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On $\triangle^m$-statistical convergence double sequences in intuitionistic fuzzy normed spaces

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Abstract

In the present paper, the basic objective of our work is to define $\triangle^m$-statistical convergence in the setup of intuitionistic fuzzy normed spaces for double sequences. We have proved some examples which shows this method of convergence is more generalized. Further, we defined the $\triangle^m$-statistical Cauchy sequences in these spaces and given the Cauchy convergence criterion for this new notion of convergence.

Keywords:*: Statistical Convergence, $\triangle^m$-Statistical Convergence, Double Sequence, Intuitionistic Fuzzy Normed Space*

AMS subject classification: *40A35; 26E50; 40G15; 46S40.*

1. Introduction

Zadeh [35] in 1965, introduced the theory on fuzzy sets and later on large number of research papers appeared in literature based on the concept of fuzzy sets/numbers. Further, fuzzification of many classical theories has produced many interesting and useful applications in different areas. Park [24] presented the important type of metric space named as intuitionistic fuzzy metric space and further along with Saadati [26] worked on its generalized concept as intuitionistic fuzzy normed space, that is, a highly motivated area of research due to its analytic properties and their generalizations for providing a tool for mathematical modelling of real life situations where fuzzy theory alone can't work. The basic terms of intuitionistic fuzzy normed space are given below:

Definition 1. *[27] A continuous t-norm is the mapping* $\otimes : [0,1] \times [0,1] \to [0,1]$ *such that*

1. $\otimes$ *is continuous, associative, commutative and with identity 1,*
2. $a_1 \otimes b_1 \leq a_2 \otimes b_2$ *whenever* $a_1 \leq a_2$ *and* $b_1 \leq b_2$, $\forall\, a_1, a_2, b_1, b_2 \in [0,1]$.

Definition 2. *[27] A continuous t-conorm is the mapping* $\odot : [0,1] \times [0,1] \to [0,1]$ *such that*

1. $\odot$ *is continuous, associative, commutative and with identity 0,*
2. $a_1 \odot b_1 \leq a_2 \odot b_2$ *whenever* $a_1 \leq a_2$ *and* $b_1 \leq b_2$, $\forall\, a_1, a_2, b_1, b_2 \in [0,1]$.

Definition 3. *[26] An intuitionistic fuzzy normed space (IFNS) is referred to the 5-tuple* $(X, \varphi, \vartheta, \otimes, \odot)$ *with vector space* X*, fuzzy sets* φ, ϑ *on* $X \times (0, \infty)$*, continuous t-norm* $\otimes$ *and continuous t-conorm* $\odot$*, if for each* $y, z \in X$ *and* $s, t > 0$*, we have*

1. $\varphi(y,t) + \vartheta(y,t) \leq 1$,
2. $\varphi(y,t) > 0$ and $\vartheta(y,t) < 1$,
3. $\varphi(y,t) = 1$ and $\vartheta(y,t) = 0$ iff $y = 0$,
4. $\varphi(\alpha y, t) = \varphi(y, \frac{t}{|\alpha|})$ *and* $\vartheta(\alpha y, t) = \vartheta(y, \frac{t}{|\alpha|})$ *for* $\alpha \neq 0$,
5. $\varphi(y,s) \otimes \varphi(z,t) \leq \varphi(y+z, s+t)$ *and* $\vartheta(y,s) \odot \vartheta(z,t) \geq \vartheta(y+z, s+t)$,
6. $\varphi(y, \circ) : (0,\infty) \to [0,1]$ *and* $\vartheta(y, \circ) : (0,\infty) \to [0,1]$ *are continuous,*

7. $\lim_{t\to\infty} \varphi(y,t) = 1$, $\lim_{t\to 0} \varphi(y,t) = 0$, $\lim_{t\to\infty} \vartheta(y,t) = 0$, *and* $\lim_{t\to 0} \vartheta(y,t) = 0$.

Then (φ,ϑ) *is known as intuitionistic fuzzy norm.*

Example 1. *[26] Let* $(X, \|\circ\|)$ *be any normed space. For every* $t > 0$ *and all* $y \in X$, *take* $\varphi(y,t) = \frac{t}{t+\|y\|}$, $\vartheta(y,t) = \frac{\|y\|}{t+\|y\|}$. *Also,* $a \otimes b = ab$ *and* $a \odot b = \min\{a+b, 1\}\ \forall\ a, b \in [0,1]$.
Then, a 5-tuple $(X, \varphi, \vartheta, \otimes, \odot)$ *is an IFNS which satisfies the above mentioned conditions.*

Definition 4. *[26] Let* $(X, \varphi, \vartheta, \otimes, \odot)$ *be an IFNS with norm* (φ,ϑ). *A sequence* $y = (y_k)$ *in* X *is called convergent to some* $\xi \in X$ *with respect to the intuitionistic fuzzy norm* (φ,ϑ) *if there exists* $k_0 \in \mathbf{N}$ *for each* $\varepsilon > 0$ *and* $t > 0$ *such that* $\varphi(y_k - \xi, t) > 1 - \varepsilon$ *and* $\vartheta(y_k - \xi, t) < \varepsilon$ *for all* $k \geq k_0$. *It is denoted by* $(\varphi,\vartheta) - \lim_{k\to\infty} y_k = \xi$.

