $2 \leq |V_i| \leq 4$. Thus, we have the following cases.

| $\|V_1\|$ | $\|V_2\|$ | $\|V_3\|$ | $\|V_4\|$ | i-deg of $V_1$ | i-deg of $V_2$ | i-deg of $V_3$ | i-deg of $V_4$ | n |
|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 8 |
| 2 | 2 | 2 | 4 | 0 | 0 | 0 | 1 | 10 |

For the first case, we have 7 possible ways of partitioning the vertices of $C_8^2$ as listed in the table. But none of the $C_8^2$ are 4.s.c. with respect to the partition $P$ of $V(C_8^2)$.

| $V_1$ | $V_2$ | $V_3$ | $V_4$ |
|---|---|---|---|
| $\{v_1, v_2\}$ | $\{v_3, v_4\}$ | $\{v_5, v_6\}$ | $\{v_7, v_8\}$ |
| $\{v_1, v_3\}$ | $\{v_2, v_4\}$ | $\{v_5, v_7\}$ | $\{v_6, v_8\}$ |
| $\{v_1, v_2\}$ | $\{v_3, v_5\}$ | $\{v_4, v_6\}$ | $\{v_7, v_8\}$ |
| $\{v_1, v_3\}$ | $\{v_2, v_4\}$ | $\{v_5, v_6\}$ | $\{v_7, v_8\}$ |
| $\{v_1, v_2\}$ | $\{v_3, v_4\}$ | $\{v_5, v_7\}$ | $\{v_6, v_8\}$ |
| $\{v_1, v_3\}$ | $\{v_2, v_8\}$ | $\{v_4, v_5\}$ | $\{v_6, v_7\}$ |
| $\{v_1, v_3\}$ | $\{v_2, v_8\}$ | $\{v_5, v_7\}$ | $\{v_4, v_6\}$ |

For the second case we have 18 possible ways of partitioning the vertices of $C_{10}^2$ as listed in the table. None of the $C_{10}^2$ are 4.s.c. with respect to the partition $P$ of $V(C_{10}^2)$.

| | $V_1$ | $V_2$ | $V_3$ | $V_4$ |
|---|---|---|---|---|
| 1 | $\{v_1, v_2, v_5, v_6\}$ | $\{v_3, v_8\}$ | $\{v_4, v_9\}$ | $\{v_7, v_{10}\}$ |
| 2 | $\{v_1, v_2, v_5, v_6\}$ | $\{v_3, v_9\}$ | $\{v_4, v_8\}$ | $\{v_7, v_{10}\}$ |
| 3 | $\{v_1, v_2, v_5, v_7\}$ | $\{v_3, v_8\}$ | $\{v_4, v_9\}$ | $\{v_6, v_{10}\}$ |
| 4 | $\{v_1, v_2, v_5, v_7\}$ | $\{v_3, v_9\}$ | $\{v_4, v_8\}$ | $\{v_6, v_{10}\}$ |
| 5 | $\{v_1, v_2, v_5, v_7\}$ | $\{v_3, v_8\}$ | $\{v_4, v_{10}\}$ | $\{v_6, v_9\}$ |
| 6 | $\{v_1, v_2, v_5, v_7\}$ | $\{v_3, v_{10}\}$ | $\{v_4, v_8\}$ | $\{v_6, v_9\}$ |
| 7 | $\{v_1, v_2, v_6, v_7\}$ | $\{v_3, v_8\}$ | $\{v_4, v_9\}$ | $\{v_5, v_{10}\}$ |
| 8 | $\{v_1, v_2, v_6, v_7\}$ | $\{v_3, v_{10}\}$ | $\{v_4, v_9\}$ | $\{v_5, v_8\}$ |
| 9 | $\{v_1, v_2, v_6, v_7\}$ | $\{v_3, v_8\}$ | $\{v_4, v_{10}\}$ | $\{v_5, v_9\}$ |
| 10 | $\{v_1, v_2, v_6, v_7\}$ | $\{v_3, v_9\}$ | $\{v_4, v_8\}$ | $\{v_5, v_{10}\}$ |
| 11 | $\{v_1, v_2, v_6, v_7\}$ | $\{v_3, v_9\}$ | $\{v_4, v_{10}\}$ | $\{v_5, v_8\}$ |
| 12 | $\{v_1, v_2, v_6, v_7\}$ | $\{v_3, v_{10}\}$ | $\{v_4, v_8\}$ | $\{v_5, v_9\}$ |
| 13 | $\{v_1, v_2, v_6, v_8\}$ | $\{v_3, v_7\}$ | $\{v_4, v_9\}$ | $\{v_5, v_{10}\}$ |
| 14 | $\{v_1, v_2, v_6, v_8\}$ | $\{v_3, v_9\}$ | $\{v_4, v_7\}$ | $\{v_5, v_{10}\}$ |
| 15 | $\{v_1, v_2, v_6, v_8\}$ | $\{v_3, v_7\}$ | $\{v_4, v_{10}\}$ | $\{v_5, v_9\}$ |
| 16 | $\{v_1, v_2, v_6, v_8\}$ | $\{v_3, v_{10}\}$ | $\{v_4, v_7\}$ | $\{v_5, v_9\}$ |
| 17 | $\{v_1, v_2, v_7, v_8\}$ | $\{v_3, v_6\}$ | $\{v_4, v_9\}$ | $\{v_5, v_{10}\}$ |
| 18 | $\{v_1, v_2, v_7, v_8\}$ | $\{v_3, v_6\}$ | $\{v_4, v_{10}\}$ | $\{v_5, v_9\}$ |

