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A new refinement of the generalized Hölder's inequality with applications

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Abstract

In this paper, we prove a further generalized refinement of the weighted arithmetic-geometric mean inequality. As application, we show a new refinement of the generalized classical Hölder's inequality and we give refinements to several inequalities for some special functions.

Keywords: *Arithmetic-geometric mean inequality; Generalized Hölder's inequality; Gamma function; (q, k)-Polygamma functions; Nielsen's β-function.*

MSC numbers: *26D07; 26D15; 33B15; 33B20.*

1. Introduction

The celebrated weighted arithmetic-geometric mean (AM-GM) inequality states as follows:

Theorem 1.1. *Let n be a positive integer. For $k = 1, 2, \dots, n$, let $a_k \geq 0$ and let $\nu_k > 0$ satisfy $\sum_{k=1}^{n} \nu_k = 1$. Then, we have*

$$\prod_{k=1}^{n} a_k^{\nu_k} \leq \sum_{k=1}^{n} \nu_k a_k. \tag{1.1}$$

One of the consequences of the weighted arithmetic-geometric mean inequality is the generalized Hölder's inequality that states as follows:

Theorem 1.2. *Let n be a positive integer and $p_k > 1$, $k = 1, \dots, n$ such that $\sum_{k=1}^{n} \frac{1}{p_k} = 1$. Let μ be a measure on a set Ω and $f_1, f_2, \dots, f_n \in \mathcal{L}^{p_k}(\mu)$. Then, we have $\prod_{k=1}^{n} f_k \in \mathcal{L}^1(\mu)$ and*

$$\int_{\Omega} \prod_{k=1}^{n} |f_k(t)| d\mu(t) \leq \prod_{k=1}^{n} \|f_k\|_{p_k}. \tag{1.2}$$

S. Furuichi [5] refined (1.1) as follows:

$$\prod_{k=1}^{n} a_k^{\nu_k} + r_0 \left(\sum_{k=1}^{n} a_k - n \sqrt[n]{\prod_{k=1}^{n} a_k} \right) \leq \sum_{k=1}^{n} \nu_k a_k, \tag{1.3}$$

where $r_0 = \min\{\nu_k : k = 1, \dots, n\}$.

A reverse of inequality (1.3), is given by the well-known inequality

$$\sum_{k=1}^{n} \nu_k a_k \leq \prod_{k=1}^{n} a_k^{\nu_k} + R_0 \left(\sum_{k=1}^{n} a_k - n \sqrt[n]{\prod_{k=1}^{n} a_k} \right), \tag{1.4}$$

where $R_0 = \max\{\nu_k : k = 1, \dots, n\}$.

Ighachane et al. [7] proved a generalized refinement of the weighted arithmetic-geometric mean inequality as follows:

Theorem 1.3. *[7] For $k = 1, 2, \dots, n$, let $a_k \geq 0$ and let $\nu_k > 0$ satisfy $\sum_{k=1}^{n} \nu_k = 1$. Then for all integers $m \geq 1$, we have*

$$\left(\prod_{k=1}^{n} a_k^{\nu_k} \right)^m + r_0^m \left(\sum_{k=1}^{n} a_k^m - n \sqrt[n]{\prod_{k=1}^{n} a_k^m} \right) \leq \left(\sum_{k=1}^{n} \nu_k a_k \right)^m,$$

where $r_0 = \min\{\nu_k : k = 1, \dots, n\}$.

This paper is organized as follows. In Section 2, we give further generalized refinements of the weighted arithmetic-geometric mean inequality. In Section 3, we apply the main result to prove some new refinements of the generalized Hölder's inequality. In Section 4, we prove some new generalized refinement for certain special functions inequalities.

2. Further generalized refinements of the weighted arithmetic-geometric mean inequality

To prove the first main result, we need the following theorem obtained by Manasrah and Kittaneh in [2].

Theorem 2.1. *[2] Let ϕ be a strictly increasing convex function defined on an interval I. If x, y, z and w are points in I such that*

$$z - w \leq x - y$$

where $w \leq z \leq x$ and $y \leq x$, then

$$(0 \leq) \quad \phi(z) - \phi(w) \leq \phi(x) - \phi(y).$$

The first main result in this section is the following theorem.

Theorem 2.2. *For $k = 1, 2, \ldots, n$, let $a_k \geq 0$ and let $\nu_k > 0$ satisfy $\sum_{k=1}^{n} \nu_k = 1$. Then for all numbers $p \geq 1$, we have*

$$\begin{aligned} & r_0^p \left(\left(\sum_{k=1}^{n} a_k \right)^p - n^p \sqrt[n]{\prod_{k=1}^{n} a_k^p} \right) \\ & \leq \left(\sum_{k=1}^{n} \nu_k a_k \right)^p - \left(\prod_{k=1}^{n} a_k^{\nu_k} \right)^p \\ & \leq R_0^p \left(\left(\sum_{k=1}^{n} a_k \right)^p - n^p \sqrt[n]{\prod_{k=1}^{n} a_k^p} \right), \end{aligned} \tag{2.1}$$

where $r_0 = \min\{\nu_k : k = 1, \ldots, n\}$ and $R_0 = \max\{\nu_k : k = 1, \ldots, n\}$.

