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Structure of a quotient ring R/P and its relation with generalized derivations of R

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Abstract

The fundamental aim of this paper is to investigate the structure of a quotient ring R/P where R is an arbitrary ring and P is a prime ideal of R. More precisely, we will characterize the commutativity of R/P via the behavior of generalized derivations of R satisfying algebraic identities involving the prime ideal P. Moreover, various well-known results characterizing the commutativity of prime (semi-prime) rings have been extended. Furthermore, examples are given to prove that the restrictions imposed on the hypothesis of the various theorems were not superfluous.

Key words: Quotient ring, prime ideal, generalized derivations, commutativity.

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1. Introduction

Throughout, the present paper R will denote an associative ring with center $Z(R)$, not necessarily with an identity element. Recall that an ideal P of R is said to be *prime* if $P \neq R$ and for all $x, y \in R$, $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. Therefore, R is called a *prime ring* if the ideal (0) is prime. The Lie product of two elements x and y of R is $[x, y] = xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. An additive mapping $d : R \to R$ is a *derivation* if d satisfies the Leibnitz'rule: $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

Recently several authors have investigated the relationship between the commutativity i.e. the structure of the ring R and some concrete additive mappings (such as derivations, automorphisms and generalized derivations) acting on appropriate subsets of the rings. Herstien [9] showed that a prime ring R with nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be a commutative integral domain if its characteristic is not two, and, if the characteristic equals two, then the ring must be commutative or an order in a simple algebra which is 4-dimensional over its center. We first recall that a mapping f of R into itself is called *centralizing* on a subset S of R if $[f(x), x] \in Z(R)$ for all $x \in S$; in the sepecial case where $[f(x), x] = 0$ for all $x \in S$, the mapping f is said to be *commuting* on S. The classical result of Posner [14] states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Mayne [11] proved the analogous result for centralizing automorphisms. A number of authors have extended these theorems of Posner and Mayne in serval ways. For example, see [6, 10, 12, 13].
One may observe that the concept of generalized derivations cover both the concepts of derivations and the left multipliers when $d = 0$. Hence it should be interesting to extend some results concerning these notions to generalized derivations. More specifically, Brešar in [7] introduce the notion of generalized derivation as follows: An additive map $F : R \longrightarrow R$ is said to be a *generalized derivation* if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Generalized derivations have been primarily studied on operator algebras. For $a, b \in R$, the mapping $F : R \to R$ defined as $F(x) = ax + xb$ for all $x \in R$ is an example of generalized derivation of R, which is called as inner generalized derivation of R. It is obvious that every derivation (or left multiplier) is a generalized derivation but the converse is not true in general.

The present paper is motivated by the previous results and we here continue this line of investigation by considering a more general concept rather than the ring R is prime or semiprime in the hypothesis of our theorems. More precisely, we will establish a relationship between the structure of quotient rings R/P and the behavior of generalized derivations satisfying algebraic identities involving prime ideals.

2. Main results

Throughout this article id_R will denote the identity map defined by $id_R(r) = r$ for all $r \in R$. We will make frequent use of the following facts whose proof will be left to the reader.

Fact 1. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \nsubseteq I$. If $aIb \subseteq P$, with $a, b \in R$, then $a \in P$ or $b \in P$.*

Remark. Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \nsubseteq I$. If $[I, I] \subseteq P$, then it is easy to show that R/P is a commutative ring.
We will use this remark whenever needed without any specific mention.

In [1, Theorem 2.2] it is proved that if P is a prime ideal of a ring R and d a derivation of R such that $\Big[[x, d(x)], y\Big] \in P$ for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is a commutative ring. Using similar arguments with some modifications, we get the following lemma which plays a crucial role in developing the proofs of our main results.

Lemma 1. *Let R be a ring, I a nonzero ideal of R and P be a prime ideal of R such that $P \nsubseteq I$. If d is a derivation of R satisfying $\overline{[d(x), x]} \in Z(R/P)$ for all $x \in I$, then either $d(R) \subseteq P$ or R/P is a commutative integral domain.*

We recall some related known results in literature: In [5] Ashraf and Rehman proved that if R is a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$, for all $u \in U$ and d a derivation which satisfies $d(u \circ v) = u \circ v$, for all $u, v \in U$, then $U \subseteq Z(R)$. Later, Quadri and al. [15], have extended the mentioned result by considering a generalized derivation F acting on a nonzero ideal I of R and without 2-torsion freeness hypothesis. More precisely, they proved that a prime ring must be commutative if it admits a generalized derivation F, associated with a nonzero derivation d, such that $F(x \circ y) = x \circ y$ for all x, y in a nonzero ideal I.

