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Stability and instability analysis for the Standing Waves for a generalized Zakharov-Rubenchik System

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Abstract

In this paper, we analyze the stability and instability of standing waves for a generalized Zakharov-Rubenchik system (or the Benney-Roskes system) in spatial dimensions $N = 2, 3$. *We show that the standing waves generated by the set of minimizers for the associated variational problem are stable, for* $N = 2$ *and* $\sigma(p-2) > 0$. *We also show that the standing waves are strongly unstable, for* $N = 3$ *and if either* $\sigma < 0$ *and* $\frac{4}{3} < p < 4$, *or* $\sigma > 0$ *and* $0 < p < 2$. *Results follow by using the variational characterization of standing waves, the concentration compactness principle due to J. Lions and the compactness lemma due to E. Lieb to solve the associated minimization problem.*

Keywords: *Standing waves, virial identity, stability, blow up.*

MSC numbers: *35Q55, 35B35, 35A15, 35B44.*

1. Introduction

In this work, we consider a generalization of the Zakharov-Rubenchik system given by

$$
(1.1)\qquad \begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi &= -\sigma_1\Delta_\perp\psi + \left(\sigma|\psi|^p + W(\rho + D\partial_z\varphi)\right)\psi, \\ \partial_t\rho + \sigma_2\partial_z\rho &= -\Delta_\perp\varphi - \partial_z^2\varphi - D\partial_z(|\psi|^2), \\ \partial_t\varphi + \sigma_2\partial_z\varphi &= -\frac{1}{M^2}\rho - |\psi|^2. \end{cases}
$$

In the case $p = 2$, this system describes the nonlinear interaction of high-frequency and low-frequency waves. The model was derived first by Benney-Roskes in the context of gravity waves [1] and also for Zakharov-Rubenchik in the context of the interaction of spectral narrow high-frequency wave packet of small amplitude with low-frequency acoustic type oscillations [24]. For a more precise description on this system, see the monographic work by D. Lannes in [14] (see also [1, 9, 12, 13, 21, 24]). We note for $p = 2$ that it is possible to obtain the well-known Zakharov system and Davey-Stewartson system from the system (1.1). In the supersonic regime $M > 1$, J. Cordero in [7] showed that solutions of the system (1.1) converge to solutions of the Zakharov system, under some restrictions without any limit layer (see also [6]). In the subsonic regime $M < 1$, it is expected that those solutions can be approximated by solutions of the system.

$$
(1.2)\qquad \begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi &= \left(\sigma|\psi|^p + W(\rho + D\partial_z\varphi)\right)\psi, \\ \sigma_2\partial_z\rho &= -\Delta_\perp\varphi - \partial_z^2\varphi - D\partial_z(|\psi|^2), \\ \sigma_2\partial_z\varphi &= -\frac{1}{M^2}\rho - |\psi|^2. \end{cases}
$$

For $p = 2$, J. Cordero and J. Quintero established the instability of ground state standing waves for a Zakharov-Rubenchik system (1.1) (see [8]). For N = 2, the result follows from M. Weinstein's approach used in the case of the Schrödinger equation. In this case, it was established a virial identity that relates the second variation of a momentum type functional with the Hamiltonian on a class of solutions for the Zakharov-Rubenchik system. For N = 3, J. Quintero *et al.* established the instability by using a scaling argument and the existence of invariant regions under the flow due to a concavity argument.

Regarding the generalized system (1.1), J. Quintero showed the following stability result: for $N = 2$ or $N = 3$ and $0 < p < \frac{N}{4}$ and $\sigma < 0$, there exists a sequence $(\omega_k)_k$ such that $\omega_0 > 0$, $\omega_k \to 0$, and φ_ω is stable (see [23]).

It is worth mentioning that the system (1.2) is related to the classical Davey-Stewartson system which appears in Fluid Mechanics, in the case of the evolution of weakly nonlinear water waves with a predominant direction of travel

$$
\left\{ \begin{array}{ll} i\partial_t\psi + \Delta\psi & = a|\psi|^p\psi + b_1\partial_z\varphi\psi, \\ \Delta\varphi & = b_2\partial_z(|\psi|^2) \end{array} \right. , \tag{1.3}
$$

for $a \in \mathbf{R}$, and $p, b_1, b_2 \in \mathbf{R}^+$.

We point out that the stability and the instability of the Davey-Stewartson type systems (1.2) and (1.3) were obtained using the fact that these systems can be reduced to a single Schrödinger type nonlinear equation of the form

$$
i\partial_t\psi + \Delta\psi = \sigma|\psi|^p\psi - \beta|\psi|^2\psi - \gamma E_0(|\psi|^2)\psi, \tag{1.4}
$$

where $\sigma \in \mathbf{R}$, $\beta, \gamma \geq 0$, and E_0 is a non-local linear operator defined via a Fourier multiplier. For instance, for the Davey-Stewartson system (1.3), R. Cipolatti showed instability the standing waves for $\beta > 0$, $\gamma > 0$, $N = 2, 3$ and $\sigma(p-2) \leq 0$ (see [5]). M. Otha showed for $\beta = 0$, $\sigma < 0$, $\gamma > 0$, $0 < p < \frac{N}{4}$, $N = 2$ or $N = 3$, that there exists a sequence $(\omega_k)_k$ such that $\omega_0 > 0$, $\omega_k \to 0$, and φ_ω is stable, assuming only for $N = 2$ that there is curve $\omega \to \varphi_\omega$ from $(0, \infty)$ into $H^1(\mathbf{R}^N)$ and that $||\varphi||_2 = ||\varphi_\omega||_2$. On the other hand, M. Ohta showed the following results for $\beta = 0$ and $\gamma > 0$: a) stability of the standing waves generated by the set of minimizers for the associated variational problem, for $N = 2$ and $\sigma(p-2) < 0$. b) strongly instability of the standing waves for $\sigma < 0$ and $\frac{4}{3} < p < 4$ or $\sigma > 0$ and $0 < p < 2$ (see [20]).

This paper is organized as follows. In section 2, we discuss the local well-posedness for the system (1.2). In section 3, we establish the existence of standing waves for the system (1.2), which is related to the existence of solutions for the single Schrödinger type nonlinear equation (1.4). In section 4, we use a variational approach to analyze stability in the case $N = 2$ and $\sigma(p-2) < 0$. In the case $N = 3$, $\sigma < 0$ and $\frac{4}{3} < p < 4$, we show that the standing waves are strongly unstable (see [20]). In the latter case, we use M. Weinstein's approach for the Schrödinger equation in [25] by showing that any solution $\Phi(t) = (u(t), v, w)$ of the system (1.2) in the regime $|\epsilon| = |\sigma_1|$ must blow up in finite time, using the following pseudo conformal identity

$$\frac{d^2}{dt^2}\int_{\mathbf{R}^3}(\sigma_1 x^2+\sigma_1 y^2+\epsilon z^2)|u(\mathbf{x},t)|^2\,d\mathbf{x}=8\epsilon^2 K(u(t)),$$

where K is an auxiliary functional.