Proof. Let $x = \sum_{k=1}^{n} \nu_k a_k$, $y = \prod_{k=1}^{n} a_k^{\nu_k}$, $z = r_0 \sum_{k=1}^{n} a_k$, $w = r_0 n \sqrt[n]{\prod_{k=1}^{n} a_k}$, $z' = R_0 \sum_{k=1}^{n} a_k$ and $w' = R_0 n \sqrt[n]{\prod_{k=1}^{n} a_k}$. Then based on inequalities (1.3) and (1.4), we have

$$z - w \leq x - y \leq z' - w'.$$

Set $\phi(t) = t^p$, $t \geq 0$. The first and the second inequalities in (2.1) follow directly by applying Theorem 2.1 to the inequalities $z - w \leq x - y$, with $w \leq z \leq x$, $y \leq x$ and $x - y \leq z' - w'$ with $y \leq x \leq z'$, $w' \leq z'$, respectively. This completes the proof. □

Lemma 2.1. *[7] Let n and m be two integers and let $a_i \geq 0$. Set $i_0 := m$, $i_n := 0$ and*

$$A := \{(i_1, \dots, i_{n-1}) : 0 \leq i_j \leq i_{j-1},\ 1 \leq j \leq n-1\}.$$

Then, we have

$$\left(\sum_{i=1}^n a_i\right)^m = \sum_{(i_1,\dots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}} a_1^{i_0-i_1} a_2^{i_1-i_2} \dots a_n^{i_{n-1}-i_n} \tag{2.2}$$

and for all $1 \leq j \leq n$, it follows that

$$ma_j \left(\sum_{i=1}^n a_i\right)^{m-1} = \sum_{(i_1,\dots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}} (i_{j-1} - i_j) a_1^{i_0-i_1} a_2^{i_1-i_2} \dots a_n^{i_{n-1}-i_n}. \tag{2.3}$$

The second main result in this section is the following theorem which says that Theorem 2.2 is a further refinement of Theorem 1.3.

Theorem 2.3. *For $k = 1, 2, \dots, n$, let $a_k \geq 0$. Then for all integers $m \geq 1$, we have*

$$\sum_{k=1}^n a_k^m - n \sqrt[n]{\prod_{k=1}^n a_k^m} \leq \left(\sum_{k=1}^n a_k\right)^m - n^m \sqrt[n]{\prod_{k=1}^n a_k^m}. \tag{2.4}$$

Proof. The inequality (2.4), is equivalent to

$$\left(\sum_{k=1}^n a_k\right)^m - \sum_{k=1}^n a_k^m \geq (n^m - n) \sqrt[n]{\prod_{k=1}^n a_k^m}.$$

By Lemma 2.1, we have the following equality

$$\left(\sum_{k=1}^n a_k\right)^m - \sum_{k=1}^n a_k^m = \sum_{(i_1,\dots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots$$

$$\binom{i_{n-2}}{i_{n-1}} a_1^{i_0-i_1} a_2^{i_1-i_2} \dots a_n^{i_{n-1}-i_n} - \sum_{k=1}^{n} a_k^m.$$

Let B be a subset of A such that

$\sum_{(i_1,\dots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}} a_1^{i_0-i_1} a_2^{i_1-i_2} \dots a_n^{i_{n-1}-i_n}$
$= \sum_{(i_1,\dots,i_{n-1})\in B} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}} a_1^{i_0-i_1} a_2^{i_1-i_2} \dots a_n^{i_{n-1}-i_n} + \sum_{k=1}^{n} a_k^m.$

Hence

$$\begin{aligned} &\tfrac{1}{n^m-n}\left(\left(\textstyle\sum_{k=1}^{n} a_k\right)^m - \textstyle\sum_{k=1}^{n} a_k^m\right) \\ &= \tfrac{1}{n^m-n} \textstyle\sum_{(i_1,\dots,i_{n-1})\in B} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}} a_1^{i_0-i_1} a_2^{i_1-i_2} \dots a_n^{i_{n-1}-i_n}. \end{aligned} \tag{2.5}$$

We have

$$\sum_{(i_1,\dots,i_{n-1})\in B} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}} = n^m - n.$$

Thus (2.5) is a convex combination of positive numbers. Therefore, by (1.1),

$$\frac{1}{n^m-n}\left(\left(\sum_{k=1}^{n} a_k\right)^m - \sum_{k=1}^{n} a_k^m\right) \geq \prod_{i=1}^{n} a_i^{\alpha_i(m)},$$

where

$$\alpha_k(m) = \frac{1}{n^m-n} \sum_{(i_1,\dots,i_{n-1})\in B} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}} (i_{k-1} - i_k)$$

for all $1 \leq k \leq n$. It is immediate that

$$\alpha_k(m) = \frac{1}{n^m-n} \sum_{(i_1,\dots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}} (i_{k-1} - i_k) - \frac{m}{n^m-n}.$$

By Lemma 2.1, we have $\alpha_k(m) = \frac{m}{n}$ for all $k = 1, 2, \dots, n$.

Thus

$$(n^m - n)\sqrt[n]{\prod_{k=1}^{n} a_k^m} \leq \left(\sum_{k=1}^{n} a_k\right)^m - \sum_{k=1}^{n} a_k^m.$$

This completes the proof. □

The following corollary is a consequence of Theorem 2.2.

Corollary 2.1. *For $k = 1, 2, \dots, n$, let $a_k \geq 0$ and let $\nu_k > 0$ satisfy $\sum_{k=1}^{n} \nu_k = 1$. Then for all numbers $p \geq 1$, we have*

$$\prod_{k=1}^{n} a_k^{\nu_k} + r_0^p \left(\left(\sum_{k=1}^{n} a_k^{\frac{1}{p}} \right)^p - n^p \sqrt[n]{\prod_{k=1}^{n} a_k} \right) \leq \left(\sum_{k=1}^{n} \nu_k a_k^{\frac{1}{p}} \right)^p \leq \sum_{k=1}^{n} \nu_k a_k,$$

where $r_0 = \min\{\nu_k : k = 1, \dots, n\}$.