In 1951, Fast [9] has introduced a new concept of convergence named as statistical convergence which is more generalized than the usual convergence. It has been studied by many researchers for various types of sequences in different setups like locally convex space [16], probabilistic normed space [13], intuitionistic fuzzy normed space [14], etc. For more work related to statistical convergence, one may refer to [1, 2, 3, 4, 12, 17, 18, 20, 21].

Although, statistical convergence of sequences was established using natural density. In fact, natural density of set A, where $A \subseteq \mathbf{N}$, has given by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \mid \{a \leq n : a \in A\} \mid,$$

provided limit exists, where $\mid . \mid$ designates the order of the enclosed set. Further, A sequence $y = (y_k)$ converges statistically to ξ, if $A(\varepsilon) = \{k \in \mathbf{N} : |y_k - \xi| > \varepsilon\}$ has natural density zero (see [10]).

Definition 5. *[14] Let* $(X, \varphi, \vartheta, \otimes, \odot)$ *be an IFNS with norm* (φ,ϑ). *A sequence* $y = (y_k)$ *in* X *is called statistically convergent to some* $\xi \in X$ *with respect to the intuitionistic fuzzy norm* (φ,ϑ) *if for each* $\varepsilon > 0$ *and* $t > 0$,

$$\delta(\{k \in \mathbf{N} : \varphi(y_k - \xi, t) \leq 1 - \varepsilon \text{ or } \vartheta(y_k - \xi, t) \geq \varepsilon\}) = 0.$$

It is denoted by $S^{(\varphi,\vartheta)} - \lim_{k\to\infty} y_k = \xi$.

A double sequence $y = (y_{jk})$ is Pringsheim's convergent if there exists $j_0 \in \mathbf{N}$ for each $\varepsilon > 0$ such that $|y_{jk} - \xi| < \varepsilon$ for all $j, k \geq j_0$ (see [25]). Further, the double natural density of set E, where $E \subseteq \mathbf{N} \times \mathbf{N}$, has defined in [23] as

$$\delta_2(E) = \lim_{m,n\to\infty} \frac{1}{mn} \mid \{j \leq m, k \leq n : (j,k) \in E\} \mid,$$

provided limit exists. The notion of statistically convergent double sequences has been introduced by Tripathy [34] and Móricz [19] independently in the year 2003. A double sequence $y = (y_{jk})$ converges statistically to ξ, if double natural density of $E(\varepsilon) = \{(j,k) \in \mathbf{N} \times \mathbf{N} : |y_{jk} - \xi| > \varepsilon\}$ is zero (also see [23]). It is denoted by $S_2 - \lim_{j,k\to\infty} y_{jk} = \xi$.

Definition 6. *[22] Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). A double sequence $y = (y_{jk})$ in X is called statistically convergent to some $\xi \in X$ with respect to the intuitionistic fuzzy norm (φ, ϑ) if for each $\varepsilon > 0$ and $t > 0$,*

$$\delta_2(\{(j,k) \in \mathbf{N} \times \mathbf{N} : \varphi(y_{jk} - \xi, t) \leq 1 - \varepsilon \text{ or } \vartheta(y_{jk} - \xi, t) \geq \varepsilon\}) = 0.$$

It is denoted by $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} y_{jk} = \xi$.

Initially, Kizmaz [15] discovered the difference sequence spaces as given below

$$Z(\triangle) = \{y = (y_k) : (\triangle y_k) \in Z\}$$

for $Z = l_\infty, c, c_0$ *i.e.* spaces of all bounded sequences, convergent sequences and null sequences respectively, where $\triangle y = (\triangle y_k) = (y_k - y_{k+1})$. In particular, $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ are also Banach spaces, relative to a norm induced by $\|y\|_\triangle = |y_1| + \sup_k |\triangle y_k|$.
Moreover, the generalized difference sequence spaces was defined as (see [7]):

$$Z(\triangle^m) = \{y = (y_k) : (\triangle^m y_k) \in Z\}$$

for $Z = l_\infty, c, c_0$ where $\triangle^m y = (\triangle^m y_k) = (\triangle^{m-1} y_k - \triangle^{m-1} y_{k+1})$ so that $\triangle^m y_k = \sum_{r=0}^{m} (-1)^r \binom{m}{r} x_{k+r}$.