2. On the Cauchy problem for the Davey-Stewartson type system

As we discussed above, we consider solutions for the system (1.1) of the form

$$\psi(\mathbf{x},t)=e^{i\omega t}u(\mathbf{x},t),\quad \rho(\mathbf{x},t)=v(\mathbf{x}),\quad \varphi(\mathbf{x},t)=w(\mathbf{x}). \tag{2.1}$$

In this case, we end up looking for solutions for the Davey-Stewartson type system (1.2). For $M<1$, we have that

$$\rho = -M^2(\sigma_2\partial_z\varphi+|\psi|^2) \tag{2.2}$$

$$\partial_z\varphi = (M^2\sigma_2-D)E(|\psi|^2), \tag{2.3}$$

where E is the non-local operator $\widehat{E(u)}(\xi)=\Gamma_M(\xi)\widehat{u}(\xi)$, with the Fourier multiplier Γ_M given by

$$\Gamma_M(\xi):=\frac{\xi_3^2}{(\xi_1^2+\xi_2^2+(1-M^2)\xi_3^2)}. \tag{2.4}$$

As we mentioned above, we see directly that system (1.2) can be written as a single like nonlinear Schrödinger equation of the form

$$i\partial_t\psi+\epsilon\partial_z^2\psi+\sigma_1\Delta_\perp\psi=\sigma|\psi|^p\psi-M^2W|\psi|^2\psi-W(D-M^2\sigma_2)^2E(|\psi|^2)\psi,$$

with $\mathcal{L}:=\epsilon\partial_z^2+\sigma_1\Delta_\perp$.

We point out that the local wellposedness and the regularity result associated with the Cauchy Problem for (1.2) follow from classical ideas in the context of the Schrödinger equation done by T. Cazenave in [3]. Results can be extended directly from the well-posedness results for the Davey-Stewarton system obtained by J. Ghidaglia and JC. Saut in [11] or by R. Cipolatti in [5] (see also J. Quintero *et. al.* [8]). More concretely, if $0<p<\frac{4}{N-2}$ and $N=2,3$, then for any $u_0\in H^1(\mathbf{R}^N)$, there exist $T_0>0$ and a unique solution $u\in C([0,T_0);H^1(\mathbf{R}^N))$ of (2) with $u(0)=u_0$. Moreover, for $0\le t<T_0$, $u(t)$ satisfies

$$\ell_\omega(u(t)) = \ell_\omega(u_0)$$
$$B(u(t)) = B(u_0),$$

where ℓ_ω and B are given by,

$$\begin{array}{ll} \ell_\omega(u) & = \frac{1}{2}I_0(u) + \frac{\omega}{B}(u) + \frac{1}{4}\Xi(u) \\ B(u) & = \|u\|_2^2 \end{array}$$

with Ξ and I_0 being defined as,

$$\begin{array}{ll} I_0(u) & = \int_{\mathbf{R}^N} \left(\sigma_1|\nabla_\perp u|^2 + \epsilon|\partial_z u|^2\right)\, d\mathbf{x}, \\ \Xi(u) & = -\int_{\mathbf{R}^N} \Big(W(D - M^2\sigma_2)^2 E(|u|^2)|u|^2 + M^2 W|u|^4 - \frac{4\sigma}{p+2}|u|^{p+2}\Big)\, d\mathbf{x}. \end{array}$$

In order to use the pseudo conformal identity for $N = 3$ to establish the instability of the standing waves, we require an existence result of solutions in the weighted Sobolev spaces

$$\Sigma := \left\{ v \in H^1(\mathbf{R}^3) \, : \, \int_{\mathbf{R}^2} \Big((1 + x^2 + y^2 + z^2)|v|^2 + |\nabla v|^2\Big)\, dx\, dy\, dz < \infty \right\},$$

such that the momentum type functional

$$M(t) = \int_{\mathbf{R}^3} (\sigma_1 x^2 + \sigma_1 y^2 + \epsilon z^2)|\psi(x, y, z, t)|^2\, dx\, dy\, dz,$$

makes sense. As done for R Cipolatti in Theorem 4.3 in [5] or in Theorem 2.4 in [4], we have that standing wave solutions of the system (1.4) belong to the set Σ. In the case of the existence result for the Cauchy Problem under the assumption that $\psi_0 \in \Sigma$, we refer for example to the work by J. Ghidaglia and J. C. Saut in the case of the Davey Stewartson systems in Theorem 3.1 and Lemma 3.1 in [11]. In particular, if $\psi_0 \in \Sigma$, then the solution (ψ, ρ, φ) for the system (1.2) satisfies that $\psi \in C([0, T^*); \Sigma)$ for $0 \le t < T^*$.

3. On the existence of standing waves

In order to look for standing waves for the system (1.2), we seek for solutions of the form

$$\psi(\mathbf{x}, t) = e^{i\omega t}u(\mathbf{x}), \quad \rho(\mathbf{x}, t) = v(\mathbf{x}), \quad \varphi(\mathbf{x}, t) = w(\mathbf{x}), \tag{3.1}$$

where $\omega > 0$. In this case (u, v, w) verifies that

$$(3.2)\quad \begin{cases} -\omega u &= -\epsilon\partial_z^2 u - \sigma_1\Delta_\perp u + (\sigma|u|^p + W(v + D\partial_z w))\, u, \\ \sigma_2\partial_z v &= -\Delta_\perp w - \partial_z^2 w - D\partial_z(|u|^2), \\ \sigma_2\partial_z w &= -\frac{1}{M^2} v - |u|^2. \end{cases}$$

As we mention above, we are interested in analyzing the stability and the instability of the standing waves for the system (1.2). In this case, if (ψ, ρ, φ) is a solution of the system (1.1) of the form (3.1), then u must satisfy the following problem

$$(3.3)\quad \omega u - \epsilon\partial_z^2 u - \sigma_1\Delta_\perp u = M^2 W|u|^2 u - \sigma|u|^p u + W(D - M^2\sigma_2)^2 E(|u|^2)u.$$

Regarding the existence of solutions (standing waves) for an equation like (3.3), we mention the work by R. Cipolatti in [4] with a similar non-local term but with only a non-linearity of order $p+1$ (no including the nonlinearity of order 3). Following Cipolatti's approach, J. Quintero in [22] obtained the existence of solutions for equation (3.3), extending Cipolatti's result in [4]. More concretely, J. Quintero established the existence of positive solutions the special elliptic equation in $\mathbf{R}^N$ with $N = 2, 3$,

$$(3.4)\qquad \omega u - \beta\Delta u + g(u) = 0,$$

where $\omega > 0$, $\beta > 0$, and the nonlinear term g is such that the model has a Hamiltonian structure. In other words, there is an operator Ξ_1 defined in $H^1(\mathbf{R}^N)$ such that $D\Xi_1(\phi) = g(\phi)$ for $\phi \in H^1(\mathbf{R}^N)$, which includes the present work with

$$g(u) = -M^2 W|u|^2 u + \sigma|u|^p u - W(D - M^2\sigma_2)^2 E(|u|^2)u, \quad \Xi_1(u) = \Xi(u).$$

In the case, for $p^* = \frac{4}{N-2}$ with $N \geq 2$, there is an existence result of solutions by imposing some restrictions on the parameters σ and $p > 0$. We begin by defining

$$\begin{aligned} a_0(p) &= \begin{cases} \infty, & \text{for } p < 2 \\ b, & \text{for } p = 2 \\ \left(\frac{p-2}{c(p+2)}\right)^{\frac{p}{p-2}} \left(\frac{(p+2)b}{2p}\right)^{\frac{p}{2}}, & \text{for } p > 2. \end{cases}, \\ a_1(p) &= \frac{b\omega(p+2)}{2\omega p - (p-2)b}, \quad \omega > b, \quad p > 2. \end{aligned}$$

We also define the sets

$$\mathcal{A}_{\omega,q,b} = \{(p, a) : 0 < p < p^*, \quad a < a_0(p)\},$$

and

$$\mathcal{B}_{\omega,q,b} = \{(p,a) : 0 < p < p^*, \quad a_1(p) < a < a_0(p)\},$$

for $\omega > b$ and $p > 2$ in the case $a_1(p) < a_0(p)$. Hereafter, we impose to $\omega > 0$ the restrictions discussed above.