Moreover, if we set $f(p) := \left(\sum_{k=1}^{n} \nu_k a_k^{\frac{1}{p}} \right)^p$, $p \geq 1$. Then f is a decreasing function and we have

$$\lim_{p \to \infty} f(p) = \prod_{k=1}^{n} a_k^{\nu_k}.$$

Proof. By Theorem 2.2, we have

$$\prod_{k=1}^{n} a_k^{\nu_k} + r_0^p \left(\left(\sum_{k=1}^{n} a_k^{\frac{1}{p}} \right)^p - n^p \sqrt[n]{\prod_{k=1}^{n} a_k} \right) \leq \left(\sum_{k=1}^{n} \nu_k a_k^{\frac{1}{p}} \right)^p.$$

It is well-known (see, e.g., [6, p. 13, p. 26]) that f is a decreasing function and that

$$\lim_{p \to \infty} f(p) = \prod_{k=1}^{n} a_k^{\alpha_k}.$$

This ends the proof. □

3. New refinements of the generalized Hölder's inequality

For the remainder of this paper, let A be as in Lemma 2.1 and $I = (i_1, \dots, i_{n-1}) \in A$, we denote C_I by $C_I := \binom{i_0}{i_1}\binom{i_1}{i_2} \dots \binom{i_{n-2}}{i_{n-1}}$, where $\binom{i_{k-1}}{i_k}$ is the binomial coefficient.

Concerning the generalized Hölder's inequality, we establish the following result.

Theorem 3.1. *Let μ be a measure on a set Ω and n be a positive integer. For $p_k > 1$, $k = 1, \dots, n$ such that $\sum_{k=1}^{n} \frac{1}{p_k} = 1$, let $f_1, f_2, \dots, f_n \in \mathcal{L}^{p_k}(\mu)$.*

Then for all integers $m \geq 2$, the inequalities

$$\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) + r_0^m \Bigg(\sum_{I \in A} C_I \prod_{k=1}^n \|f_k\|_{p_k}^{1-\frac{p_k(i_{k-1}-i_k)}{m}}$$

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_{k-1}-i_k)}{m}} d\mu(t) - n^m \prod_{k=1}^n \|f_k\|_{p_k}^{1-\frac{p_k}{n}} \int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t) \Bigg)$$

$$\leq \sum_{I \in A} C_I \frac{1}{p_1^{i_0-i_1} \cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \|f_k\|_{p_k}^{1-\frac{p_k(i_{k-1}-i_k)}{m}}$$

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_{k-1}-i_k)}{m}} d\mu(t) \leq \prod_{k=1}^n \|f_k\|_{p_k}$$

hold, where $r_0 = \min\{\frac{1}{p_k} : k = 1, \ldots, n\}$.

Proof. Choose for $k = 1, \ldots, n$, $a_k = \frac{|f_k(t)|^{p_k}}{\|f_k\|_{p_k}^{p_k}}$, and $\nu_k = \frac{1}{p_k}$. Then by Corollary 2.1, we have

$$\frac{\prod_{k=1}^n |f_k(t)|}{\prod_{k=1}^n \|f_k\|_{p_k}} + r_0^m \left(\left(\sum_{k=1}^n \frac{|f_k(t)|^{\frac{p_k}{m}}}{\|f_k\|_{p_k}^{\frac{p_k}{m}}} \right)^m - n^m \sqrt[n]{\prod_{k=1}^n \frac{|f_k(t)|^{p_k}}{\|f_k\|_{p_k}^{p_k}}} \right) \leq$$

$$\left(\sum_{k=1}^n \frac{1}{p_k} \frac{|f_k(t)|^{\frac{p_k}{m}}}{\|f_k\|_{p_k}^{\frac{p_k}{m}}} \right)^m \leq \sum_{k=1}^n \frac{1}{p_k} \frac{|f_k(t)|^{p_k}}{\|f_k\|_{p_k}^{p_k}}.$$

Therefore, we have

$$\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) + r_0^m \prod_{k=1}^n \|f_k\|_{p_k} \left(\left(\int_\Omega \left(\sum_{k=1}^n \frac{|f_k(t)|^{\frac{p_k}{m}}}{\|f_k\|_{p_k}^{\frac{p_k}{m}}} \right)^m d\mu(t) \right.\right.$$

$$\left. - n^m \prod_{k=1}^n \|f_k\|_{p_k}^{\frac{-p_k}{n}} \int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t) \right) \leq$$

$$\prod_{k=1}^n \|f_k\|_{p_k} \left(\int_\Omega \left(\sum_{k=1}^n \frac{1}{p_k} \frac{|f_k(t)|^{\frac{p_k}{m}}}{\|f_k\|_{p_k}^{\frac{p_k}{m}}} \right)^m d\mu(t) \right) \leq \prod_{k=1}^n \|f_k\|_{p_k}.$$

Then by Lemma 2.1, we get

$$\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) + r_0^m \Bigg(\sum_{I\in A} C_I \prod_{k=1}^n \|f_k\|_{p_k}^{1-\frac{p_k(i_{k-1}-i_k)}{m}}$$

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_{k-1}-i_k)}{m}} d\mu(t) - n^m \prod_{k=1}^n \|f_k\|_{p_k}^{1-\frac{p_k}{n}} \int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t) \Bigg)$$

$$\leq \sum_{I\in A} C_I \frac{1}{p_1^{i_0-i_1} \cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \|f_k\|_{p_k}^{1-\frac{p_k(i_{k-1}-i_k)}{m}} \int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_{k-1}-i_k)}{m}}$$

$$d\mu(t) \leq \prod_{k=1}^n \|f_k\|_{p_k}.$$

This ends the proof. □

For the discrete case, we have the following theorem.