Thus, $C_n^2$ is not 4-s.c. for any partition $P$ of $V(C_n^2)$.

**5-s.c $C_n^2$:**

From the Equation (2.1), $5 \leq n \leq 10$. But from (ii) of Theorem 1.1, $n \geq 8$. Thus, $8 \leq n \leq 10$ and $1 \leq |V_i| \leq 5$. As $k$ is odd, $|V_i|,\ 1 \leq i \leq 5$ are either all odd or all even. We have the following cases.

| $\|V_1\|$ | $\|V_2\|$ | $\|V_3\|$ | $\|V_4\|$ | $\|V_5\|$ | i-deg of $V_1$ | i-deg of $V_2$ | i-deg of $V_3$ | i-deg of $V_4$ | i-deg of $V_5$ | n |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 10 |
| 1 | 1 | 1 | 1 | 5 | 0 | 0 | 0 | 0 | 2 | 9 |
| 1 | 1 | 1 | 3 | 3 | 0 | 1 | 1 | 1 | 1 | 9 |

For the second case, the unique way of partitioning the vertices is $V_1 = \{v_3\}, V_2 = \{v_5\}, V_3 = \{v_7\}, V_4 = \{v_9\}, V_5 = \{v_1, v_2, v_4, v_6, v_8\}$. For this partition, $(C_9^2)_5^P$ is regular with regularity 4, but is not 5-s.c.
The last case is not possible, as $|V_4| = 3$ and $i$-degree of all the vertices of $V_4$ is 1.
For the first case, $(C_{10}^2)_5^P$ is 5-s.c with respect to the partition $P$ as shown in Figure 7.

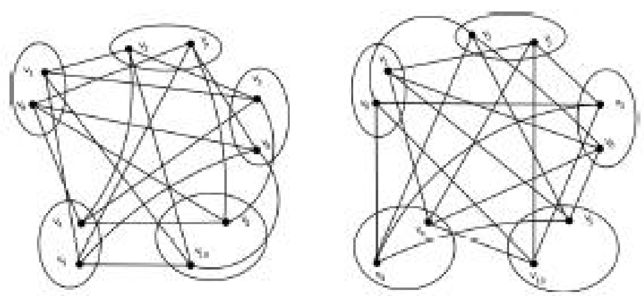

Figure 7: $C_{10}^2$ and $(C_{10}^2)_5^P$ with .

**2.1. $k$-co self complementary $C_n^m$**

The graph $C_n^m$ is $k$-co s.c if $(C_n^m)_k^P \cong \overline{C_n^m}$. In $\overline{C_n^m}$, degree of all the vertices is $(n - 2m - 1)$. We obtain bounds for number of vertices in terms of $k$ and $m$. We conclude this section by proving that there are no $C_n^2$ which are $k$-co-s.c. for any partition $P$ of $V(C_n^2)$. First we prove some necessary results.