Definition 3.1. *We say that a set $\Omega \subset H^1(\mathbf{R}^N)$ is stable, if for a given $\eta > 0$, there exists $\delta > 0$ with the following property: if $u_0 \in H^1(\mathbf{R}^N)$ and the solution $u(t)$ of (1.2) with $u(0) = u_0$ satisfies $\inf_{\varphi\in\Omega} ||u_0 - \varphi||_{H^1(\mathbf{R}^N)} < \delta$, then for any $t \in [0,\infty)$*

$$\inf_{\varphi\in\Omega} ||u(t) - \varphi||_{H^1(\mathbf{R}^N)} < \eta.$$

Otherwise, Ω is said to be unstable. Moreover, for $\varphi \in \mathcal{G}_\omega$, we shall say at the standing wave $u_\omega(t) = e^{it\omega}\varphi$ is stable, if the set $\mathcal{G}_\omega$ is stable, and that u_ω is unstable if the orbit $\mathcal{O}_\omega$ is unstable, where

$$\mathcal{O}_\omega = \{e^{i\theta}\varphi(\cdot + y) : \theta \in \mathbf{R},\ y \in \mathbf{R}^N\}.$$

Furthermore, we shall say that u_ω is strongly unstable if for any $\eta > 0$, there exists $u_0 \in H^1(\mathbf{R}^N)$ such that $||u_0 - \varphi||_{H^1(\mathbf{R}^N)} < \eta$ and the solution $u(t)$ of (1.2) with $u(0) = u_0$ blows up in a finite time.

4. On the stability and instability analysis

We follow the approach of H. Nawa in the case of the blow-up solutions of the nonlinear Schrödinger equation with critical power nonlinearity (see [18] and M. Ohta in the analysis of stability and instability of standing waves for the generalized Davey-Stewartson system (see [20])).

Before we go further, we set for $\mu > 0$,

$$\Sigma_\mu = \left\{ v \in H^1(\mathbf{R}^N) : I_\mu = \ell_0(v),\ ||v||_2 = \sqrt{\mu} \right\},$$
$$\mu_0 = \inf\{||v||_2,\ v \in H^1(\mathbf{R}^N) \setminus \{0\}, E_0(v) \le 0\},$$

where I_μ and E_0 are defined as

$$I_\mu = \inf\{\ell_0(v) :\ v \in H^1(\mathbf{R}^N),\ \ ||v||_2 = \sqrt{\mu}\},$$
$$E_0(u) = \tfrac{1}{2} I_0(u) - \tfrac{1}{4} R(u),$$
$$R(u) = M^2 W ||v||_4^4 + W(D - M^2\sigma_2)^2 \int_{\mathbf{R}^2} E(|u|^2)|u|^2\, d\mathbf{x}.$$

4.1. Stability results for $N = 2$

Lemma 4.1. *Let $N = 2$. 1) If $\sigma < 0$ and $0 < p < 2$, then for $0 < \mu < \mu_0$, the set Σ_μ is not empty and stable.*
2) If $\sigma > 0$ and $p > 2$, then for $\mu > \mu_0$, the set Σ_μ is not empty and stable.

Proof. The first step is to show that there is $v_0 \in H^1(\mathbf{R}^2)$ such that $v_0 \neq 0$, $E_0(v_0) = 0$ and $||v_0||_2^2 = \mu_0$. So, let $(v_j)_j$ be a minimizing sequence for $\sum_\mu$ and consider the normalization $u_j(x) = v_j(y)$ with $y = \lambda_j x$ and $\lambda_j^2 R(v_j) = 1$. Clearly, we have that

$$||u_j||_2 = ||v_j||_2 \to \mu_0, \ R(u_j) = 1 \quad E_0(u_j) = \lambda_j^2 E_0(v_j) \leq 0,$$

meaning that the sequence $(u_j)_j$ is bounded in $H^1(\mathbf{R}^2)$. So, from measure theory results by J. Fröhlich, E. Lieb and M. Loss in [10] and by E. Lieb in [15], there are a subsequence (denoted the same), a sequence of $y_j \in \mathbf{R}^2$ and $v_0 \in H^1(\mathbf{R}^2)$ such that

$$u_j(\cdot + y_j) \rightharpoonup v_0 \text{ weakly in } H^1(\mathbf{R}^2), \ |u_j(\cdot + y_j)|^2 \rightharpoonup |v_0|^2 \ \text{(weakly in } L^2(\mathbf{R}^2).$$

Moreover, we also have that $u_j(\cdot + y_j) \to v_0$ almost everywhere in $\mathbf{R}^2$. On the other hand, from the Brézis-Lieb result in [2], we also have that

$$\begin{array}{ll} E_0(u_j(\cdot + y_j)) - E_0(u_j(\cdot + y_j) - v_0) - E_0(v_0) & \to 0, \\ ||u_j(\cdot + y_j)||_2^2 - ||u_j(\cdot + y_j) - v_0||_2^2 - ||v_0||_2^2 & \to 0. \end{array}$$

We claim that $E_0(v_0) = 0$. First, assume that $E_0(v_0) > 0$. From previous fact and using that $E_0(u_j) \leq 0$, we conclude for j large enough that

$$E_0(u_j(\cdot + y_j) - v_0) \leq 0,$$

which implies that

$$\mu_0 \leq ||u_j(\cdot + y_j) - v_0||_2^2.$$

Using that $||u(\cdot + y_j)||_2^2 \to \mu_0$, we conclude that $||v_0||_2 = 0$, which is a contradiction since we are assuming that $E_0(v_0) > 0$. Then we have that $E_0(v_0) \leq 0$. Moreover, for Fatou's lemma, we have that

$$\mu_0 \leq ||v_0||_2^2 \leq \liminf ||u_j||_2^2.$$

If we had $E_0(v_0) < 0$, then there is $0 < \theta < 1$ such that $E_0(\theta v_0) = 0$. From this, we have that

$$\mu_0 \leq ||\theta v_0||_2^2 = \theta^2 ||v_0||_2^2 < ||v_0||_2^2 = \mu_0,$$

which is a contradiction. In other words, we conclude that $E_0(v_0) = 0$. Finally, we claim that

$$\mu_0 = \inf \left\{ \frac{2||v||_2^2 I_0(v)}{R(v)}, \ v \in H^1(\mathbf{R}^N) \setminus \{0\} \right\} := \nu_0.$$