Theorem 3.2. *Let n, N be two integers and $\{Q_{j,k}\} \subset \mathbf{R}$, where $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, N$. Let $p_k > 1$ such that $\sum_{k=1}^n \frac{1}{p_k} = 1$, then the inequalities*

$$\sum_{j=1}^N \left| \prod_{k=1}^n Q_{j,k} \right| + r_0^m \Bigg(\sum_{I\in A} C_I \prod_{k=1}^n \left(\sum_{j=1}^N |Q_{j,k}|^{p_k} \right)^{\frac{1}{p_k} - \frac{(i_{k-1}-i_k)}{m}}$$

$$\sum_{j=1}^N \prod_{k=1}^n |Q_{j,i}|^{\frac{p_k(i_{k-1}-i_k)}{m}} - n^m \prod_{k=1}^n \left(\sum_{j=1}^N |Q_{j,k}|^{p_k} \right)^{\frac{1}{p_k} - \frac{1}{n}}$$

$$\sum_{j=1}^N \prod_{k=1}^n |Q_{j,k}|^{\frac{p_k}{n}}$$

$$\leq \sum_{I\in A} C_I \frac{1}{p_1^{i_0-i_1} \cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \left(\sum_{j=1}^N |Q_{j,k}|^{p_k} \right)^{\frac{1}{p_k} - \frac{(i_{k-1}-i_k)}{m}}$$

$$\sum_{j=1}^N \prod_{k=1}^n |Q_{j,i}|^{\frac{p_k(i_{k-1}-i_k)}{m}}$$

$$\leq \prod_{k=1}^n \left(\sum_{j=1}^N |Q_{j,k}|^{p_k} \right)^{\frac{1}{p_k}}$$

are valid, where $r_0 = \min\{\frac{1}{p_k} : k = 1, \ldots, n\}$.

4. Applications to refine certain inequalities for well-known special functions.

In this section, we apply Theorems 3.1 and 3.2 to refine some inequalities for the extended Gamma function, the Nielsen's β-function, the s-extension of Nielsen's β-function, the derivatives of the s-extension of Nielsen's β-function, and the (q, s)-Polygamma functions.

4.1. Refinement to the extended Gamma function inequality

In 1994, Chaudhy and Zubair [3] introduced the extended gamma function by setting

$$\Gamma_\omega(x) := \int_0^{+\infty} t^{x-1} e^{-t-wt^{-1}} dt, \quad Re(x) > 0, \quad w \in (0, +\infty).$$

Akkouchi and Ighachane [1], proved the following theorem.

Theorem 4.1. *Let $p, q > 1$, be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for all real number $x, y \in [0, \infty)$, and all integers $m \geq 2$ we have*

$$\begin{aligned} \Gamma_\omega\left(\tfrac{x}{p} + \tfrac{y}{q}\right) &\leq \left(\tfrac{1}{p^m} + \tfrac{1}{q^m}\right)\Gamma_\omega(x)^{\frac{1}{p}}\Gamma_\omega(y)^{\frac{1}{q}} \\ &+ \textstyle\sum_{k=1}^{m-1} \binom{k}{m} \frac{1}{p^k q^{m-k}} \Gamma_\omega(x)^{\frac{1}{p}-\frac{k}{m}} \Gamma_\omega(y)^{\frac{1}{q}-\frac{(m-k)}{m}} \times \Gamma_\omega\left(\tfrac{k}{m}x + \tfrac{m-k}{m}y\right) \\ &\leq \Gamma_\omega(x)^{\frac{1}{p}}\Gamma_\omega(y)^{\frac{1}{q}}. \end{aligned}$$

By using Theorem 3.1, we obtain the following inequalities, for the extended gamma function.

Theorem 4.2. *Let $p_k > 1$ for $k = 1, 2, \ldots, n$ with $\sum_{i=1}^n \frac{1}{p_k} = 1$ and $x_k \geq 0$. Then for all integers $m \geq 2$, we have*

$$\begin{aligned} &\Gamma_\omega\left(\textstyle\sum_{k=1}^n \frac{1}{p_k} x_k\right) + r_0^m \left(\textstyle\sum_{I\in A} C_I \prod_{k=1}^n \Gamma_\omega(x_k)^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}}\right. \\ &\left.\Gamma_\omega\left(\textstyle\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m}\right) - n^m \prod_{k=1}^n \Gamma_\omega(x_k)^{\frac{1}{p_k}-\frac{1}{n}} \Gamma_\omega\left(\textstyle\sum_{k=1}^n \frac{1}{n} x_k\right)\right) \\ &\leq \textstyle\sum_{I\in A} C_I \frac{1}{p_1^{i_0-i_1} \cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \Gamma_\omega(x_k)^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}} \Gamma_\omega\left(\textstyle\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m}\right) \\ &\leq \textstyle\prod_{k=1}^n \Gamma_\omega(x_k)^{\frac{1}{p_k}}, \end{aligned}$$

where $r_0 = \min\{\frac{1}{p_k},\ k = 1, \ldots, n\}$.

