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Spectral operation in locally convex algebras

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Abstract

We show that if A is a spectrally bounded algebra, then all functions operate spectrally on A if and only if $Sp_A x$ is finite for every $x \in A$. We also prove that if A is a commutative Q-l.m.c.a, then all functions operate spectrally on A if and only if $A/RadA$ is algebraic. Furthermore, if A is a semi-simple commutative Q-l.m.c.a. which is a Baire space, all functions operate spectrally on A if and only if it is isomorphic to $\mathbf{C}^n$ for some $n \in \mathbf{N}$. A structure result concerning semi-simple commutative complete m-convex algebras of countable dimension is also given.

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1. Preliminaries

A locally convex algebra (*l.c.a.*) is a locally vector space which is an algebra with separately continuous multiplication. A locally convex algebra is said to be m-convex (*l.m.c.a.*) if its topology can be given by a family of submultiplicative seminorms. An algebra A is said to be F-algebra if A is an F-space with continuous multiplication. An F-algebra is called Fréchet algebra if it is m-convex. A locally convex algebra A with unit is called a Q-algebra if the group $G(A)$ of its invertible elements is open. Let A be an algebra with unit e. The spectrum of an element x of A, denoted by $Sp_A x$, is defined by:

$$Sp_A x = \{\lambda \in \mathbf{C} : x - \lambda e \notin G(A)\}$$

The spectral radius $\rho_A(x)$ of x is given by:

$$\rho_A(x) = \sup\{|\lambda| : \lambda \in Sp_A x\}.$$

We say that an algebra A is spectrally bounded if the spectrum of every element of A is bounded. An element of an algebra A is said to be algebraic if it is the root of a non-zero complex polynomial. The algebra A is said to be algebraic if each element of A is algebraic. A commutative algebra A with unit is said to be an integral domain if the product of any two nonzero elements of A is nonzero. The Jacobson radical of an algebra A with unit, denoted by $Rad(A)$, is the intersection of all left maximal ideals of A. If $Rad(A) = \{0\}$, we say that A is semi-simple. Let A be a semi-simple commutative Banach algebra and D be a subset of $\mathbf{C}$. We say that a function $f : D \to \mathbf{C}$ operates on A, in the sense of Katznelson ([6]), if for every $x \in A$ such that $Sp_A x \subset D$, there exists a unique element in A denoted by $f(x)$ such that $\widehat{f(x)} = f \circ \widehat{x}$, where $\widehat{x}$ denotes the Fourier-Gelfand transform of x. In the sequel, $H(\mathbf{C})$ denotes the algebra of all complex-valued functions endowed with the family of semi-norms $(p_k)_{k\geq 1}$ where $p_k(f) = \sup\limits_{|z|\leq k} |f(z)|$. It is a Fréchet algebra. This topology is also defined by the family of semi-norms

$$q_k\left(\sum_{n\geq 0} a_n z^n\right) = \sup_{|z|\leq k}\left(\sum_{n\geq 0} |a_n z^n|\right).$$

Let (A, τ) be a complex *l.c.a* with unit. We will say that an entire function $f = \sum\limits_{n\geq 0} a_n z^n \in H(\mathbf{C})$ operates in (A, τ) if for every $x \in A$ the series $\sum\limits_{n\geq 0} a_n x^n$ converges in (A, τ).

For a detailed account of the basic property of m-convex and Q-algebras see [9]. All algebras considered here are complex and with unit e.

2. Introduction

In [6], Y. Katznelson showed that if A is a semi-simple commutative Banach algebra, then a necessary and sufficient condition for all functions to be operate on A, in the sense of Katznelson ([6]), is that A is finite dimensional. In [3], the second author extended the previous result to Banach algebras which are not necessarily semi-simple and commutative. To do this, he introduced the notion of spectral operation namely if A is an algebra, D be a subset of $\mathbf{C}$ and $f : D \to \mathbf{C}$ be a function; then we say that f operates spectrally on A if, for every $x \in A$ such that $Sp_A x \subset D$, there exists a unique element in A denoted by $f(x)$ such that $Sp_A f(x) = f(Sp_A x)$. He showed that a necessary and sufficient condition for all functions operate spectrally on a Banach algebra is that its Jacobson's radical is of finite codimension.