Let $(v_j)_j$ be a minimizing sequence for the infimum ν_0. We define $u_j(x) = v_j(y)$ with $x = \lambda_j y$ and $\lambda_j^2 R(v_j) = 2I_0(v_j)$. We see directly that

$$E_0(u_j) = 0, \quad ||u_j||_2^2 = \frac{2||v_j||_2^2 I_0(v_j)}{R(v_j)} \geq \mu_0,$$

which implies that $\mu_0 \leq \nu_0$. On the other hand, using that $E_0(v_0) = 0$, we also have that

$$\nu_0 \leq \frac{2||v_0||_2^2 I_0(v_j)}{R(v_0)} = ||v_0||_2^2 = \mu_0.$$

1) Suppose that $\sigma < 0$ and $0 < p < 2$. From previous fact and the Gagliardo–Nirenberg inequality, we see for $\mu = ||u||_2^2 < \mu_0$ that

$$\begin{aligned} \ell_0(u) &\geq \tfrac{1}{2}\left(1 - \tfrac{\mu}{\mu_0}\right) I_0(u) - \tfrac{C|\sigma|}{p+2}(I_0(u))^{\frac{p}{2}}||u||_2^2, \\ &\geq \tfrac{1}{2}\left(1 - \tfrac{\mu}{\mu_0}\right) I_0(u) - C_1(I_0(u))^{\frac{p}{2}}\mu_0, \end{aligned}$$

meaning that $I_\mu > -\infty$ for $0 < \mu < \mu_0$. Now, we note that any minimizing sequence $(v_j)_j$ for $\sum_\mu$ is bounded in $H^1(\mathbf{R}^2)$. In fact, assume that $\ell_0(v_j) \to I_\mu$. For previous inequality (4.1), we have that

$$\frac{\ell_0(v_j)}{I_0(v_j)} \geq \frac{1}{2}\left(1 - \frac{\mu}{\mu_0}\right) - C_1(I_0(v_j))^{\frac{p-2}{2}}\mu_0.$$

If the sequence minimizing sequence $(v_j)_j$ for $\sum_\mu$ were unbounded in $H^1(\mathbf{R}^2)$, so taking limit as $j \to \infty$, we conclude that

$$\frac{1}{2}\left(1 - \frac{\mu}{\mu_0}\right) \leq 0,$$

which is a contradiction. We also see that $I_\mu < 0$. In fact, for $v \in H^1(\mathbf{R}^2)$ with $||v||_2^2 = \mu$, we consider $u^\lambda(x) = \lambda v(y)$ with $y = \lambda x$. Then we see that

$$\ell_0(u_\lambda) = \frac{\lambda^2}{2} I_0(v) - \frac{\lambda^2}{4} R(v) + \frac{\sigma \lambda^p}{p+2}||v||_{p+2}^{p+2}.$$

So, taking $\lambda > 0$ large enough, we see that $\ell_0(u_\lambda) < 0$ and so $I_\mu < 0$.
On the other hand, for fixed $0 < \mu < \mu_0$ and $v \in H^1(\mathbf{R}^2)$ with $||v||_2^2 = \mu$, we consider $u(x) = v(y)$ with $x = \sqrt{\lambda} y$. Then we see that $||u||_2^2 = \lambda ||v||_2^2 = \mu\lambda$. So, we have that

$$\begin{aligned} I_{\mu\lambda} &\leq \ell_0(u), \\ &\leq \tfrac{1}{2} I_0(v) - \tfrac{\lambda}{4} R(v) + \tfrac{\sigma\lambda}{p+2}||v||_{p+2}^{p+2}, \\ &\leq \ell_0(v) + (1 - \lambda)\left(\tfrac{1}{4} R(v) - \tfrac{\sigma\lambda}{p+2}||v||_{p+2}^{p+2}\right), \\ &< \ell_0(v), \end{aligned}$$

since we have that $\sigma < 0$ and $\mu < \lambda\mu < \mu_0$. In particular, $I_{\mu\lambda} < \lambda I_\mu$ for $\mu < \lambda\mu < \mu_0$ and we conclude for $0 < \lambda < \mu < \mu_0$ that,

$$I_\mu < I_\lambda + I_{\mu-\lambda}.$$

In fact, in the case $\lambda > \mu - \lambda$ note that

$$I_\mu = I_{(\frac{\mu}{\lambda})\lambda} < \left(\frac{\mu}{\lambda}\right) I_\lambda = I_\lambda + \left(\frac{\mu-\lambda}{\lambda}\right) I_\lambda \leq I_\lambda + I_{\mu-\lambda}.$$

For $\lambda < \mu - \lambda$, we get the same conclusion. Now, we set the measure ν_n with density $\rho(v_n)$ with respect to the Lebesgue measure given by

$$\rho(v) = |\nabla v|^2 + |v|^2.$$

We know in this case that

$$\int_{\mathbf{R}^2} d\nu_n = \int_{\mathbf{R}^2} \rho(v_n)\, dV := \sigma_n \to \sigma_0, \quad n \to \infty.$$

From Lion's concentration-compactness principle in [16], after a translation, we conclude that any minimizing sequence for $\sum_\mu$ converges strongly in $H^1(\mathbf{R}^2)$. In other words, the set Σ_μ is not empty.

Now, we want to establish that Σ_μ is stable. If not, there is $\epsilon_0 > 0$ and $(v_j)_j \subset H^1(\mathbf{R}^2)$ such that

$$\inf_{\psi\in\Sigma_\mu} ||v_j^0 - \psi||_{H^1} \to 0 \text{ and } \inf_{\psi\in\Sigma_\mu} ||v_j(t) - \psi||_{H^1} > \epsilon_0, \quad t \geq 0, \tag{4.1}$$

where $v_j(t)$ is a solution of (1.2) with $v_j(0) = v_j^0$. So, we can take $t_j > 0$ such that

$$\inf_{\psi\in\Sigma_\mu} ||v_j(t_j) - \psi||_{H^1} \geq \epsilon_0.$$

From the conserved quantities, we have that

$$||v_j(t_j)||_2^2 = ||v_j^0||_2^2 \to \mu, \quad \ell_0(v_j(t_j)) = \ell_0(v_j^0) \to I_\mu.$$

Using previous discussion, there is a subsequence of $(v_j(t_j))_j$ (denoted the same) and a sequence $(y_j)_j \subset \mathbf{R}^2$ and $v_0 \in \Sigma_\mu$ such that

$$v_j(t_j)(\cdot + y_j) \to v_0 \text{ in } H^1(\mathbf{R}^2),$$

contradicting the condition (4.1).