Proof. To apply Theorem 3.1, we set $\Omega := (0, +\infty)$ and take the measure $d\mu(t) := e^{-t-\omega t^{-1}} dt$. Then we choose $f_k(t) = t^{\frac{1}{p_k}(x_k-1)}$, for $k = 1, 2, \ldots, n$. Hence we have the following equalities:

$$\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) = \Gamma_\omega\left(\sum_{k=1}^n \frac{1}{p_k} x_k\right),$$

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t) = \Gamma_\omega\left(\sum_{k=1}^n \frac{1}{n} x_k\right),$$

$$\|f_k\|_{p_k} = \Big[\Gamma_\omega(x_k)\Big]^{1/p_k}$$

and

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_{k-1}-i_k)}{m}} d\mu(t) = \Gamma_\omega\Big(\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m}\Big).$$

By virtue of Theorem 3.1, we have

$$\begin{aligned}
&\Gamma_\omega\Big(\textstyle\sum_{k=1}^n \frac{1}{p_k}x_k\Big) + r_0^m\Big(\textstyle\sum_{I\in A} C_I \prod_{k=1}^n \Gamma_\omega(x_k)^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}}\\
&\Gamma_\omega\Big(\textstyle\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m}\Big) - n^m \prod_{k=1}^n \Gamma_\omega(x_k)^{\frac{1}{p_k}-\frac{1}{n}}\Gamma_\omega\Big(\textstyle\sum_{k=1}^n \frac{1}{n}x_k\Big)\Big)\\
&\le \textstyle\sum_{I\in A} C_I \frac{1}{p_1^{i_0-i_1}\cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \Gamma_\omega(x_k)^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}}\\
&\Gamma_\omega\Big(\textstyle\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m}\Big)\\
&\le \textstyle\prod_{k=1}^n \Gamma_\omega(x_k)^{\frac{1}{p_k}}.
\end{aligned}$$

This ends the proof. □

4.2. Refinement to the Nielsen's β-function inequality

Recall that the Nielsen's β-function [9] may be defined by

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt, \quad x > 0.$$

Nantomah [9] proved the following theorem involving the function β.

Theorem 4.3. *Let $p, q > 1$, be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for all real numbers $x, y \in (0, \infty)$,*

$$\beta\Big(\frac{x}{p} + \frac{y}{q}\Big) \le \beta(x)^{\frac{1}{p}}\beta(y)^{\frac{1}{q}}. \tag{4.1}$$

By using Theorem 3.1, we obtain the following inequalities, for the Nielsen's β-function.

Theorem 4.4. *Let $p_k > 1$ for $k = 1, 2, \ldots, n$ with $\sum_{i=1}^n \frac{1}{p_k} = 1$ and $x_k \ge 0$. Then for all integers $m \ge 2$, we have*

$$\beta\Big(\sum_{k=1}^n \tfrac{1}{p_k} x_k \Big) + r_0^m \Big(\sum_{I\in A} C_I \prod_{k=1}^n \beta(x_k)^{\frac{1}{p_k} - \frac{i_{k-1}-i_k}{m}}$$
$$\beta\Big(\sum_{k=1}^n \tfrac{(i_{k-1}-i_k)x_k}{m} \Big) - n^m \prod_{k=1}^n \beta(x_k)^{\frac{1}{p_k} - \frac{1}{n}} \beta\Big(\sum_{k=1}^n \tfrac{1}{n} x_k \Big) \Big)$$
$$\leq \sum_{I\in A} C_I \tfrac{1}{p_1^{i_0-i_1} \cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \beta(x_k)^{\frac{1}{p_k} - \frac{(i_{k-1}-i_k)}{m}} \beta\Big(\sum_{k=1}^n \tfrac{(i_{k-1}-i_k)x_k}{m} \Big)$$
$$\leq \prod_{k=1}^n \beta(x_k)^{\frac{1}{p_k}},$$

where $r_0 = \min\{\frac{1}{p_k},\ k = 1, \dots, n\}$.

Proof. To apply Theorem 3.1, we set $\Omega := (0,1)$ and take the measure $d\mu(t) := \frac{1}{t(t+1)} dt$. Then we choose $f_k(t) = t^{\frac{1}{p_k} x_k}$, for $k = 1, 2, \dots, n$. After easy computations, we have the following equalities

$$\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) = \beta\Big(\sum_{k=1}^n \frac{1}{p_k} x_k \Big),$$

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t) = \beta\Big(\sum_{k=1}^n \frac{1}{n} x_k \Big),$$

$$\|f_k\|_{p_k} = \Big[\beta(x_k) \Big]^{1/p_k}$$

and

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_{k-1}-i_k)}{m}} d\mu(t) = \beta\Big(\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m} \Big).$$

By using Theorem 3.1, we have

$$\beta\Big(\sum_{k=1}^n \tfrac{1}{p_k} x_k \Big) + r_0^m \Big(\sum_{I\in A} C_I \prod_{k=1}^n \beta(x_k)^{\frac{1}{p_k} - \frac{i_{k-1}-i_k}{m}}$$

$$\beta\Big(\sum_{k=1}^n \tfrac{(i_{k-1}-i_k)x_k}{m} \Big) - n^m \prod_{k=1}^n \beta(x_k)^{\frac{1}{p_k} - \frac{1}{n}} \beta\Big(\sum_{k=1}^n \tfrac{1}{n} x_k \Big) \Big)$$

$$\leq \sum_{I\in A} C_I \tfrac{1}{p_1^{i_0-i_1} \cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \beta(x_k)^{\frac{1}{p_k} - \frac{(i_{k-1}-i_k)}{m}} \beta\Big(\sum_{k=1}^n \tfrac{(i_{k-1}-i_k)x_k}{m} \Big)$$

$$\leq \prod_{k=1}^n \beta(x_k)^{\frac{1}{p_k}}.$$

Which is the desired inequalities. □

4.3. Refinements to the s-extension of Nielsen's β-function inequality

The s-extension of Nielsen's β-function [9], is given by

$$\beta_s(x) = \int_0^1 \frac{t^{\frac{x}{s}-1}}{1+t} dt, \quad x > 0.$$

Nantomah et al. [9] proved the following theorem involving the function $\beta_s(x)$.