Tripathy [29] developed the concept of difference spaces for double sequences as

$$Z(\triangle) = \{y = (y_{jk}) = (y_{j,k}) : (\triangle y_{jk}) \in Z\}$$

for $Z = l_\infty^2, c^2, c_0^2$, where $\triangle y = (\triangle y_{jk}) = (y_{j,k} - y_{j+1,k} - y_{j,k+1} + y_{j+1,k+1})$. Also, these above described spaces are Banach spaces, relative to a norm

induced by $\|y\|_\triangle = \sup_j |y_{j,1}| + \sup_k |y_{1,k}| + \sup_{j,k} |\triangle y_{jk}|$.
The generalized difference double sequence spaces can be approximated (see [5]) as:

$$Z(\triangle^m) = \{y = (y_{jk}) : (\triangle^m y_{jk}) \in Z\}$$

for $Z = l_\infty^2, c^2, c_0^2$ where $\triangle^m y = (\triangle^m y_{jk}) = (\triangle^{m-1} y_{jk} - \triangle^{m-1} y_{j,k+1})$ so that $\triangle^m y_{j,k} = \sum_{r=0}^m \sum_{s=0}^m (-1)^{r+s} \binom{m}{r}\binom{m}{s} x_{j+r,k+s}$.

The $\triangle^m$-statistical convergence was defined by Et and Nuray [8]. Further, many researchers worked and discussed this topic in different setups [5, 6, 11, 28, 29, 30, 31, 33, 32, 33]. In the next section we extend this concept in IFNS for difference double sequences.

2. Main Results

In order to explain the $\triangle^m$-statistical convergence and related concepts according to the setup of intuitionistic fuzzy normed space(IFNS) of double sequences, we first propose the following terms.

Definition 7. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). A double sequence $y = (y_{jk})$ in X is called $\triangle^m$-statistically convergent to some $\xi \in X$ with respect to the intuitionistic fuzzy norm (φ, ϑ) if for each $\varepsilon > 0$ and $t > 0$,*

$$\delta_2(\{(j,k) \in \mathbf{N} \times \mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) \leq 1 - \varepsilon \text{ or } \vartheta(\triangle^m y_{jk} - \xi, t) \geq \varepsilon\}) = 0.$$

It is denoted by $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$.

Definition 8. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). A double sequence $y = (y_{jk})$ in X is called $\triangle^m$-statistically Cauchy with respect to the intuitionistic fuzzy norm (φ, ϑ) if there exists $j_0, k_0 \in \mathbf{N}$ for each $\varepsilon > 0$ and $t > 0$ such that for all $j, r \geq j_0$ and $k, s \geq k_0$, we have*

$$\delta_2(\{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \triangle^m y_{rs}, t) \leq 1 - \varepsilon \text{ or } \vartheta(\triangle^m y_{jk} - \triangle^m y_{rs}, t) \geq \varepsilon\}) = 0.$$

Lemma 1. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). Then the following statements are equivalent for double sequence $y = (y_{jk})$ in X whenever $\varepsilon > 0$ and $t > 0$,*

1. $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$,

2. $\delta_2(\{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) > 1-\varepsilon\}) = \delta_2(\{(j,k) \in \mathbf{N}\times\mathbf{N} : \vartheta(\triangle^m y_{jk} - \xi, t) < \varepsilon\}) = 1$,

3. $\delta_2(\{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) \leq 1-\varepsilon\}) = \delta_2(\{(j,k) \in \mathbf{N}\times\mathbf{N} : \vartheta(\triangle^m y_{jk} - \xi, t) \geq \varepsilon\}) = 0$,

4. $S_2 - \lim_{j,k\to\infty} \varphi(\triangle^m y_{jk} - \xi, t) = 1$ *and* $S_2 - \lim_{j,k\to\infty} \vartheta(\triangle^m y_{jk} - \xi, t) = 0$.