In the following theorem we will consider the situation of a ring R having a generalized derivation F satisfying the more general condition $\overline{F(x \circ y)} \in Z(R/P)$ for all $x, y \in I$. Further, our goal is to confirm that there is a relationship between the structure of the ring R/P and generalized derivations of R, where P is a prime ideal of R.

Theorem 1. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \nsubseteq I$ and $char(R/P) \neq 2$. If F is a generalized derivation of R associated with a derivation d satisfies the condition $\overline{F(x \circ y)} \in Z(R/P)$ for all $x, y \in I$, then $F(R) \subseteq P$ or R/P is a commutative integral domain.*

Proof. We are given that

$$\overline{F(x \circ y)} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{1.1}$$

If $Z(R/P) = \{\overline{0}\}$, then R/P is non-commutative and the relation (1.1) becomes

$$F(x \circ y) \in P \quad \text{for all } x, y \in I. \tag{1.2}$$

Replacing y by yx in (1.2), we find that

$$F(x \circ y)x + (x \circ y)d(x) \in P \quad \text{for all } x, y \in I$$

in such a way that

$$(x \circ y)d(x) \in P \quad \text{for all } x, y \in I. \tag{1.3}$$

Substituting ry for y in (1.3), one can easily verify that

$$r(x \circ y)d(x) + [x, r]yd(x) \in P \quad \text{for all } x, y, r \in I$$

which implies that

$$[x, r]Id(x) \subseteq P \quad \text{for all } x, r \in I. \tag{1.4}$$

Applying Fact 1, it follows that either $[x, I] \subseteq P$ or $d(x) \in P$ holds for all $x \in I$. Let us set $H = \{x \in I \,/\, [x, I] \subseteq P\}$ and $K = \{x \in I \,/\, d(x) \in P\}$. Then it can be seen that H and K are two additives subgroups of I whose union is I. Using Brauer's trick we have either $I = H$ or $I = K$. Because of R/P is a non-commutative ring, we get necessarily $d(R) \subseteq P$. In this case replacing y by yr in (1.2), we obtain

$$F(y)[r, x] \in P \quad \text{for all } \ x, y, r \in I.$$

Hence the last expression proves that $F(y)I[r, x] \subseteq P$, by view of the primeness of P we conclude that $F(I) \subseteq P$ and thus $F(R) \subseteq P$.
Now if $Z(R/P) \neq \{\overline{0}\}$, then there exists $\overline{z} \in Z(R/P)$ such that $\overline{z} \neq \overline{0}$. Substituting yz for y in (1.1), we can obviously get

$$\overline{F\Big((x \circ y)z - y[x, z]\Big)} \in Z(R/P) \quad \text{for all } \ x, y \in I$$

which reduces to

$$\overline{(x \circ y)d(z) - y[x, d(z)]} \in Z(R/P) \quad \text{for all } \ x, y \in I. \tag{1.5}$$

Putting ry instead of y in (1.5), it is obvious to verify that

$$\overline{r\Big((x \circ y)d(z) - y[x, d(z)]\Big) + [x, r]yd(z)} \in Z(R/P) \quad \text{for all } \ x, y, r \in I$$

we readily see from the above relation that

$$\Big[[x, r]yd(z), r\Big] \in P \quad \text{for all } \ x, y, r \in I \tag{1.6}$$

this equation can be rewritten as

$$[x, r]yd(z)r - r[x, r]yd(z) \in P \quad \text{for all } \ x, y, r \in I. \tag{1.7}$$

Writing xt instead of x in (1.7) and using it, we arrive at

$$x[t, r]yd(z)r - rx[t, r]yd(z) \in P \quad \text{for all } \ x, y, r, t \in I. \tag{1.8}$$

On the other hand, if we replace x by t in (1.7) and then left multiplying it by x, we obtain

$$x[t, r]yd(z)r - xr[t, r]yd(z) \in P \quad \text{for all } \ x, y, r, t \in I. \tag{1.9}$$

Using (1.8) together with (1.9), we can also write

$$[x, r][t, r]yd(z) \in P \quad \text{for all } \ x, y, r, t \in I$$

in particular

$$[x, r]I[x, r]Id(z) \subseteq P \quad \text{for all } \ x, r \in I.$$

In light of primeness, we get either $[x,r]\in P$ for all $x,r\in I$ or $d(z)\in P$ and thus, we conclude that R/P is an integral domain or $\overline{d(z)}=\overline{0}$. Now if we take $y=z$ in (1.1), then we can see that

$$\overline{\Big(F(x)+d(x)\Big)z+F(z)x}\in Z(R/P) \quad \text{for all } x\in I. \tag{1.10}$$