2) We consider the case $\sigma > 0$ and $p > 2$. From the Gagliardo–Nirenberg inequality, we see for $\mu = ||u||_2^2 > \mu_0$ that

$$\begin{aligned}\ell_0(u) &\geq \tfrac{1}{2}I_0(u) + \tfrac{\sigma}{p+2}||u||_{p+2}^{p+2} - C_1\left(I_0(u) + (I_0(u))^{\frac{p}{2}}\right)||u||_2^2,\\ &\geq I_0(u)\left(\tfrac{1}{2} - C_2\mu(1+(I_0(u))^{\frac{p-2}{2}}\right) + \tfrac{\sigma}{p+2}||u||_{p+2}^{p+2},\end{aligned}$$

meaning that $I_\mu > -\infty$ for $\mu > 0$. Moreover, any minimizing sequence $(v_j)_j$ for $\sum_\mu$ is bounded in $H^1(\mathbf{R}^2)$. We also see that $I_\mu < 0$ for $\mu > \mu_0$. In fact, take $v_0 \in H^1(\mathbf{R}^2)$ with $||v||_2^2 = \mu_0$ and $E_0(v_0) = 0$. So, we have for $\theta = \frac{\mu}{\mu_0}$ that

$$E_0(\theta v_0) = \frac{\theta^2}{2}I_0(v_0) - \frac{\theta^4}{4}R(v_0) = \frac{\theta^2}{4}(1-\theta^2)R(v_0) < 0.$$

Now, we define $u = \theta v_0$ and we consider $u_\lambda(x) = \lambda u(y)$ with $y = \lambda x$. Then we see that

$$\ell_0(u_\lambda) = \frac{\lambda^2}{2}I_0(u) - \frac{\lambda^2}{4}R(u) + \frac{\sigma\lambda^p}{p+2}||u||_{p+2}^{p+2} = \lambda^2 E_0(u) + \frac{\sigma\lambda^p}{p+2}||u||_{p+2}^{p+2} < 0,$$

for $\lambda > 0$ small enough and $\mu > \mu_0$ and so, $I_\mu < 0$ for $\mu > \mu_0$.
Using the fact that $I_\mu < 0$ for $\mu > \mu_0$, we see directly that

$$I_\mu = \inf\left\{\ell_0(v): \ v \in H^1(\mathbf{R}^N), \ \ ||v||_2 = \sqrt{\mu}, \ \ R(u) - \frac{4\sigma}{p+2}||v||_{p+2}^{p+2} > 0\right\}.$$

On the other hand, for fixed $\mu > \mu_0$ and $v \in H^1(\mathbf{R}^2)$ with $||v||_2^2 = \mu$ and $\frac{1}{4}R(v) - \frac{\sigma}{p+2}||v||_{p+2}^{p+2} > 0$, we consider $u(x) = v(y)$ with $x = \sqrt{\lambda}y$. Then, we see that

$$||u||_2^2 = \lambda||v||_2^2 = \mu\lambda.$$

So, we have that

$$\begin{aligned}I_{\mu\lambda} &\leq \ell_0(u).\\ &\leq \tfrac{1}{2}I_0(v) - \tfrac{\lambda}{4}R(v) + \tfrac{\sigma\lambda}{p+2}||v||_{p+2}^{p+2}.\\ &\leq \ell_0(v) + (1-\lambda)\left(\tfrac{1}{4}R(v) - \tfrac{\sigma\lambda}{p+2}||v||_{p+2}^{p+2}\right)\\ &< \ell_0(v).\end{aligned}$$

In particular, we have that

$$I_{\mu\lambda} < \lambda I_\mu, \ \ \mu > \mu_0, \ \ \lambda > 1.$$

As in previous case, we conclude that

$$I_\mu < I_\lambda + I_{\mu-\lambda}, \ \ \mu > \mu_0, \ \ \mu > \lambda > 0.$$

As in case 1), from Lion's concentration-compactness principle in [16] and previous inequality, we conclude that any minimizing sequence for $(\sum_\mu)$, after a translation, converges strongly in $H^1(\mathbf{R}^2)$. In other words, the set Σ_μ is not empty. In a similar fashion, one gets that Σ_μ is stable. □

4.2. Instability Results for $N = 3$

In order to analyze the instability, we first establish a psuedo conformal identity associated with time variations of a momentum functional M.

Hereafter, we use the following auxiliary functional

$$K(\psi) := I_0(\psi) - \frac{3}{4}R(\psi) + \frac{3p\sigma}{2(p+2)}||\psi||_{p+2}^{p+2},$$

where I_0 and $R(u)$ are given by

$$\begin{aligned} I_0(\psi) &= \int_{\mathbf{R}^3} \left(\sigma_1|\psi_x|^2 + \sigma_1|\psi_y|^2 + \epsilon|\psi_z|^2\right)\, dx\, dy\, dz, \\ R(\psi) &= \int_{\mathbf{R}^3} \left(W(M^2\sigma - D)^2 E(|\psi|^2)|\psi|^2 + WM^2|\psi|^4\right)\, dx\, dy\, dz. \end{aligned}$$

We define the set of solutions $\mathcal{X}$ and ground state solutions $\mathcal{G}_\omega$ for (3.3)

$$\begin{aligned} \mathcal{X}_\omega &= \left\{\psi \in H^1(\mathbf{R}^3) : K(\psi) = 0\right\}, \\ \mathcal{G}_\omega &= \left\{\psi \in \mathcal{X} : \ell_\omega(\psi) \le \ell_\omega(\phi) \text{ for all } \phi \in \mathcal{X}_\omega\right\}. \end{aligned}$$

We will see below that $\ell'_\omega = K$.

Theorem 4.2. *Let $N = 3$, $|\epsilon| = |\sigma_1| > 0$ and $\Phi(t) = (\psi(t), \rho, \varphi) \in \mathcal{X}$ be the solution of the (1.2) system when $\psi_0 \in \Sigma$. If M is the momentum type functional*

$$M(t) = \int_{\mathbf{R}^3} (\sigma_1 x^2 + \sigma_1 y^2 + \epsilon z^2)|\psi(x,y,z,t)|^2\, dx\, dy\, dz,$$

then we have the pseudo conformal identity

$$\frac{d^2}{dt^2}M(t) = 8\epsilon^2 K(\psi(t)). \tag{4.2}$$

Proof. Following exactly the same computations in the case $N = 2$ done by J. Quintero *et. al.* in [8], we see for $N = 3$ that

$$\begin{aligned} \tfrac{d^2}{dt^2}M(t) &= 8I_0(\psi) + 8\int_{\mathbf{R}^3} \Big[\tfrac{\sigma p(\epsilon^2+2\sigma_1^2)}{2(p+2)}|\psi|^{p+2} - \tfrac{(\epsilon^2+2\sigma_1^2)W}{4M^2}\rho^2 + \\ &\quad \tfrac{(\epsilon^2+2\sigma_1^2)WD}{2}\varphi_z|\psi|^2 + \tfrac{(\epsilon^2+2\sigma_1^2)W}{4}\varphi_z^2 + \tfrac{(-\epsilon^2+4\sigma_1^2)W}{4}(\varphi_x^2 + \varphi_y^2)\Big]\, dxdydz. \end{aligned}$$

Using that $|\sigma_1| = |\epsilon|$, we have that

$$\frac{\epsilon^2 + 2\sigma_1^2}{4\epsilon^2} = \frac{3}{4}, \quad \frac{4\sigma_1^2 - \epsilon^2}{4\epsilon^2} = \frac{3}{4}, \qquad \frac{\sigma_1^3}{\epsilon^2} = \sigma_1.$$

From these definitions, we obtain that

$$\begin{aligned} \tfrac{d^2}{dt^2}M(t) &= 8\epsilon^2 \int_{\mathbf{R}^3} \Big(\sigma_1|\psi_x|^2 + \sigma_1|\psi_y|^2 + \epsilon|\psi_z|^2 + \tfrac{3\sigma p}{2(p+2)}|\psi|^{p+2} - \tfrac{3W}{4M^2}\rho^2 \\ &\quad + \tfrac{3WD}{2}\varphi_z|\psi|^2 + \tfrac{3W}{4}\varphi_z^2 + \tfrac{3W}{4}\varphi_y^2\Big)\, dx\, dy\, dz. \end{aligned}$$