In this paper, we consider the spectral operation in a more general framework. We show (Proposition 3.2) that if A is a spectrally bounded algebra (in particular Q-algebra), then all functions operate spectrally on A if and only if $Sp_A x$ is finite for every $x \in A$. Unlike the classical case, we show (Proposition 3.4) that spectral operation does not comes necessarily from morphism of algebras. We also prove (Proposition 3.11) that if A is a commutative Q-$l.m.c.a$, then all functions operate spectrally on A if and only if $A/RadA$ is algebraic. If furthermore A is a semi-simple commutative Q-$l.m.c.a.$ which is a Baire space, all functions operate spectrally on A if and only if it is isomorphic to $\mathbf{C}^n$ for some $n \in \mathbf{N}$ (Theorem 3.14). Finally if (A, τ) is a semi-simple commutative unitary $l.c.a.$ of countable dimension which admits a topology such that $\mathbf{H}(\mathbf{C})$ operate on A, we establish (Theorem 4.2) that there exists a complete m-convex topology τ' on A finer than τ such that the topological algebras (A, τ') and $(\mathbf{S}, \tau_{\mathbf{S}})$ are isomorphic, where $(\mathbf{S}, \tau_{\mathbf{S}})$ is the algebra of stationary complex sequences and $\tau_{\mathbf{S}}$ is the locally convex topology defined by the inductive system $(\mathbf{S}_k)_{k\geq 1}$ with

$$\mathbf{S}_k = \{(x_n)_n \in \mathbf{S} : \ x_n = x_k \text{ for every } n \geq k\} .$$

As a consequence, we obtain (Corollary 4.3) the structure of a commutative complete m-convex algebras of countable dimension i.e., which are of countable Hamel basis.

3. Spectral operation

We are first interested in the spectral operation in any algebra which is not necessarily topological. We start with the following result which often allows us to consider the semi-simple case.

Proposition 3.1. *Let A be an algebra. The following assertions are equivalent:*
1) *All functions operate spectrally on A.*
2) *All functions operate spectrally on $A/Rad(A)$.*

Proof. It follows from the fact that for every $x \in A$,

$$Sp_A x = Sp_{A/Rad(A)} s(x),$$

where s denotes the canonical surjection of A on $A/Rad(A)$. □

Here is now a characterization of the spectral operation in terms of spectrum.

Proposition 3.2. *Let A be a spectrally bounded algebra. The following assertions are equivalent:*
1) *All functions operate spectrally on A.*
2) *$Sp_A x$ is finite for every $x \in A$.*

Proof. **1)$\Longrightarrow$2)** Suppose that there exists $x \in A$ such that $Sp_A x$ is infinite. Let $(\lambda_n)_n \subset Sp_A x$ be a sequence of distinct elements. Consider $f : \mathbf{C} \longrightarrow \mathbf{C}$ defined by $f(\lambda) = n$ if $\lambda = \lambda_n$ and $f(\lambda) = 0$ otherwise. By **1)**, there exists $f(x) \in A$ such that $Sp_A f(x) = f(Sp_A x)$. As $n = f(\lambda_n) \in f(Sp_A x)$, we have $n \in Sp_A f(x)$ for every $n \in \mathbf{N}$; which contradicts the fact that $Sp_A f(x)$ is bounded.
2)$\Longrightarrow$1) Suppose that $Sp_A x$ is finite for every $x \in A$. Let $f : D \longrightarrow \mathbf{C}$ be a complex function and $x \in A$. If $Sp_A x =$, we put $f(x) = x$. If $Sp_A x = \{\lambda_1, ..., \lambda_n\}$, then by Lagrange's interpolation theorem, there exists $P \in \mathbf{C}[X]$ such that $P(\lambda_i) = f(\lambda_i)$ for every $i = 1, ..., n$. Put $f(x) = P(x) \in A$. Then one has $Sp_A f(x) = f(Sp_A x)$. Hence f operates spectrally on A. □

Remark 3.3. *There are several equivalent formulations of the concept of "spectrally bounded" ([10], proposition 10, p. 123) namely:*

1. $Sp_A x$ is closed for every $x \in A$ and $Sp_A x \nsubseteq \mathbf{C}$.