Theorem 1. *Let* $(X, \varphi, \vartheta, \otimes, \odot)$ *be an IFNS with norm* (φ, ϑ). *If* $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$, *then* ξ *is unique.*

Proof. Let if possible, $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi_1$ and $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi_2$.
For given $\varepsilon \in (0,1)$ and $t > 0$, take $\rho > 0$ such that $(1-\rho)\otimes(1-\rho) > 1-\varepsilon$ and $\rho \odot \rho < \varepsilon$. Consider

$$K_{1,\varphi}(\rho, t) = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \xi_1, t/2) \leq 1-\rho\},$$

$$K_{2,\varphi}(\rho, t) = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \xi_2, t/2) \leq 1-\rho\},$$

$$K_{3,\vartheta}(\rho, t) = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \vartheta(\triangle^m y_{jk} - \xi_1, t/2) \geq \rho\},$$

$$K_{4,\vartheta}(\rho, t) = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \vartheta(\triangle^m y_{jk} - \xi_2, t/2) \geq \rho\}.$$

Using Lemma 1 we have

$$\delta_2(K_{1,\varphi}(\rho, t)) = \delta_2(K_{3,\vartheta}(\rho, t)) = 0.$$

$$\delta_2(K_{2,\varphi}(\rho, t)) = \delta_2(K_{4,\vartheta}(\rho, t)) = 0.$$

Let $K_{\varphi,\vartheta}(\rho, t) = [K_{1,\varphi}(\rho, t) \bigcup K_{2,\varphi}(\rho, t)] \bigcap [K_{3,\vartheta}(\rho, t) \bigcup K_{4,\vartheta}(\rho, t)]$. Clearly,

$$\delta_2(K_{\varphi,\vartheta}(\rho, t)) = 0.$$

Whenever $(j,k) \in \mathbf{N}\times\mathbf{N} - K_{\varphi,\vartheta}(\rho, t)$, we have two possibilities, either $(j,k) \in \mathbf{N}\times\mathbf{N} - [K_{1,\varphi}(\rho, t) \bigcup K_{2,\varphi}(\rho, t)]$ or $(j,k) \in \mathbf{N}\times\mathbf{N} -$

$[K_{1,\vartheta}(\rho,t)\bigcup K_{2,\vartheta}(\rho,t)]$.
First we consider $(j,k)\in \mathbf{N}\times\mathbf{N}-[K_{1,\varphi}(\rho,t)\bigcup K_{2,\varphi}(\rho,t)]$. Then

$$\begin{aligned}\varphi(\xi_1-\xi_2,t) &\geq \varphi(\triangle^m y_{jk}-\xi_1,t/2)\otimes\varphi(\triangle^m y_{jk}-\xi_2,t/2)\\ &> (1-\rho)\otimes(1-\rho)\\ &> 1-\varepsilon.\end{aligned}$$

As given $\varepsilon\in(0,1)$ was arbitrary, then $\varphi(\xi_1-\xi_2,t)=1$ for all $t>0$, then $\xi_1=\xi_2$.

Similarly, if $(j,k)\in \mathbf{N}\times\mathbf{N}-[K_{3,\vartheta}(\rho,t)\bigcup K_{4,\vartheta}(\rho,t)]$,

$$\begin{aligned}\vartheta(\xi_1-\xi_2,t) &\leq \vartheta(\triangle^m y_{jk}-\xi_1,t/2)\odot\vartheta(\triangle^m y_{jk}-\xi_2,t/2)\\ &< \rho\odot\rho\\ &< \varepsilon.\end{aligned}$$

since $\varepsilon\in(0,1)$ was arbitrary, then $\vartheta(\xi_1-\xi_2,t)=0$ for all $t>0$, i.e., $\xi_1=\xi_2$. Therefore, $S_2^{(\varphi,\vartheta)}-\lim\limits_{j,k\to\infty}\triangle^m y_{jk}$ exists uniquely. □

Theorem 2. *Let $(X,\varphi,\vartheta,\otimes,\odot)$ be an IFNS with norm (φ,ϑ). If $(\varphi,\vartheta)-\lim\limits_{j,k\to\infty}\triangle^m y_{jk}=\xi$, then $S_2^{(\varphi,\vartheta)}-\lim\limits_{j,k\to\infty}\triangle^m y_{jk}=\xi$. But converse may be not true.*

Proof. Let $(\varphi,\vartheta)-\lim\limits_{j,k\to\infty}\triangle^m y_{jk}=\xi$. Then, there exists $j_0,k_0\in\mathbf{N}$ for given $\varepsilon>0$ and any $t>0$ such that for all $j\geq j_0$ and $k\geq k_0$ we have $\varphi(\triangle^m y_{jk}-\xi,t)>1-\varepsilon$ and $\vartheta(\triangle^m y_{jk}-\xi,t)<\varepsilon$,. Further, the set $A(\varepsilon,t)=\{(j,k)\in\mathbf{N}\times\mathbf{N}:\varphi(\triangle^m y_{jk}-\xi,t)\leq 1-\varepsilon$ or $\vartheta(\triangle^m y_{jk}-\xi,t)\geq\varepsilon\}$, contains only finite number of elements. We know that natural density of any finite set is always zero. Therefore, $\delta_2(A(\varepsilon,t))=0$ *i.e.* $S_2^{(\varphi,\vartheta)}-\lim\limits_{j,k\to\infty}\triangle^m y_{jk}=\xi$.