Using the relationship between ρ and φ in conditions (2.2) and (2.3), we see that

$$\begin{aligned}&\int_{\mathbf{R}^3}\left(-\tfrac{1}{2M^2}\rho^2 + D\varphi_z|\psi|^2 + \tfrac{1}{2}|\nabla\varphi|^2\right)\,dx\,dy\,dz\\ &= -\tfrac{1}{2}\int_{\mathbf{R}^3}\left((M^2\sigma - D)^2E(|\psi|^2)|\psi|^2 + M^2|\psi|^4\right)\,dx\,dy\,dz,\end{aligned}$$

which leads to the desired estimate

$$\frac{d^2}{dt^2}M(t) = 8\epsilon^2 K(\psi(t)).$$

□

Theorem 4.3. *Let assume that either* $\sigma < 0$ *and* $\frac{4}{3} < p < 4$, *or* $\sigma > 0$ *and* $0 < p < 2$. *If* $u \in M_\omega$ *satisfies* $\ell_\omega(u) = m$, *then we have that* $\ell'_\omega(u) = 0$

Proof. If $\ell_\omega(u) = m$ with $u \in M_\omega$, then there is $\beta \in \mathbf{R}$ such that for $v \in H^1(\mathbf{R}^3)$ we have,

$$\ell'_\omega(u)(v) = \beta K'(u)(v).$$

Replacing v for u in previous equation, we see directly that u satisfies the equation

$$\frac{(p-4+3\beta p^2)}{3p}I_0(u) + \frac{(p-2)(3\beta p+2)}{4p}R(u) = 0,$$

where we are using that $K(u) = 0$. We note for $\frac{4}{3} < p \leq 2$ and $\beta \geq \frac{1}{2}$ that

$$p - 4 + 3\beta p^2 > 0, \quad (p-2)(3\beta p + 2) > 0,$$

which is a contradiction, and so we already have that $\beta < \frac{1}{2}$. On the other hand, we also have that

$$\frac{d}{d\lambda}(\ell_\omega(u) - \beta K'(u))(v_\lambda)_{|\lambda=1} = K(u) - \beta\left(2I_0(u) - \frac{9}{4}R(u) + \frac{9p^2\sigma}{4(p+2)}||u||_{p+2}^{p+2}\right) = 0.$$

If $\beta \neq 0$, then the fact $K(u) = 0$, previous equation and the restrictions on p give that

$$0 = \frac{(4-3p)}{2}I_0(u) + \frac{9(p-2)}{8}R(u) < 0,$$

implying necessarily that $\beta = 0$.
In the cases either $\sigma < 0$ and $2 < p < 4$ or $\sigma > 0$ and $0 < p < 2$, we get from a similar analysis that,

$$\begin{aligned}\tfrac{d}{d\lambda}(\ell_\omega(u) - \beta K'(u))(v_\lambda)_{|\lambda=1} &= K(u) - \beta\left(2I_0(u) - \tfrac{9}{4}R(u) + \tfrac{9p^2\sigma}{4(p+2)}||u||_{p+2}^{p+2}\right),\\ &= 0.\end{aligned}$$

From the fact $K(u)=0$, we have that,

$$2I_0(u)-\frac{9}{4}R(u)+\frac{9p^2\sigma}{4(p+2)}||u||_{p+2}^{p+2}=-I_0(u)+\frac{9p\sigma(p-2)}{4(p+2)}||u||_{p+2}^{p+2}<0,$$

implying again necessarily that $\beta=0$. □

Before we go further, we define the following functional

$$\begin{aligned}\ell^1(u) &= \ell_\omega(u)-\tfrac{2}{3p}K(u)=\tfrac{(3p-4)}{6p}I_0(u)+\tfrac{\omega}{2}B(u)+\tfrac{1}{2p}R(u),\\ \ell^2(u) &= \ell_\omega(u)-\tfrac{1}{3}K(u)=\tfrac{1}{6}I_0(u)+\tfrac{\omega}{2}B(u)+\tfrac{\sigma(2-p)}{2(p+2)}||u||_{p+2}^{p+2}.\end{aligned}$$

With these functionals, we are able to consider the following minimization problems

$$m_k=\inf\left\{\ell^k(u):\ u\in H^1(\mathbf{R}^3),\ u\neq 0,\ K(u)\leq 0\right\}, \tag{4.3}$$

for $k=1,2,3$ with $\ell^2=\ell^3$.

Theorem 4.4. *1) Let $\sigma<0$ and $\frac{4}{3}<p\leq 2$. Then m_1 is obtained at some $v\in\mathcal{M}_\omega$.*
2) Let either $\sigma<0$ and $2<p<4$, or $\sigma>0$ and $0<p<2$. Then m_k is obtained at some $v\in\mathcal{M}_\omega$, for $k=2,3$.

Proof. **1)** Let $(v_j)_j\subset H^1(\mathbf{R}^3)$ be a minimizing sequence for m_1. In other words, $K(v_j)\leq 0$ and

$$\lim_{j\to\infty}\ell^1(v_j)=m_1.$$

From the restriction $K(v_j)\leq 0$ and the Galiardo–Nirenberg–Sobolev's inequality, we have that

$$1\leq C_1(||v_j||_{H^1}^{p+2}+||v_j||_{H^1}^4).$$

Using that the sequence $(v_j)_j$ is bounded in $H^1(\mathbf{R}^3)$, we have from measure theory results by J. Fröhlich, E. Lieb and M. Loss in [10] and by E. Lieb in [15], that there are a subsequence of $(v_j)_j$ (denoted the same), a sequence of $y_j\in\mathbf{R}^3$ and $v_0\in H^1(\mathbf{R}^3)$ such that

$$v_j(\cdot+y_j)\rightharpoonup v_0\neq 0 \text{ weakly in } H^1(\mathbf{R}^2),\ |v_j(\cdot+y_j)|^2\rightharpoonup|v_0|^2\ \text{(weakly in } L^2(\mathbf{R}^3)).$$

Moreover, we also have that $v_j(\cdot+y_j)\to v_0$ almost everywhere in $\mathbf{R}^3$. On the other hand, from the Brézis-Lieb result in [2], we also have for $q\geq 2$ that

$$\begin{aligned} K(v_j(\cdot+y_j)) - K(v_j(\cdot+y_j)-v_0) - K(v_0) &= o(1),\\ \ell^1(v_j(\cdot+y_j)) - \ell^1(v_j(\cdot+y_j)-v_0) - \ell^1(v_0) &= o(1),\\ ||v_j(\cdot+y_j)||_q^q - ||v_j(\cdot+y_j)-v_0||_q^q - ||v_0||_q^q &= o(1). \end{aligned}$$