Theorem 4.5. *Let $p, q > 1$, be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for all real numbers $x, y \in (0, \infty)$,*

$$\beta_s\left(\frac{x}{p} + \frac{y}{q}\right) \leq \beta_s(x)^{\frac{1}{p}} \beta_s(y)^{\frac{1}{q}}. \tag{4.2}$$

By using Theorem 3.1, we obtain the following inequalities, of the s-extension Nielsen's β-function.

Theorem 4.6. *Let $p_k > 1$ for $k = 1, 2, \ldots, n$ with $\sum_{i=1}^n \frac{1}{p_k} = 1$ and $x_k \geq 0$. Then for all integers $m \geq 2$, we have*

$$\beta_s\bigg(\textstyle\sum_{k=1}^n \frac{1}{p_k} x_k \bigg) + r_0^m \bigg(\sum_{I \in A} C_I \prod_{k=1}^n \beta_s(x_k)^{\frac{1}{p_k} - \frac{i_{k-1}-i_k}{m}}$$

$$\beta_s\bigg(\textstyle\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m} \bigg) - n^m \prod_{k=1}^n \beta_s(x_k)^{\frac{1}{p_k} - \frac{1}{n}} \beta_s\bigg(\sum_{k=1}^n \frac{1}{n} x_k \bigg)\bigg)$$

$$\leq \textstyle\sum_{I \in A} C_I \frac{1}{p_1^{i_0-i_1} \cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \beta_s(x_k)^{\frac{1}{p_k} - \frac{i_{k-1}-i_k}{m}} \beta_s\bigg(\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m} \bigg)$$

$$\leq \textstyle\prod_{k=1}^n \beta_s(x_k)^{\frac{1}{p_k}},$$

where $r_0 = \min\{\frac{1}{p_k},\ k = 1, \ldots, n\}$.

Proof. To apply Theorem 3.1, we set $\Omega := (0, 1)$ and take the measure $d\mu(t) := \frac{1}{t(t+1)} dt$. Then we choose $f_k(t) = t^{\frac{x_k}{sp_k}}$, for $k = 1, 2, \ldots, n$. Hence we have the following equalities:

$$\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) = \beta_s\bigg(\sum_{k=1}^n \frac{1}{p_k} x_k \bigg),$$

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)|^{\frac{p_k}{n}} d\mu(t) = \beta_s\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big),$$

$$\|f_k\|_{p_k} = \Big[\beta_s(x_k)\Big]^{1/p_k}$$

and

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)|^{\frac{p_k(i_{k-1}-i_k)}{m}} d\mu(t) = \beta_s\Big(\sum_{k=1}^{n} \frac{(i_{k-1}-i_k)x_k}{m}\Big).$$

By virtue of Theorem 3.1, we have

$$\beta_s\Big(\textstyle\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big) + r_0^m \Big(\textstyle\sum_{I\in A} C_I \prod_{k=1}^{n} \beta_s(x_k)^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}}$$

$$\beta_s\Big(\textstyle\sum_{k=1}^{n} \frac{(i_{k-1}-i_k)x_k}{m}\Big) - n^m \textstyle\prod_{k=1}^{n} \beta_s(x_k)^{\frac{1}{p_k}-\frac{1}{n}} \beta_s\Big(\textstyle\sum_{k=1}^{n} \frac{1}{n} x_k\Big)\Big)$$

$$\leq \textstyle\sum_{I\in A} C_I \frac{1}{p_1^{i_0-i_1}\cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \beta_s(x_k)^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}} \beta_s\Big(\textstyle\sum_{k=1}^{n} \frac{(i_{k-1}-i_k)x_k}{m}\Big)$$

$$\leq \textstyle\prod_{k=1}^{n} \beta_s(x_k)^{\frac{1}{p_k}}.$$

This ends the proof. □

4.4. Refinement of inequality involving derivatives of the s-extension of Nielsen's β-function

The derivatives of the s-extension of Nielsens β-function [9] is given by

$$\beta_s^{(N)}(x) = \frac{(-1)^N}{s^N} \int_0^{+\infty} \frac{t^N e^{\frac{-xt}{s}}}{1+e^{-t}} dt, \quad x > 0.$$

Nantomah et al. [9] proved the following theorem involving the function β_s.

Theorem 4.7. *Let N be a positive integer, and let $p, q > 1$, be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for all real numbers $x, y \in (0, \infty)$,*

$$\Big|\beta_s^{(N)}\Big(\frac{x}{p} + \frac{y}{q}\Big)\Big| \leq \Big|\beta_s^{(N)}(x)\Big|^{\frac{1}{p}} \Big|\beta_s^{(N)}(y)\Big|^{\frac{1}{q}}. \tag{4.3}$$

By using Theorem 3.1, we obtain the following inequalities of derivatives of the s-extension Nielsens β-function.