**Theorem 2.17.** *Let $C_n^m$ be k-co-s.c. with respect to the partition $P = \{V_1, V_2, \ldots, v_k\}$. Then, all the vertices of $V_i$, $1 \le i \le k$ are of same i-degree.*

**Proof.** Let $v, w \in V_i, 1 \leq i \leq k$ and $i$-degrees of $v$ and $w$ be $r$ and $s$, respectively. Therefore, outside $V_i$, the vertex $v$ is adjacent to $(2m - r)$ vertices and $w$ is adjacent to $(2m - s)$ vertices. As $C_n^m$ is $k$-co s.c., degree of $v$ in $k$- complement of $C_n^m$ is $(n - 2m - 1)$. The vertex $v$ is non adjacent to exactly $(n - 2m - 1 - r)$ vertices outside $V_i$, which implies
$|V_1| + \ldots + |V_{i-1}| + |V_{i+1}| + \ldots |V_k| = 2m - r + n - 2m - 1 - r = n - 2r - 1.$
Similarly, as $i$-degree of $w$ is $s$, we have
$|V_1| + \ldots + |V_{i-1}| + |V_{i+1}| + \ldots |V_k| = 2m - s + n - 2m - 1 - s = n - 2s - 1.$
Therefore $r = s$. □

**Theorem 2.18.** *Let $C_n^m$ be $k$-co s.c. with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(C_n^m)$. Then $|V_i|, 1 \leq i \leq k$, is an odd number.*

**Proof.** From Theorem 2.17, if $i$-degree of $v$ is $r$, for $v \in V_i$, then
$|V_1| + \ldots + |V_{i-1}| + |V_{i+1}| + \ldots |V_k| = n - 2r - 1$. Therefore $|V_i| = 2r + 1$, an odd number. □

Let $C_n^m$ be $k$-co s.c. with respect to a partition $P$ of $V(C_n^m)$. Then the following remarks hold.

**Remark 2.19.** *When $k$ is odd, $n$ is also odd and when $k$ is even, $n$ is also even.*

**Remark 2.20.** *As $|V_i|$ is an odd number, $i$-degree of every vertex is even and so is $o$-degree.*

**Theorem 2.21.** *Let $C_n^m$ be 2-co s.c. with respect to the partition $P = \{V_1, V_2\}$ of $V(C_n^m)$. Let $v \in V_i, i = 1, 2$, then $i$-degree of $v$ is at least 2.*

**Proof.** If $i$-degree of $v \in V_i$ is 0, then $|V_i| = 1$. The vertex $v$ is adjacent to $2m$ vertices and non adjacent to $2m$ vertices in $V_j, j \neq i$. Therefore the vertices of $V_j$, has different $i$-degree. Hence, $i$-degree of $v$ is at least 2. □

**Theorem 2.22.** *Let $C_n^m$ be $k$-co s.c. with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$. If $2r_1, 2r_2, \ldots, 2r_k$ $(0 \leq r_i \leq m - 1,\ 1 \leq i \leq k)$ are $i$-degrees of the vertices of the partite sets $V_1, V_2, \ldots, V_k$ respectively, then $n = 4 \sum_{i=1}^{k} r_i + k$ and $k + 4 \leq n \leq 4km - 3k$.*

**Proof.** From Theorem 2.18, $|V_i| = 4r_i + 1, \;\; 1 \leq i \leq k$. Thus $n = 4(r_1 + r_2 + \ldots + r_k) + k$. But, if all $r_i = 0$, then $|V_i| = 1$ and $k = n$. Therefore at least one $r_i \geq 1$, which implies $n \geq k + 4$.
If all $r_i = m - 1$, then $n \leq 4km - 3k$. □

**Remark 2.23.** *From Theorem 2.22, it is clear that $n$ can take the values $k+4$ or $k+8$ or $\ldots$ or $4km - 3k$.*

Next we show that there are no $C_n^2$ which are $k$-co-s.c. for any partition $P$ of $V(C_n^2)$.