2. $Sp_A x$ is compact for every $x \in A$.

For other versions see [10].

Let $\mathbf{F}(\mathbf{C})$ be the algebra of all complex functions defined on $\mathbf{C}$. If $\mathbf{F}(\mathbf{C})$ operates spectrally on A at a point $x \in A$, then $Sp_A f(x) = f(Sp_A x)$ for every $f \in \mathbf{F}(\mathbf{C})$. But there does not necessarily exist a unitary morphism of algebras φ_x of $\mathbf{F}(\mathbf{C})$ in A such that $\varphi_x(f) = f(x)$ for every $f \in \mathbf{F}(\mathbf{C})$. The following result provides more information on this fact:

Proposition 3.4. *Let A be an integral domain algebra and $x \in A$ such that $\mathbf{F}(\mathbf{C})$ operates spectrally on A at x. If there exists a unitary morphism of algebras φ_x of $\mathbf{F}(\mathbf{C})$ in A such that $\varphi_x(f) = f(x)$ for every $f \in \mathbf{F}(\mathbf{C})$, then $Sp_A x$ is reduced to a point λ. Furthermore we have $\varphi_x(f) = f(\lambda)e$ for every $f \in \mathbf{F}(\mathbf{C})$.*

Proof. The algebra $\mathbf{F}(\mathbf{C})/\ker\varphi_x$ is isomorphic to the sub-algebra $\operatorname{Im}\varphi_x$ of A. It follows that $\mathbf{F}(\mathbf{C})/\ker\varphi_x$ is an integral domain algebra. We deduce that $\ker\varphi_x$ is a prime ideal. Then $\ker\varphi_x$ is maximal. Indeed $f \in \mathbf{F}(\mathbf{C})$ and $f \notin \ker\varphi_x$. We must show that

$$\ker\varphi_x + \mathbf{F}(\mathbf{C})f = \mathbf{F}(\mathbf{C}).$$

Consider $g \in \mathbf{F}(\mathbf{C})$ defined by $g(z) = 0$ if $f(z) = 0$ and $g(z) = 1/f(z)$ if $f(z) \neq 0$, so that

$$(\mathbf{1} - gf)f = 0 \in \ker\varphi_x$$

where $\mathbf{1}$ denotes the function identically equal to 1. As $\ker\varphi_x$ is prime ideal and $f \notin \ker\varphi_x$, then $\mathbf{1} - gf \in \ker\varphi_x$, so $\mathbf{1} \in \ker\varphi_x + \mathbf{F}(\mathbf{C})f$ and consequently $\ker\varphi_x + \mathbf{F}(\mathbf{C})f = \mathbf{F}(\mathbf{C})$. So $\ker\varphi_x$ is a maximal ideal. Furthermore, $\ker\varphi_x$ is contained in the ideal of $\mathbf{F}(\mathbf{C})$ constituted of functions f which are equal to zero on $Sp_A x$. So this last ideal is exactly equal to $\ker\varphi_x$. Using the fact that $\ker\varphi_x$ is a maximal ideal, we obtain that $Sp_A x$ is reduced to a point λ and moreover $\ker\varphi_x$ is of codimension 1. On the other hand, we have $\varphi_x(\mathbf{1}) = e$, where $\mathbf{1}$ is the unit of $\mathbf{F}(\mathbf{C})$. Hence $\varphi_x(f) = f(\lambda)e$ for every $f \in \mathbf{F}(\mathbf{C})$. $\square$

Remark 3.5. *The result of the previous proposition is not valid in the case where the algebra is not an integral domain. Indeed, consider the algebra $\mathbf{F}(\mathbf{C})$ and an element $x \in \mathbf{F}(\mathbf{C})$ for which the spectrum is equal to $\mathbf{C}$. We have $\mathbf{F}(\mathbf{C})$ operates on itself at point x. It suffices to take $f(x) = f$ for* *every $f \in \mathbf{F}(\mathbf{C})$. And we see that neither the spectrum of x is not reduced to a single point nor the morphism associated with this operation which is equal to the identity is not of the form given by Proposition 3.4.*