Theorem 4.8. *Let $p_k > 1$ for $k = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \frac{1}{p_k} = 1$ and $x_k \geq 0$. Then for all integers $m \geq 2$ and $N \geq 1$, we have*

$$\left|\beta_s^{(N)}\left(\sum_{k=1}^{n} \frac{1}{p_k} x_k\right)\right| + r_0^m \left(\sum_{I \in A} C_I \prod_{k=1}^{n} \left|\beta_s^{(N)}(x_k)\right|^{\frac{1}{p_k} - \frac{i_{k-1}-i_k}{m}} \right|$$

$$\beta_s^{(N)}\left(\sum_{k=1}^{n} \frac{(i_{k-1}-i_k)x_k}{m}\right)\Bigg| - n^m \prod_{k=1}^{n} \left|\beta_s^{(N)}(x_k)\right|^{\frac{1}{p_k} - \frac{1}{n}} \left|\beta_s^{(N)}\left(\sum_{k=1}^{n} \frac{1}{n} x_k\right)\right|\Bigg)$$

$$\leq \sum_{I \in A} C_I \frac{1}{p_1^{i_0-i_1} \cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \left|\beta_s^{(N)}(x_k)\right|^{\frac{1}{p_k} - \frac{i_{k-1}-i_k}{m}} \left|\beta_s^{(N)}\left(\sum_{k=1}^{n} \frac{i_{k-1}-i_k)x_k}{m}\right)\right|$$

$$\leq \prod_{k=1}^{n} \left|\beta_s^{(N)}(x_k\right|^{\frac{1}{p_k}},$$

where $r_0 = \min\{\frac{1}{p_k},\ k = 1, \ldots, n\}$.

Proof. To apply Theorem 3.1, we set $\Omega := (0, +\infty)$ and take the measure $d\mu(t) := \frac{t^N}{s^N(1+e^{-t})} dt$. Then we choose $f_k(t) = e^{\frac{-x_k t}{s p_k}}$ for $k = 1, 2, \ldots, n$. Hence we have the following identities:

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)| d\mu(t) = \left|\beta_s^{(N)}\left(\sum_{k=1}^{n} \frac{1}{p_k} x_k\right)\right|,$$

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)|^{\frac{p_k}{n}} d\mu(t) = \left|\beta_s^{(N)}\left(\sum_{k=1}^{n} \frac{1}{n} x_k\right)\right|,$$

$$\|f_k\|_{p_k} = \left|\beta_s^{(N)}(x_k)\right|^{1/p_k}$$

and

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)|^{\frac{p_k(i_{k-1}-i_k)}{m}} d\mu(t) = \left|\beta_s^{(N)}\left(\sum_{k=1}^{n} \frac{(i_{k-1}-i_k)x_k}{m}\right)\right|.$$

By virtue of Theorem 3.1, we have

$$\left|\beta_s^{(N)}\left(\sum_{k=1}^n \frac{1}{p_k}x_k\right)\right| + r_0^m \left(\sum_{I\in A} C_I \prod_{k=1}^n \left|\beta_s^{(N)}(x_k)\right|^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}}\right.$$
$$\left.\left|\beta_s^{(N)}\left(\sum_{k=1}^n \frac{(i_{k-1}-i_k)x_k}{m}\right)\right| - n^m \prod_{k=1}^n \left|\beta_s^{(N)}(x_k)\right|^{\frac{1}{p_k}-\frac{1}{n}} \left|\beta_s^{(N)}\left(\sum_{k=1}^n \frac{1}{n}x_k\right)\right|\right)$$
$$\leq \sum_{I\in A} C_I \frac{1}{p_1^{i_0-i_1}\cdots p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \left|\beta_s^{(N)}(x_k)\right|^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}} \left|\beta_s^{(N)}\left(\sum_{k=1}^n \frac{i_{k-1}-i_k)x_k}{m}\right)\right|$$
$$\leq \prod_{k=1}^n \left|\beta_s^{(N)}(x_k\right|^{\frac{1}{p_k}}.$$

This ends the proof. □

4.5. Refinement to the Turan-type inequality for the (q, s)-Polygamma functions

The (q, s)-analogue of the gamma function, $\Gamma_{q,s}(x)$ is defined for $x > 0$, $q \in (0, 1)$ and $s > 0$ (see [4], [8] and the references therein).

$$\Gamma_{q,s}(x) = \frac{1}{(1-q)^{\frac{x}{s}-1}} \prod_{k=0}^{+\infty} \frac{1-q^{(k+1)s}}{1-q^{ks+x}},$$

and the (q, s)-Polygamma functions, $\psi_{q,s}^{(N)}(x)$ are defined as follows (see [9]).

$$\psi_{q,s}^{(N)}(x) = \ln(q)^{N+1} \sum_{k=1}^{\infty} \frac{(ks)^N q^{ksx}}{1-q^{ks}} := \xi_N(x),$$

where $N \in \mathbf{N}$, and the function

$$\xi_\beta(x) := \ln(q)^{\beta+1} \sum_{k=1}^{\infty} \frac{(ks)^\beta q^{ksx}}{1-q^{ks}}$$

is defined for all number $\beta \geq 1$ and for all positive number $x > 0$.

Nantomah [10] proved the following Turan-type inequality involving the function $\psi_{q,k}^{(N)}(x)$.

Theorem 4.9. *For $k = 1, 2, \ldots, n$, let $p_k > 1$, $\sum_{k=1}^n \frac{1}{p_k} = 1$, $n_k \in \mathbf{N}$, and $\sum_{k=1}^n \frac{n_k}{p_k} \in \mathbf{N}$. Then the inequality*

$$\psi_{q,s}^{(\sum_{k=1}^n \frac{n_k}{p_k})}\left(\sum_{k=1}^n \frac{1}{p_k}x_k\right) \leq \prod_{k=1}^n \left(\psi_{q,s}^{(n_k)}(x_k)\right)^{\frac{1}{p_k}}, \tag{4.4}$$

holds for $x_k > 0$.

By using Theorem 3.2, we obtain the following refinements of the inequality (4.4).