We claim that $K(v_0) \leq 0$. For instance, assume that $K(v_0) > 0$. From previous fact and using that $K(v_j) \leq 0$, we conclude for j large enough that $K(v_j(\cdot+y_j)-v_0) \leq 0$, which implies that $\ell^1(v_0) < 0$, since

$$0 \geq m_1 - \ell^1(v_j(\cdot+y_j)-v_0) = \ell^1(v_j(\cdot+y_j)) - \ell^1(v_j(\cdot+y_j)-v_0) + o(1),$$

which is a contradiction due to the fact that $\ell^1 \geq 0$ for $\frac{4}{3} < p < 2$. So, we have that $K(v_0) \leq 0$, which implies from Fatou's lemma that

$$m_1 \leq \ell^1(v_0) \liminf_{j\to\infty} \ell^1(v_j(\cdot+y_j)) = m_1.$$

On the other hand, we see that $K(v_0) = 0$. In fact, first consider the auxiliary function $v_\lambda = \lambda^{\frac{3}{2}} v(y)$ with $y = \lambda x$. If we had $K(v_0) < 0$, then there is $0 < \lambda < 1$ such that $K((v_0)_\lambda) = 0$, since

$$K((v_0)_\lambda) = \lambda^2 \left(I_0(v_0) - \frac{3\lambda}{4} R(v_0) + \frac{3p\sigma\lambda^{\frac{3p-4}{2}}}{2(p+2)} ||v_0||_{p+2}^{p+2} \right) := \lambda^2 f(\lambda),$$

with $f(0) > 0$ and $f(1) < 0$. So, we conclude that

$$m_1 \leq \ell^1((v_0)_\lambda) = \frac{(3p-4)\lambda^2}{6p} I_0(u) + \frac{\omega}{2} B(u) + \frac{\lambda^3}{2p} R(u) < \ell^1(v_0) = m_1,$$

which is a contradiction. In other words, we conclude that $K(v_0) = 0$.
2) We see directly for $k = 2, 3$ and $\sigma(p-2) < 0$ that

$$\ell^2(u) = \frac{1}{6} I_0(u) + \frac{\omega}{2} B(u) + \frac{\sigma(2-p)}{2(p+2)} ||u||_{p+2}^{p+2} \geq 0.$$

From this fact, we have get the result in the same fashion as case (1). □

As a consequence of previous result and the fact that $K(v) = 0$ for any $v \in \mathcal{X}_\omega$, we have the following result,

Theorem 4.5. *Let $N = 3$. If either $\sigma < 0$ and $\frac{4}{3} < p < 4$ or $\sigma > 0$ and $0 < p < 2$, then ψ is a standing wave for (1.2) if and only if $\psi \in \mathcal{X}_\omega$ and $\ell_\omega(\psi) = m$, where*

$$m = \inf \{\ell_\omega(u) : u \in \mathcal{X}_\omega\}.$$

The final step to get the instability result is related with the invariant set

$$\mathcal{R} = \left\{ v \in H^1(\mathbf{R}^3) : \ \ell_\omega(v) < m, \quad K(v) < 0 \right\},$$

under the flow of the system (1.2).

Lemma 4.6. *If $u_0 \in \mathcal{R}$ and $u(t)$ is the solution of the system (1.2) with $u(0) = u_0$, then we have $K(u(t)) \leq \ell_\omega(u_0) - m$ for any $t \in [0, T^*(u_0))$.*

Proof. We know that $\ell_\omega(u(t)) = \ell_\omega(u_0) < m$, as long as the solution exists ($0 \leq t < T^*$). Moreover, we have that $K(u(t)) < 0$ for $0 \leq t < T^*$. In fact, if there is a first $0 \leq t_1 < T^*$ such that $K(u(t_1)) = 0$, then we have that

$$\begin{aligned} m = m_k \leq \ell^k(u(t)) &= \ell_\omega(u(t)) - \alpha_k K(u(t)), \\ &= \ell_\omega(u(t)) - K(u(t)) + (1-\alpha_k)K(u(t)). \end{aligned}$$

But we have that $\alpha_k = \frac{2p}{3}$ for $\frac{4}{3} < p < 2$ and $\alpha_k = \frac{1}{3}$ for $2 < p < 4$, then $1 - \alpha_k > 0$. Using that $K(u(t)) < 0$, we conclude that

$$m < \ell_\omega(u(t)) - K(u(t)),$$

as desired. □

As a direct consequence of the pseudo conformal identity, we establish that solutions for the Cauchy problem associated with the system (1.2) necessarily blow up in finite time.

Theorem 4.7. *Let $N = 3$, $\epsilon = \sigma_1 > 0$. If either $\sigma < 0$ and $\frac{4}{3} < p < 4$ or $\sigma > 0$ and $0 < p < 2$, then the standing wave $u(t) = e^{i\omega t}\psi_\omega$, is strongly unstable for any $\omega \in (0, \infty)$.*

Proof. The first observation is that if $\psi \in \mathcal{G}_\omega$ then we have that $\psi_\lambda \in \mathcal{A}$ for any $\lambda > 1$, where $\psi_\lambda(x) = \lambda^{\frac{3}{2}}\psi(y)$ with $y = \lambda x$. In fact, note that,

$$\begin{aligned} \frac{d}{d\lambda}\ell_\omega(\psi_\lambda) &= \lambda \left(\frac{3(\lambda^2-\lambda)}{4} R(u) + \frac{3p\sigma(\lambda^{\frac{3p-2}{2}} - \lambda)}{2(p+2)} ||\psi||_{p+2}^{p+2} \right), \\ &= \lambda^2 \left((\lambda^{-1} - 1) I_0(\psi) + \frac{3p\sigma(\lambda^{\frac{3(p-2)}{2}} - 1)}{2(p+2)} \right). \end{aligned}$$

We see directly that $\frac{d}{d\lambda}\ell_\omega(\psi_\lambda) < 0$ for $\lambda > 1$ in either case $\sigma > 0$ and $\frac{4}{3} < p < 2$ or $\sigma(p-2) < 0$. Then we have for $\lambda > 1$ that

$$\ell_\omega(\psi_\lambda) < \ell_\omega(\psi) = m.$$

On the other hand, we also have for $\lambda > 1$ that $\frac{d}{d\lambda}\ell_\omega(\psi_\lambda) = \lambda K(\psi_\lambda) < 0$ for $\lambda > 1$, which implies that $\psi_\lambda \in \mathcal{A}$ for $\lambda > 1$. So, let $u(t)$ be the solution of the system (1.2) with initial condition $u(0) = \psi_\lambda$. Now, we now that

$$\int_{\mathbf{R}^3} (x^2 + y^2 + z^2)|\psi_\lambda|^2 \, dx\, dy\, dz < \infty.$$

From the pseudo conformal identity and previous result, we have for $t \in [0, T^*(\psi_\lambda))$ that

$$\frac{d^2}{dt^2} \int_{\mathbf{R}^3} \epsilon(x^2 + y^2 + z^2)|u(t)|^2 \, dx\, dy\, dz = 8\epsilon^2 K(u(t)) < \ell_\omega(\psi_\lambda) - m < 0.$$

On the other hand, we note that

$$\begin{aligned} M(t) &= M(0) + M'(0)t + \tfrac{1}{2}M''(t_1)t^2, \\ &= M(0) + M'(0)t + 4\epsilon^2 K(u(t_1))t^2, \end{aligned}$$

which implies form the fact $K(u(t_1)) < \ell_\omega(\psi_\lambda) - m < 0$, that there is T_* such that

$$\lim_{t \uparrow T_*} \int_{\mathbf{R}^3} (x^2 + y^2 + z^2)|u(t)|^2 \, dx\, dy\, dz = 0.$$