But, converse of the above result does not hold, this can be justified with the next example.

Example 2. *Let $(\mathbf{R},|.|)$ be the real normed space under the usual norm. Define $a\otimes b=ab$ and $a\odot b=\min\{a+b,1\}$ $\forall$ $a,b\in[0,1]$. Also for every $t>0$ and all $y\in\mathbf{R}$, consider $\varphi(y,t)=\frac{t}{t+|y|}$ $\vartheta(y,t)=\frac{|y|}{t+|y|}$.*
Then, clearly $(\mathbf{R},\varphi,\vartheta,\otimes,\odot)$ is an IFNS.
Define the sequence

$$\triangle^m y_{jk}=\begin{cases} jk & j \text{ and } k \text{ are squares}\\ 0 & \text{otherwise}\end{cases}$$

By given $\varepsilon > 0$ and any $t > 0$, we obtain the below set for $\xi = 0$.

$$\begin{aligned} K(\varepsilon, t) &= \{(j,k) \in \mathbf{N} \times \mathbf{N} : \varphi(\triangle^m y_{jk}, t) \leq 1 - \varepsilon \text{ or } \vartheta(\triangle^m y_{jk}, t) \geq \varepsilon\} \\ &= \{(j,k) \in \mathbf{N} \times \mathbf{N} : |\triangle^m y_{jk}| \geq \tfrac{\varepsilon t}{1-\varepsilon} > 0\} \\ &= \{(j,k) \in \mathbf{N} \times \mathbf{N} : |\triangle^m y_{jk}| = jk\} \\ &= \{(j,k) \in \mathbf{N} \times \mathbf{N} : j \text{ and } k \text{ are squares}\} \end{aligned}$$

Thus, $\frac{1}{mn}|K(\varepsilon, t)| \leq \frac{\sqrt{mn}}{mn}$. $\Rightarrow \lim_{m,n\to\infty} \frac{1}{mn}|K(\varepsilon, t)| = 0.$

Hence, $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = 0.$

By the above defined sequence $(\triangle^m y_{jk})$, we get

$$\varphi(\triangle^m y_{jk}, t) = \begin{cases} \frac{t}{t+|jk|} & j \text{ and } k \text{ are squares} \\ \\ 0 & \text{otherwise} \end{cases}$$

$$i.e.\ \varphi(\triangle^m y_{jk}, t) \leq 1,\ \forall\ j,k$$

and

$$\vartheta(\triangle^m y_{jk}, t) = \begin{cases} \frac{|jk|}{t+|jk|} & j \text{ and } k \text{ are squares} \\ \\ 0 & \text{otherwise} \end{cases}$$

$$i.e.\ \vartheta(\triangle^m y_{jk}, t) \geq 0,\ \forall\ j,k.$$

This shows that $(\varphi, \vartheta) - \lim_{j,k\to\infty} \triangle^m y_{jk} \neq 0.$

□

Next, we find the algebraic characterization in an IFNS for $\triangle^m$-statistically convergent double sequences.

Theorem 3. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). Let $y = (y_{jk})$ and $z = (z_{jk})$ be any two double sequences in X. Then*
(i) If $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$ *then* $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m \alpha y_{jk} = \alpha\xi\ ;\ \alpha \in \mathbf{R}$,
(ii) If $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi_1$ *and* $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m z_{jk} = \xi_2$ *then* $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m (y_{jk} + z_{jk}) = \xi_1 + \xi_2$.

Proof. Proof is obvious so we leave it. □

Theorem 4. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). Then $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$ iff there exists a set $J = \{(j_p, k_q) : p, q = 1, 2, 3, ...\} \subseteq \mathbf{N} \times \mathbf{N}$ such that $\delta(J) = 1$ and $(\varphi, \vartheta) - \lim_{j_p,k_q\to\infty} \triangle^m y_{j_p k_q} = \xi$.*

Proof. Necessary part: Assume $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$. For $t > 0$ and $\rho \in \mathbf{N}$, we take

$$M(\rho, t) = \{(j, k) \in \mathbf{N} \times \mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) > 1 - 1/\rho \text{ and } \vartheta(\triangle^m y_{jk} - \xi, t) < 1/\rho\},$$

and

$$K(\rho, t) = \{(j, k) \in \mathbf{N} \times \mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) \leq 1 - 1/\rho \text{ or } \vartheta(\triangle^m y_{jk} - \xi, t) \geq 1/\rho\}.$$