Theorem 4.10. *For $k = 1, 2, \dots, n$, let $p_k > 1$, $\sum_{k=1}^{n} \frac{1}{p_k} = 1$, $m_k \in \mathbf{N}$, and $\sum_{k=1}^{n} \frac{n_k}{p_k} \in \mathbf{N}$. Then for all integers $m \geq 2$, and $x_k > 0$, we have*

$$
\begin{aligned}
&\psi_{q,s}^{(\sum_{k=1}^{n} \frac{n_k}{p_k})}\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k \Big) + r_0^m \Bigg(\sum_{I \in A} C_I \prod_{k=1}^{n} \Big(\psi_{q,s}^{(n_k)}(x_k) \Big)^{\frac{1}{p_k} - \frac{i_{k-1} - i_k}{m}} \\
&\xi_{(\sum_{k=1}^{n} \frac{(i_{k-1}-i_k) n_k}{m})}\Big(\sum_{k=1}^{n} \frac{(i_{k-1}-i_k) x_k}{m} \Big) - n^m \prod_{k=1}^{n} \Big(\psi_{q,k}^{(n_k)}(x_k) \Big)^{\frac{1}{p_k} - \frac{1}{n}} \\
&\xi_{(\sum_{k=1}^{n} \frac{n_k}{n})}\Big(\sum_{k=1}^{n} \frac{x_k}{n} \Big) \\
&\leq \sum_{I \in A} C_I \frac{1}{p_1^{i_0 - i_1} \cdots p_n^{i_{n-1} - i_n}} \prod_{k=1}^{n} \Big(\psi_{q,s}^{(n_k)}(x_k) \Big)^{\frac{1}{p_k} - \frac{(i_{k-1} - i_k)}{m}} \\
&\times \xi_{(\sum_{k=1}^{n} \frac{(i_{k-1}-i_k) n_k}{m})}\Big(\sum_{k=1}^{n} \frac{(i_{k-1}-i_k) x_k}{m} \Big) \\
&\leq \prod_{k=1}^{n} \Big(\psi_{q,s}^{(n_k)}(x_k) \Big)^{\frac{1}{p_k}},
\end{aligned}
$$

where $r_0 = \min\{\frac{1}{p_k},\ k = 1, \dots, n\}$.

Proof. To apply Theorem 3.2, let $Q_{j,k} = \frac{|\ln q|^{\frac{(n_k+1)}{p_k}} (ks)^{\frac{(n_k)}{p_k}} q^{\frac{jsx_k}{p_k}}}{(1-q^{js})^{\frac{1}{p_k}}}$, for $k = 1, 2, \dots, n$. After easy computations, we have the following equalities

$$
\prod_{k=1}^{n} \Big(\sum_{j=1}^{\infty} |Q_{j,k}|^{p_k} \Big)^{\frac{1}{p_k}} = \prod_{k=1}^{n} \Big(\psi_{q,s}^{(n_k)}(x_k, z) \Big)^{\frac{1}{p_k}},
$$

$$
\|Q_{j,k}\|_{p_k}^{\frac{p_k}{m}} = \Big(\psi_{q,s}^{(n_k)}(x_k, z) \Big)^{\frac{1}{m}},
$$

$$
\sum_{j=1}^{+\infty} \Big| \prod_{k=1}^{n} Q_{j,k} \Big| = \psi_{q,s}^{(\sum_{k=1}^{n} \frac{n_k}{p_k})}\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k, z \Big),
$$

$$
\sum_{j=1}^{N} \prod_{k=1}^{n} |Q_{j,k}|^{\frac{p_k}{n}} = \xi_{(\sum_{k=1}^{n} \frac{n_k}{n})}\Big(\sum_{k=1}^{n} \frac{x_k}{n} \Big)
$$

and

$$
\sum_{j=1}^{+\infty} \prod_{k=1}^{n} |Q_{j,k}|^{\frac{p_k (i_k - i_{k-1})}{m}} = \xi_{(\sum_{k=1}^{n} \frac{(i_{k-1}-i_k) n_k}{m})}\Big(\sum_{k=1}^{n} \frac{(i_{k-1} - i_k) x_k}{m} \Big).
$$

By virtue of Theorem 3.2, we have

$$\psi_{q,s}^{(\sum_{k=1}^{n}\frac{n_k}{p_k})}\left(\sum_{k=1}^{n}\frac{1}{p_k}x_k\right)+r_0^m\left(\sum_{I\in A}C_I\prod_{k=1}^{n}\left(\psi_{q,s}^{(n_k)}(x_k)\right)^{\frac{1}{p_k}-\frac{i_{k-1}-i_k}{m}}\right.$$
$$\xi_{(\sum_{k=1}^{n}\frac{(i_{k-1}-i_k)n_k}{m})}\left(\sum_{k=1}^{n}\frac{(i_{k-1}-i_k)x_k}{m}\right)-n^m\prod_{k=1}^{n}\left(\psi_{q,k}^{(n_k)}(x_k)\right)^{\frac{1}{p_k}-\frac{1}{n}}$$
$$\xi_{(\sum_{k=1}^{n}\frac{n_k}{n})}\left(\sum_{k=1}^{n}\frac{x_k}{n}\right)$$
$$\leq\sum_{I\in A}C_I\frac{1}{p_1^{i_0-i_1}\cdots p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}\left(\psi_{q,s}^{(n_k)}(x_k)\right)^{\frac{1}{p_k}-\frac{(i_{k-1}-i_k)}{m}}$$
$$\times\xi_{(\sum_{k=1}^{n}\frac{(i_{k-1}-i_k)n_k}{m})}\left(\sum_{k=1}^{n}\frac{(i_{k-1}-i_k)x_k}{m}\right)$$
$$\leq\prod_{k=1}^{n}\left(\psi_{q,s}^{(n_k)}(x_k)\right)^{\frac{1}{p_k}}.$$

This ends the proof. □

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