On the other hand, we have from the Weyl-Heisenberg's inequality in $\mathbf{R}^3$ (see for example M. Weinstein [25]) that

$$\|f\|_2^2 \leq \frac{2}{3}||\nabla f||_2 || |z| f||_2,$$

whenever $|z|f \in L^2(\mathbf{R}^3)$ and $|\nabla f| \in L^2(\mathbf{R}^3)$. Using that $||u(t)||_2 = ||\psi_\lambda||_2$ as long as the solution exists, we conclude that

$$\|\psi_\lambda\|_2^2 \leq \frac{2}{3}||\nabla u(t)||_2 \left(\int_{\mathbf{R}^3} (x^2 + y^2 + z^2)|u(t)|^2 \, dx\, dy\, dz \right)^{\frac{1}{2}},$$

which implies that

$$\lim_{t \uparrow T_*} \int_{\mathbf{R}^3} |\nabla u(t)|^2 \, dx\, dy\, dz = +\infty,$$

meaning that the solution must blows up in finite time. On the other hand, the fact

$$\lim_{\lambda \to 1^-} ||\psi_\lambda - \psi||_{H^1(\mathbf{R}^3)} = 0,$$

implies that the standing wave ψ is strongly unstable, as desired. □

As a consequence of the proof of previous result, we also have that solutions for the Cauchy problem associated with the system (1.2) necessarily blow up in finite time, in the case of initial data having negative energy K. More precisely,

Corollary 4.1. *Let* $N = 3$, $\epsilon = \sigma_1 > 0$, $(\psi_0, \rho) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$, $\partial_j \varphi \in L^2(\mathbf{R}^3)$ *for* $j = 1, 2, 3$, *and* $\psi_0 \in \Sigma$. *If* $K(\psi_0) < 0$, *then the maximal existence time* $T_* > 0$ *for the unique solution* $\Phi(t) = (\psi(t), \rho, \varphi)$ *of the system (1.2) with initial data* $\Phi(\psi_0, \rho_0, \varphi_0)$ *is finite. More exactly,* $T_* > 0$ *is such that*

$$\lim_{t \uparrow T_*} \|\nabla \psi(t)\|_{L^2(\mathbf{R}^3)} = +\infty.$$

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References

[1] D. Benney and G. Roskes, “Wave Instability”, *Studies in Applied Mathematics*, vol. 48, pp. 455-472, 1969.

[2] H. Brézis and E. Lieb, “A relation between pointwise convergence of functions and convergence of functionals”, *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983. doi: 10.2307/2044999

[3] T. Cazenave, *An introduction to nonlinear Schrodinger equations.* Universidade Federal do Rio de Janeiro, 1996.

[4] R. Cipolatti, “On the existence of standing waves for a Davey–Stewartson System”, *Communications in Partial Differential Equations*, vol. 17, no. 5-6, pp. 967–988, 1992. doi: 10.1080/03605309208820872

[5] R. Cipolatti, “On the instability of ground states for a Davey-Stewartson system”, *Annales de l'Institut Henri Poincaré, physique théorique*, vol. 58, pp. 84-104, 1993.

[6] J. Cordero, "Sudsonic and supersonic limits for the Zakharov-Rubenchik system", Tese de Doutorado, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 2010.

[7] J. C. Cordero, “Supersonic limit for the Zakharov–Rubenchik System”, *Journal of Differential Equations*, vol. 261, no. 9, pp. 5260–5288, 2016. doi: 10.1016/j.jde.2016.07.022

[8] J. C. Cordero and J. R. Quintero, “Instability of the standing waves for a benney-roskes/zakharov-rubenchik system and blow-up for the zakharov equations”, *Discrete and Continuous Dynamical Systems - B,* vol. 25, no. 4, pp. 1213–1240, 2020. doi: 10.3934/dcdsb.2019217

[9] A. Davey and K. Stewartson, “On three-dimensional packets of surface waves”, *Proceedings of the Royal Society of London A,* vol. 338, no. 1613, pp. 101–110, 1974. doi: 10.1098/rspa.1974.0076

[10] J. Fröhlich, E. H. Lieb, and M. Loss, “Stability of coulomb systems with magnetic fields”, *Communications in Mathematical Physics,* vol. 104, no. 2, pp. 251–270, 1986. doi: 10.1007/bf01211593

[11] J.-M. Ghidaglia and J.-C. Saut, “On the initial value problem for the Davey-Stewartson Systems”, *Nonlinearity,* vol. 3, no. 2, pp. 475–506, 1990. doi: 10.1088/0951-7715/3/2/010

[12] J. Ghidaglia and J. Saut, “On the Zakharov-Schulman equations”, In: *Nonlinear Dispersive Waves,* L. Debnath Ed., World Scientific, 1992.

[13] E. Kuznetsov and V. Zakharov, “Hamiltonian formalism for systems of hydrodynamics type”, *Mathematical Physics Review, Soviet Scientific Reviews,* vol. 4, pp. 167-220, 1984.

[14] D. Lannes, *Water waves: mathematical theory and asymptotics.* Providence: AMS, 2013.

[15] E. H. Lieb, “On the lowest eigenvalue of the laplacian for the intersection of two domains”, *Inventiones Mathematicae,* vol. 74, no. 3, pp. 441–448, 1983. doi: 10.1007/bf01394245

[16] P. L. Lions, “The concentration-compactness principle in the calculus of variations. the locally compact case, part 1”, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire,* vol. 1, no. 2, pp. 109–145, 1984. doi: 10.1016/s0294-1449(16)30428-0

[17] P. L. Lions, “The concentration-compactness principle in the calculus of variations. the locally compact case, part 2”, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire,* vol. 1, no. 4, pp. 223–283, 1984. doi: 10.1016/s0294-1449(16)30422-x

[18] H. Nawa, “Asymptotic profiles of blow-up solutions of the nonlinear Schrödinger equation with critical power nonlinearity”, *Journal of the Mathematical Society of Japan,* vol. 46, no. 4, pp. 557–586, 1994. doi: 10.2969/jmsj/04640557

[19] M. Ohta, “Stability of standing waves for the generalized Davey-Stewartson System”, *Journal of Dynamics and Differential Equations,* vol. 6, no. 2, pp. 325–334, 1994. doi: 10.1007/bf02218533

[20] M. Ohta, "Stability and instability of standing vaves for the generalized Davey-Stewartson System", *Differential and Integral Equations,* vol. 8, no. 7, pp. 1775-1788, 1995.

[21] G. Ponce and J. C. Saut, "Well-posedness for the Benney-Roskes/ Zakharov-Rubenchik system", *Discrete and Continuous Dynamical Systems,* vol. 13, no. 3, pp. 811-825, 2005. doi: 10.3934/dcds.2005.13.811

[22] J. Quintero, "On the existence of positive solutions for a nonlinear elliptic class of equations in $\mathbb{R}^2$ and $\mathbb{R}^3$", *Journal of advances in mathematics,* vol. 20, pp. 188–210, 2021. doi: 10.24297/jam.v20i.9044

[23] J. Quintero, "Stability of Standing Waves for a generalized Zakharov-Rubenchik System". *Preprint,* 2020. [Online]. Available: https://bit.ly/3t4JWkV

[24] A. Rubenchik and V. Zakharov, "Nonlinear Interaction of High-Frequency and Low-Frequency Waves", *Journal of Applied Mechanics and Technical Physics,* vol. 13, pp. 669-681, 1972.

[25] M. I. Weinstein, "Nonlinear schrödinger equations and sharp interpolation estimates," *Communications in Mathematical Physics,* vol. 87, no. 4, pp. 567–576, 1983. doi: 10.1007/bf01208265

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