Remark 3.6. *Let (A, τ) be a l.c.a. If $\mathbf{H}(\mathbf{C})$ operates at a point $x \in A$ i.e., if entire series operate on A at x, then we easily verify that there exists a unitary morphism of algebras φ_x of $\mathbf{H}(\mathbf{C})$ in A such that $\varphi_x(f) = f(x)$ for every $f \in \mathbf{H}(\mathbf{C})$.*

In what follows, we examine the notion of spectral operation in topological algebras. As any Q-algebra is spectrally bounded, the following result is an immediate consequence of Proposition 3.2.

Proposition 3.7. *Let A be a Q-algebra. The following assertions are equivalent:*
1) *All functions operate spectrally on A.*
2) *$Sp_A x$ is finite for every $x \in A$.*

As a consequence, we obtain the following result which shows that the spectral operation considered in Banach algebras is a strong notion:

Corollary 3.8. *Let $(A, \|.\|)$ be a Banach algebra. The following assertions are equivalent:*
1) *All functions operate spectrally on A.*
2) $\dim(A/RadA) < +\infty$.

Proof. **1)$\Longrightarrow$2)** By Proposition 3.7, $Sp_A x$ is finite for every $x \in A$. And we conclude by a Kaplansky result ([5]).
2)$\Longrightarrow$1) Suppose that $\dim(A/RadA) < +\infty$, then $Sp_{A/RadA} x$ is finite for every $x \in A/RadA$. And by Proposition 3.2, all functions operate spectrally on $A/RadA$ and therefore all functions operate spectrally on A. □

Corollary 3.9. *([6], Theorem 1). Let $(A, \|.\|)$ be a semi-simple commutative Banach algebra. A necessary and sufficient condition for any function to operate in the sense of [6] on A is that A is of finite dimension.*

In a complete Q-*l.m.c.a.* in which all functions operate spectrally, Jacobson's radical is not necessarily of finite codimension as shown in the following example:

Example 3.10. *Let* $\mathbf{S}$ *be the algebra of stationary complex sequences. For* $k \geq 1$ *define* $\mathbf{S}_k$ *by:*

$$\mathbf{S}_k = \{(x_n)_n \in \mathbf{S} : \ x_n = x_k \text{ for every } n \geq k\}.$$

It is clear that $(\mathbf{S}_k)_k$ *is an increasing sequence of finite dimensional sub-algebras of* $\mathbf{S}$ *such that* $\mathbf{S} = \bigcup_{k \geq 1} \mathbf{S}_k$*. We endowed* $\mathbf{S}$ *with the locally convex topology* $\tau_{\mathbf{S}}$ *defined by the inductive system* $(\mathbf{S}_k)_{k \geq 1}$*. By a result of Arosio ([1]), the algebra* $\mathbf{S}$ *is* m*-convex algebra. Furthermore it is complete ([7]), proposition 9, p. II. 35]). Moreover* $\mathbf{S}_k$ *is barrelled for every* $k \geq 1$*. Then by ([2], corollary 3, p. III.23],* $\mathbf{S}$ *is barrelled. On the other hand the spectrum of every element of* $\mathbf{S}$ *is bounded. It follows from ([12], Corollary 3, p. 296) that* $\mathbf{S}$ *is a* Q*-algebra. Thus* $\mathbf{S}$ *is a semi-simple complete* Q*-l.m.c.a. of infinite dimension satisfying* $Sp_{\mathbf{S}}x$ *is finite for every* $x \in \mathbf{S}$*. By Proposition 3.7, all functions operate spectrally on* $\mathbf{S}$*.*

The algebra of example 3.10 is a complete *Q-l.m.c.a.* in which all functions operate spectrally and its Jacobson's radical is not of finite codimension. However, it is algebraic. Thus, we have the following result:

Proposition 3.11. *Let* A *be a commutative Q-l.m.c.a. The following assertions are equivalent:*
1) *All functions operate spectrally on* A*.*
2) $A/RadA$ *is algebraic.*