As $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$, then $\delta_2(K(\rho, t)) = 0$.
Also for any $t > 0$ and $\rho \in \mathbf{N}$, evidently we get $M(\rho, t) \supset M(\rho + 1, t)$, and

$$\delta_2(M(\rho, t)) = 1. \tag{2.1}$$

For $(j, k) \in M(\rho, t)$, we prove $(\varphi, \vartheta) - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$.
We prove this contrary. Suppose that double sequence $y = (y_{jk})$ is not $\triangle^m$-convergent to ξ for all $(j, k) \in M(\rho, t)$. So, there exists some $\alpha > 0$ and $k_0 \in \mathbf{N}$ such that

$$\varphi(\triangle^m y_{jk} - \xi, t) \leq 1 - \alpha \text{ or } \vartheta(\triangle^m y_{jk} - \xi, t) \geq \alpha \text{ for all } j, k \geq k_0.$$
$$\Rightarrow \varphi(\triangle^m y_{jk} - \xi, t) > 1 - \alpha \text{ and } \vartheta(\triangle^m y_{jk} - \xi, t) < \alpha \text{ for all } j, k < k_0.$$

Therefore, $\delta_2(\{(j, k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) > 1 - \alpha \text{ and } \vartheta(\triangle^m y_{jk} - \xi, t) < \alpha\}) = 0$. *i.e.* $\delta_2(M(\alpha, t)) = 0$. Since $\alpha > \frac{1}{\rho}$, then $\delta_2(M(\rho, t)) = 0$ as $M(\rho, t) \subset M(\alpha, t)$, which is a contradiction to (2.1). This shows that there exists a set $M(\rho, t)$ for which $\delta_2(M(\rho, t)) = 1$ and the double sequence $y = (y_{jk})$ is statistically $\triangle^m$-convergent to ξ.

Sufficient Part: Suppose there exists a subset $J = \{(j_p, k_q) : p, q = 1, 2, 3, ...\} \subseteq \mathbf{N} \times \mathbf{N}$ with $\delta_2(J) = 1$ and $(\varphi, \vartheta) - \lim_{j_p, k_q \to \infty} \triangle^m y_{j_p k_q} = \xi$, *i.e.* for given $\alpha > 0$ and any $t > 0$ we have $N_0 \in \mathbf{N}$ which gives

$$\varphi(\triangle^m y_{jk} - \xi, t) > 1 - \alpha \text{ and } \vartheta(\triangle^m y_{jk} - \xi, t) < \alpha \text{ for all } j, k \geq N_0.$$

Now, let

$$K(\alpha, t) = \{(j, k) \in \mathbf{N} \times \mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) \leq 1 - \alpha \text{ or } \vartheta(\triangle^m y_{jk} - \xi, t) \geq \alpha\}.$$

Then,

$$K(\alpha, t) \subseteq \mathbf{N} - \{(j_{N_0+1}, k_{N_0+1}), (j_{N_0+2}, k_{N_0+2}), ...\}.$$

As $\delta_2(J) = 1 \Rightarrow \ \delta_2(K(\alpha, t)) \leq 0$. Hence, $S_2^{(\varphi,\vartheta)} - \lim_{j,k \to \infty} \triangle^m y_{jk} = \xi$. □

Theorem 5. *Let $y = (y_{jk})$ be any double sequence in an IFNS $(X, \varphi, \vartheta, \otimes, \odot)$ with norm (φ, ϑ). Then $S_2^{(\varphi,\vartheta)} - \lim_{j,k \to \infty} \triangle^m y_{jk} = \xi$ iff there is a double sequence $x = (x_{jk})$ such that $(\varphi, \vartheta) - \lim_{j,k \to \infty} \triangle^m x_{jk} = \xi$ and $\delta_2(\{(j, k) \in \mathbf{N} \times \mathbf{N} : \triangle^m y_{jk} = \triangle^m x_{jk}\}) = 1$.*

Proof. Necessary part: Let $S_2^{(\varphi,\vartheta)} - \lim_{j,k \to \infty} \triangle^m y_{jk} = \xi$. By Theorem 4 we get a set $J = \{(j_p, k_q) : p, q = 1, 2, 3, ...\} \subseteq \mathbf{N} \times \mathbf{N}$ with $\delta_2(J) = 1$ and $(\varphi, \vartheta) - \lim_{j_p, k_q \to \infty} \triangle^m y_{j_p k_q} = \xi$.
Consider the sequence

$$\triangle^m x_{jk} = \begin{cases} \triangle^m y_{jk} & (j, k) \in J \\ \xi & \text{otherwise} \end{cases}$$

which gives the required result.