**Theorem 2.24.** *$C_n^2$ is not $k$-co-s.c with respect to any partition $P$ of $V(C_n^2)$, if $k \geq 7$.*

**Proof.** We have $k + 4 \leq n$. Thus, for $k \geq 7$, we have $n \geq 11$. And $i$-degree of all the vertices are either 0 or 2. Also, $i$-degree of all the vertices in at least one of the partite sets is 2 and all other may be zero. Therefore, at least one $\langle V_i \rangle$, is $C_5$. But, in $C_n^2, n \geq 11$, the length of the cycle is at least $\lceil \frac{n}{2} \rceil$ , i.e. at least $C_6$. Since it is not possible to decompose $C_n^2, n \geq 11$, into disjoint union of at least one $C_5$ and other $K_1$'s, the theorem follows. □

**Corollary 2.25.** *The only possibility for $C_n^2$ to be $k$-co-s.c. with respect to any partition $P$ of $V(C_n^2)$ is when $8 \leq n \leq 10$.*

**Proof.** From Theorem 2.24, $n \leq 10$. But for $n \leq 7$, we cannot decompose $C_n^2$ into disjoint union of at least one $C_5$ and other $K_1$' s. □

$^\dagger If C_n^2$ is $k$-co-s.c. with respect to a partition $P$ of $V(C_n^2)$ then $8 \leq n \leq 10$ and $2 \leq k \leq 6$.

**Theorem 2.26.** *The graph $C_n^2$ is not $k$-co-s.c. with respect to any partition $P$ of $V(C_n^2)$.*

**Proof.** From the Note 2.1, we consider the following cases.

**Case I:** When $k = 2$ and $n = 10$, $|V_1| = |V_2| = 5$, and $i$-degree of all the vertices are 2. Therefore $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are $C_5$'s which is not possible.

**Case II:** When $k = 3$, either $|V_1| = 5, |V_2| = |V_3| = 1 \Rightarrow n = 7$ or $|V_1| = |V_2| = 5, |V_3| = 1 \Rightarrow n = 11$. This is not possible.

**Case III:** When $k = 4$ and $n = 8$, $|V_1| = 5, |V_2| = |V_3| = |V_4| = 1$, and $i$-degree of all the vertices of $V_1$ is 2 and of the remaining vertices is zero. Thus, the unique way of partitioning the vertices is $V_1 = \{v_1, v_2, v_4, v_5, v_7\}, V_2 = \{v_3\}, V_3 = \{v_6\}, V_4 = \{v_8\}$, which is not 4-co-s.c.

**Case IV:** When $k = 5$ and $n = 9$, $|V_1| = 5, |V_2| = |V_3| = |V_4| = |V_5| = 1$, and $i$-degree of all the vertices of $V_1$ is 2 and remaining vertices is zero. Thus, the unique way of partitioning is $V_1 = \{v_1, v_2, v_4, v_6, v_8\}, V_2 = \{v_3\}, V_3 = \{v_5\}, V_4 = \{v_7\}, V_5 = \{v_9\}$ which is not 5-co-s.c.

**Case V:** When $k = 6$ and $n = 10$, $|V_1| = 5, |V_2| = |V_3| = |V_4| = |V_5| = |V_6| = 1$, and $i$-degree of all the vertices of $V_1$ is 2 and of the remaining vertices is zero. The unique way of partitioning is $V_1 = \{v_1, v_3, v_5, v_7, v_9\}, V_2 = \{v_2\}, V_3 = \{v_4\}, V_4 = \{v_6\}, V_5 = \{v_8\}, V_6 = \{v_{10}\}$, which is not 6-k-co-s.c.
For all these 5 cases, we get regular graphs, but they are not $k$-co- s.c. for any partition $P$ of $V(C_n^2)$. Therefore none of the $C_n^2$ is $k$-co- s.c. with respect to any partition $P$ of $V(C_n^2)$. □

## 3. Conclusions:

We obtained few results related to bounds on number of vertices of $m^{th}$ powers of cycle graph $C_n$ on $n$ vertices in terms of $m$, when they are $k$-self complementary and $k$-co-self complementary with respect to some partition $P$ of its vertex set. Also we obtained all $C_n^2$ which are $k$-self complementary and the respective partition $P$ of $V(C_n^2)$ and proved that there are no $C_n^2$ which are $k$-co-self complementary for any partition $P$ of $V(C_n^2)$.

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**K. Arathi Bhat**
Department of Mathematics,
Manipal Institute of Technology
Manipal Academy of Higher Education,
Manipal
Karnataka, 576104
India
e-mail: arathi.bhat@manipal.edu

and

**G. Sudhakara**
Department of Mathematics,
Manipal Institute of Technology
Manipal Academy of Higher Education,
Manipal
Karnataka, 576104
India
e-mail: sudhakara.g@manipal.edu
Corresponding Author