Proof. **1)**$\Longrightarrow$**2)** Without loss of generality, we suppose that the algebra A is semi-simple. Moreover $Sp_A x \neq$ for every $x \in A$. Let $\lambda_1, ..., \lambda_n$ be the spectral values of x. Put $\prod_{i=1}^{n} (x - \lambda_i e) = a$. Then $Sp_A a = \{0\}$, so $a \in RadA = \{0\}$. Hence $P(x) = 0$, where $P = \prod_{i=1}^{n} (X - \lambda_i)$.
2)$\Rightarrow$**1)** Let $x \in A$. Since x is algebraic, $Sp_A x$ is finite. And we conclude with Proposition 3.2. □

Remark 3.12. *In the previous result, the Q-property can not be dropped from hypothesis as the following example shows: Let* A *be the algebra of complex sequences* $\mathbf{C}^{\mathbf{N}}$ *endowed with the product topology. It is a semi-simple commutative Fréchet algebra in which all functions operate spectrally and which is not algebraic.*

Remark 3.13. *In Proposition 3.11, the m-convexity of the algebra is not superfluous as shown the algebra of rational fractions $\mathbf{C}(X)$ provided with the Williamson topology ([11]). It is a metrizable locally convex algebra with continuous multiplication. Furthermore, it is a Q-algebra which is not l.m.c.a. This algebra verifies assertion* **1)** *of proposition 3.11. but not assertion* **2)**.

The previous proposition shows, in particular, that a semi-simple Q-*l.m.c.a.* on which all functions operate spectrally is algebraic and not necessarily finite dimension. However, if in addition the algebra is a Baire space, then one has the following result:

Theorem 3.14. *Let A be a semi-simple commutative Q-l.m.c.a. which is a Baire space. The following assertions are equivalent:*
1)*All functions operate spectrally on A.*
2) $A \simeq \mathbf{C}^n$ *for an integer $n \in \mathbf{N}$.*

Proof. **1)$\Longrightarrow$2)** By Proposition 3.7, $Sp_A x$ is finite for every $x \in A$. Let us show that A admits a finite number of characters. Consider $(\chi_n)_n$ a sequence of non zero distinct characters of A. For $m \neq n$, put:

$$A_{m,n} = \{x \in A : \chi_n(x) = \chi_m(x)\}.$$

The $A_{m,n}$ are closed vector subspaces of A. Furthermore

$$A = \bigcup_{m \neq n} A_{m,n}$$

As A is a Baire space, there exists $(m_0, n_0) \in \mathbf{N}^2$ $(m_0 \neq n_0)$ such that $int(A_{m_0,n_0}) \neq$. So $A_{m_0,n_0} = A$. Hence $\chi_{n_0} = \chi_{m_0}$; which contradicts the fact that the characters $(\chi_n)_n$ are distinct. Let us note χ_1, χ_2,..., χ_n the non zero characters of A. So we have

$$A / \bigcap_{i=1}^{n} \ker \chi_i \simeq \prod_{i=1}^{n} A / \ker \chi_i \simeq \mathbf{C}^n.$$

□

Remark 3.15. *The result of the previous theorem is not valid without "Baire" hypothesis. Indeed, the algebra considered in example 3.10 is a semi-simple commutative complete Q-l.m.c.a. on which all functions operate spectrally and which is of infinite dimension.*

We end this section with a result on the operation in the sense of Katznelson ([6]).

Proposition 3.16. *Let A be a Q-l.m.c.a. in which all functions operate in the sense of Katznelson ([6]). Then*
1) *A is algebraic, semi-simple and commutative.*
2) *If moreover A is a Baire space, then that A is finite dimensional.*