Sufficient Part: Consider $x = (x_{jk})$ and $z = (z_{jk})$ in X with $(\varphi, \vartheta) - \lim_{j,k \to \infty} \triangle^m x_{jk} = \xi$ and $\delta_2(\{(j, k) \in \mathbf{N} \times \mathbf{N} : \triangle^m y_{jk} = \triangle^m x_{jk}\}) = 1$. Then for each $\varepsilon > 0$ and $t > 0$,

$$\{(j, k) \in \mathbf{N} \times \mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) \leq 1 - \varepsilon \text{ or } \vartheta(\triangle^m y_{jk} - \xi, t) \geq \varepsilon\} \subseteq A \cup B,$$

where $A = \{(j, k) \in \mathbf{N} \times \mathbf{N} : \varphi(\triangle^m x_{jk} - \xi, t) \leq 1 - \varepsilon \text{ or } \vartheta(\triangle^m x_{jk} - \xi, t) \geq \varepsilon\}$ and $B = \{(j, k) \in \mathbf{N} \times \mathbf{N} : (\triangle^m y_{jk} \neq \triangle^m x_{jk})\}$.

Since $(\varphi,\vartheta) - \lim_{j,k\to\infty} \triangle^m x_{jk} = \xi$ then the set A contains at most finitely many terms.
Also $\delta_2(B) = 0$ as $\delta_2(B^c) = 1$ where $B^c = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \triangle^m y_{jk} = \triangle^m x_{jk}\}$. Therefore

$$\delta_2(\{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t) \leq 1-\varepsilon \text{ or } \vartheta(\triangle^m y_{jk} - \xi, t) \geq \varepsilon\}) = 0.$$

We get $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$. □

Theorem 6. *Let $y = (y_{jk})$ be a double sequence in an IFNS $(X,\varphi,\vartheta,\otimes,\odot)$. Then $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$ if and only if there exists two double sequences $z = (z_{jk})$ and $x = (x_{jk})$ in X such that $\triangle^m y_{jk} = \triangle^m z_{jk} + \triangle^m x_{jk}$ for all $(j,k) \in \mathbf{N}\times\mathbf{N}$ where $(\varphi,\vartheta) - \lim_{j,k\to\infty} \triangle^m z_{jk} = \xi$ and $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m x_{jk} = \xi$.*

Proof. Necessary part: Let $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$. By Theorem 4 we get a set $J = \{(j_p,k_q) : p,q = 1,2,3,....\} \subseteq \mathbf{N}\times\mathbf{N}$ with $\delta_2(J) = 1$ and $(\varphi,\vartheta) - \lim_{j_p,k_q\to\infty} \triangle^m y_{j_pk_q} = \xi$.
Consider the double sequences $z = (z_{jk})$ and $x = (x_{jk})$

$$\triangle^m z_{jk} = \begin{cases} \triangle^m y_{jk} & (j,k) \in J \\ \xi & \text{otherwise} \end{cases}$$

and

$$\triangle^m x_{jk} = \begin{cases} 0 & (j,k) \in J, \\ \triangle^m y_{jk} - \xi & \text{otherwise} \end{cases}$$

which gives the required result.

Sufficient Part: Consider $x = (x_{jk})$ and $z = (z_{jk})$ in X with $\triangle^m y_{jk} = \triangle^m z_{jk} + \triangle^m x_{jk}$ for all $(j,k) \in \mathbf{N}\times\mathbf{N}$ where $(\varphi,\vartheta) - \lim_{j,k\to\infty} \triangle^m z_{jk} = \xi$ and $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m x_{jk} = \xi$. Then we get result using Theorem 2 and Theorem 3. □

Theorem 7. *Let $(X,\varphi,\vartheta,\otimes,\odot)$ be an IFNS with norm (φ,ϑ). Then subsequence of a double sequence which is $\triangle^m$-statistically convergent, is also $\triangle^m$-statistically convergent with respect to (φ,ϑ).*

Proof. *Proof is obvious so we leave it.* □

In the next result we establish the Cauchy criterion for $\triangle^m$-statistically convergent sequences in IFNS.