Proof. **1)** Using the fact that the function $Z : \lambda \longmapsto \lambda$ operates in the sense of Katznelson ([6]), we show that A is without nonzero nilpotent elements. As in the proof of Proposition 3.2, one has $Sp_A x$ is finite for every $x \in A$. Let $x \in A$ and put $Sp_A x = \{\lambda_1, ..., \lambda_r\}$. Then

$$Sp_A \prod_{i=1}^{r} (x - \lambda_i e) = 0.$$

So $\prod_{i=1}^{r} (x - \lambda_i e) = 0$. This implies that A is algebraic. Let us show that A is semi-simple and commutative. Let x be a non-zero element of A. Then there exists $n \geq 1$ and $P \in \mathbf{C}[X]$ with non-zero constant term such that

$$x^n P(x) = 0 \qquad (1)$$

If x is a non-zero element of $Rad(A)$, then $P(x)$ is invertible and so $x^n = 0$, which is impossible. Whence, A is semi-simple. Consider minimal n which satisfies equality (1) and show that $n = 1$. Suppose that $n \geq 2$. One has $(x^{n-1}P(x)x)^2 = 0$. So $x^{n-1}P(x)x = 0$. Similar considerations apply to $x^{n-1}P(x)$ which also satisfies $(x^{n-1}P(x))^2 = 0$ gives $x^{n-1}P(x) = 0$. This contradicts the choice of n. Whence $xP(x) = 0$. As the constant term of P is non zero, the last equality implies that

$$x = ax^2 \qquad (2)$$

where $a \in A$. Let us now show that $xA \subset Ax$ for every $x \in A$. Let $y \in A$. Then $yx = yax^2$ by (2). An easy computation shows that $(xy - xyax)^2 = 0$. So $xy = xyax \in Ax$ for every $y \in A$. It follows that any left maximal ideal M of A is two-sided ideal of A and therefore A/M is a field. But A/M is algebraic and $\mathbf{C}$ is an algebraically closed field. Whence, A/M is isomorphic to $\mathbf{C}$. It follows that $A = A/Rad(A)$ is commutative.
2) It follows from the same method as in theorem 3.14. □

As a consequence, we obtain the following result.

Corollary 3.17. *Let A be a Fréchet Q-algebra. The following assertions are equivalent:*
1) *All functions operate on A in the sense of Katznelson ([6]).*
2) *A is semi-simple commutative and is finite dimensional.*

4. Structure of some countable dimensional algebras

The algebra of stationary complex sequences $\mathbf{S}$ given by example 3.10 is a semi-simple commutative complete Q-*l.m.c.a.* of countable dimension on which all functions operate spectrally and its Jacobson's radical is not of finite codimension. One would be tempted to look for other examples. In fact, as we will see the algebra $\mathbf{S}$ is, apart from isomorphism, the only complete semi-simple commutative *l.m.c.a.* of countable dimensional.

The following result establishes a fundamental algebraic property of *l.c.a.* of countable dimensional on which entire series operate.

Proposition 4.1. *Let (A,τ) be a l.c.a. not necessarily commutative of countable dimensional. The following assertions are equivalent:*
1) *Entire series operate on A.*
2) *A is algebraic.*

Proof. **1)$\Longrightarrow$2)** Suppose entire series operate on A. Let $x \in A$ and $f \in \mathbf{H}(\mathbf{C})$. Then $f(x) \in A$. Let φ_x be the mapping defined of $\mathbf{H}(\mathbf{C})$ in A by $\varphi_x(f) = f(x)$. It is an algebra morphism. So we have the following algebraic isomorphism:

$$\mathcal{H}(\mathbf{C})/\ker\varphi_x \simeq \ Im\varphi_x.$$

Let us show that φ_x is continuous. Let $(p_\lambda)_{\lambda\in\Lambda}$ be a family of seminorms defining τ. By lemma of [4], for every $x \in A$,

$$M_\lambda = \sup_n \left[p_\lambda\left(x^n\right)\right]^{\frac{1}{n}} < +\infty \text{ for every } \lambda \in \Lambda.$$

Now let $\lambda \in \Lambda$ and $f(z) = \sum\limits_{k=0}^{+\infty} \alpha_k z^k \in \mathbf{H}(\mathbf{C})$. Then one has

$$p_\lambda\left(f(x)\right) \leq \sum_{k=0}^{+\infty} p_\lambda\left(\alpha_k x^k\right) \leq \sum_{k=0}^{+\infty} |\alpha_k|\, M_\lambda^k \leq q_r(f)$$

where $r \in \mathbf{N}$ such that $r > M_\lambda$. Whence φ_x is continuous. It follows that $\ker\varphi_x$ is a closed ideal of $\mathbf{H}(\mathbf{C})$. Endowed with the quotient topology,

the algebra $\mathbf{H}(\mathbf{C})/\ker\varphi_x$ is a Fréchet algebra. It is necessarily of finite dimension since A is of countable dimension. Consequently $\ker\varphi_x$ which contains $\mathbf{C}[x]$ is of finite dimensional. Hence the algebraicity of A.
2)$\Longrightarrow$1) If A is algebraic, then $\mathbf{C}[x]$ is of finite dimension for every $x \in A$. So entire series operate at x on $\mathbf{C}[x]$ and therefore on A. □

Here is the main result of this paragraph.

Theorem 4.2. *Let (A, τ) be a semi-simple commutative l.c.a. of countable dimensional. We assume that A has a topology such that $\mathbf{H}(\mathbf{C})$ operate on A. Then*
1) *There exists a continuous isomorphism between the algebras $(\mathbf{S}, \tau_{\mathbf{S}})$ and (A, τ).*
2) *There exists a complete m-convex topology τ' on A finer than τ such that the topological algebras (A, τ') and $(\mathbf{S}, \tau_{\mathbf{S}})$ are isomorphic.*

Proof. **1)** By Proposition 4.1, A is algebraic. Consider $(e_n)_{n\geq 1}$ a Hamel basis of A and for every $n \geq 1$ let us put: $A_n = \mathbf{C}[e_1, ..., e_n]$ be the subalgebra of A generated by the family $\{e_1, ..., e_n\}$. As A is algebraic and commutative, the A_n are of finite dimension and $A = \bigcup_{n\geq 1} A_n$. Furthermore, for every $n \geq 1$, there exists an integer $k_n \geq 1$ such that A_n is isomorphic to the algebra $\mathbf{S}_{k_n}$. Now let's show the existence of a continuous isomorphism from $(\mathbf{S}, \tau_{\mathbf{S}})$ to (A, τ). The algebra A_n being semi simple commutative and of finite dimension, therefore there exists an isomorphism φ_n from A_n to $\mathbf{C}^{k_n}$. Moreover one has $\mathbf{C}^{k_n} = \mathbf{C}[u_n]$ where $u_n = (1, 2, ..., k_n)$. It follows that $A_n = \mathbf{C}[v_n]$ where $Spv_n = \{1, 2, ..., k_n\}$ $(v_n = \varphi_n^{-1}(u_n))$. On the other hand, consider the injection denoted by i_n from $\mathbf{C}^{k_n}$ into $\mathbf{S}$ given by:

$$i_n : \mathbf{C}^{k_n} \longrightarrow \mathbf{S} \text{ where } i_n(x_1, ..., x_{k_n}) = (x_1, ..., x_{k_n}, x_{k_n}, ...)$$

Finally $\psi_n = i_n \circ \varphi_n$ is a morphism of algebras from A_n in $\mathbf{S}$. Moreover the restriction of ψ_{n+1} to A_n is exactly ψ_n. Whence the existence of the morphism ψ which is also bijective between the algebras (A, τ) and $(\mathbf{S}, \tau_{\mathbf{S}})$. The isomorphism ψ^{-1} is continuous since the topology $\tau_{\mathbf{S}}$ of $\mathbf{S}$ is the finest of the locally convex topologies of $\mathbf{S}$.
2) Let τ' be the locally convex inductive limit topology on A defined by the inductive system $(A_n)_n$, where $(A_n)_n$ is an increasing sequence. Then by ([1]) the algebra (A, τ') is m-convex. It is complete since the A_n are of finite dimensional ([7]). Thus ψ is an algebraic and topological isomorphism between (A, τ') and $(\mathbf{S}, \tau_{\mathbf{S}})$. □

As a consequence we get the following result:

Corollary 4.3. *Let (A, τ) be a semi-simple commutative complete l.m.c.a. of countable dimensional. Then there exists a complete m-convex topology τ' on A finer than τ such that the topological algebras (A, τ') and $(\mathbf{S}, \tau_{\mathbf{S}})$ are isomorphic.*

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