Theorem 8. *A double sequence $y = (y_{jk})$ in IFNS $(X, \varphi, \vartheta, \otimes, \odot)$ is $\triangle^m$-statistically convergent with respect to (φ, ϑ) if and only if it is $\triangle^m$-statistically Cauchy with respect to (φ, ϑ).*

Proof. Let $S_2^{(\varphi,\vartheta)} - \lim_{j,k\to\infty} \triangle^m y_{jk} = \xi$. Then, for $\varepsilon > 0$ and $t > 0$, take $\rho > 0$ such that $(1-\rho)\otimes(1-\rho) > 1-\varepsilon$ and $\rho \odot \rho < \varepsilon$. Let $K(\rho, t) = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \xi, t/2) \leq 1-\rho \text{ or } \vartheta(\triangle^m y_{jk} - \xi, t/2) \geq \rho\}$, therefore $\delta_2(K(\rho,t)) = 0$ and $\delta_2([K(\rho,t)]^c) = 1$.

Let $M(\varepsilon, t) = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \triangle^m y_{rs}, t) \leq 1-\varepsilon \text{ or } \vartheta(\triangle^m y_{jk} - \triangle^m y_{rs}, t) \geq \varepsilon\}$.
Now, we prove $M(\varepsilon,t) \subset K(\rho,t)$, for this if $(j,k) \in M(\varepsilon,t) - K(\rho,t)$. Then we get

$$\varphi(\triangle^m y_{jk} - \xi, t/2) \leq 1-\rho \text{ or } \vartheta(\triangle^m y_{jk} - \xi, t/2) \geq \rho. \text{ Also}$$

$$\begin{aligned} 1-\varepsilon \geq \varphi(\triangle^m y_{jk} - \triangle^m y_{rs}, t) &\geq \varphi(\triangle^m y_{jk} - \xi, t/2) \otimes \varphi(\triangle^m y_{rs} - \xi, t/2) \\ &> (1-\rho)\otimes(1-\rho) \\ &> 1-\varepsilon \end{aligned}$$

and

$$\begin{aligned} \varepsilon \leq \vartheta(\triangle^m y_{jk} - \triangle^m y_{rs}, t) &\leq \vartheta(\triangle^m y_{jk} - \xi, t/2) \odot \vartheta(\triangle^m y_{rs} - \xi, t/2) \\ &< \rho \odot \rho \\ &< \varepsilon. \end{aligned}$$

which is impossible. Therefore $M(\varepsilon,t) \subset K(\rho,t)$ and $\delta_2(M(\varepsilon,t)) = 0$ *i.e.* $y = (y_{jk})$ is $\triangle^m$-statistically Cauchy with respect to (φ, ϑ).

Conversely, assume that $y = (y_{jk})$ is $\triangle^m$-statistically Cauchy with respect to (φ, ϑ) but not $\triangle^m$-statistically convergent with respect to (φ, ϑ). Thus for $\varepsilon > 0$ and $t > 0$, $\delta_2(M(\varepsilon,t)) = 0$, where

$$M(\varepsilon, t) = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk} - \triangle^m y_{j_0k_0}, t) \leq 1-\varepsilon \text{ or } \vartheta(\triangle^m y_{jk} - \triangle^m y_{j_0k_0}, t) \geq \varepsilon\}.$$

Take $\rho > 0$ such that $(1-\rho) \otimes (1-\rho) > 1-\varepsilon$ and $\rho \odot \rho < \varepsilon$. Also, $\delta_2(K(\rho,t)) = 0$, where

$$K(\rho,t) = \{(j,k) \in \mathbf{N}\times\mathbf{N} : \varphi(\triangle^m y_{jk}-\xi, t/2) > 1-\rho \text{ and } \vartheta(\triangle^m y_{jk}-\xi, t/2) < \rho\}.$$

Now

$$\begin{aligned} \varphi(\triangle^m y_{jk} - \triangle^m y_{j_0k_0}, t) &\geq \varphi(\triangle^m y_{jk} - \xi, t/2) \otimes \varphi(\triangle^m y_{j_0k_0} - \xi, t/2) \\ &> (1-\rho) \otimes (1-\rho) \\ &> 1-\varepsilon \end{aligned}$$

and

$$\begin{aligned} \vartheta(\triangle^m y_{jk} - \triangle^m y_{j_0k_0}, t) &\leq \vartheta(\triangle^m y_{jk} - \xi, t/2) \odot \vartheta(\triangle^m y_{j_0k_0} - \xi, t/2) \\ &< \rho \odot \rho \\ &< \varepsilon. \end{aligned}$$

Therefore, $\delta_2([M(\varepsilon,t)]^c) = 0$ i.e. $\delta_2(M(\varepsilon,t)) = 1$, which is a contradiction as $y = (y_{jk})$ is $\triangle^m$-statistically Cauchy.
Hence, $y = (y_{jk})$ is $\triangle^m$-statistically convergent with respect to (φ, ϑ). □

Conflict of Interest/Competing interests:
The authors declare that there are no conflicts of interest.

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