The substitution xy for x in (1.10) gives $2\overline{[xd(y),y]}\overline{z}=\overline{0}$ for all $x,y\in I$. Putting rx instead of x, it follows that

$$2\overline{[r,y]xd(y)}\overline{z}=\overline{0} \quad \text{for all } x,y,r\in I.$$

Using 2-torsion freeness, we find that $[r,y]Id(y)\subseteq P$ for all $y,r\in I$. Therefore, we get $d(R)\subseteq P$ or R/P is commutative. By the first case, our hypothesis leads to

$$\overline{F(x)y+F(y)x}\in Z(R/P) \quad \text{for all } x,y\in I. \tag{1.11}$$

Replacing y by yr in (1.11), one can verify that

$$\overline{\Big(F(x)y+F(y)x\Big)r-F(y)[x,r]}\in Z(R/P) \quad \text{for all } x,y,r\in I$$

thereby obtaining

$$\Big[F(y)[x,r],r\Big]\in P \quad \text{for all } x,y,r\in I.$$

Putting $yF(y)$ instead of y, we get

$$F(y)\Big[F(y)[x,r],r\Big]+[F(y),r]F(y)[x,r]\in P \quad \text{for all } x,y,r\in I$$

which leads to $[F(y),r]F(y)[x,r]\in P$ for all $x,y,r\in I$. Replacing x by tx, we obtain

$$[F(y),r]F(y)t[x,r]\in P \quad \text{for all } x,y,r,t\in I.$$

Substituting $xF(y)$ for x, we deduce that

$$[F(y),r]F(y)tx[F(y),r]\in P \quad \text{for all } x,y,r,t\in I.$$

On the other hand, we have $[F(y),r]F(y)tx[F(y),r]F(y)\in P$. As a special case of the latter expression, we may write

$$[F(y),r]F(y)I[F(y),r]F(y)I[F(y),r]F(y)\subseteq P \quad \text{for all } y,r\in I.$$

According to Fact 1, it follows that $[F(y), r]F(y) \in P$ for all $y, r \in I$. Substituting rt for r in the last relation, we obtain

$$[F(y), r]t[F(y), r] \in P \quad \text{for all} \;\; y, r, t \in I.$$

Hence the above equation assures that $[F(y), r] \in P$ for all $y, r \in I$. Now writing yt instead of y, we find that $F(y)[t, r] \in P$ and therefore we conclude that $F(R) \subseteq P$ or R/P is commutative. Consequently, in any cases, it follows that $F(R) \subseteq P$ or R/P is a commutative integral domain. $\square$

Letting R be a prime ring in the previous theorem, then $P = (0)$ is a prime ideal of R, in this case we obtain the commutativity criteria for category of prime rings.

Corollary 1. *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a nonzero generalized derivation F associated with a derivation d satisfies $F(x \circ y) \in Z(R)$ for all $x, y \in I$, then R is a commutative integral domain.*

As an application of Corollary 1 we have the following result.

Corollary 2. *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F \neq id_R$ and $F(x \circ y) - x \circ y \in Z(R)$ $\Big($resp. $F \neq -id_R$ and $F(x \circ y) + x \circ y \in Z(R)\Big)$ for all $x, y \in I$, then R is a commutative integral domain.*

Proof. Assume that $F \neq \pm id_R$, then $\mathcal{F} = F - id_R$ (resp. $\mathcal{F} = F + id_R$) is also a nonzero generalized derivation satisfying the condition $\mathcal{F}(x \circ y) \in Z(R)$ $\Big($resp. $\mathcal{F}(x \circ y) \in Z(R)\Big)$ for all $x, y \in I$. However by virtue of Corollary 1, we conclude that R is commutative. $\square$

With no further assumption to the characteristic of the considered ring, the following proposition gives an improved version of some known results obtained in [5] and [15] for semiprime ring.

Proposition 1. *Let R be a semiprime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d, satisfies one of the following properties:*
1) *$F(x \circ y) = 0$ for all $x, y \in I$*
2) *$F(x \circ y) \pm x \circ y = 0$ for all $x, y \in I$*
then R contains a nonzero central ideal.

Proof. (1) Suppose that $F(x \circ y) = 0$ for all $x, y \in I$. The ring R is semiprime then there exists a family of prime ideals $\mathcal{P} = \{P_\alpha \,/\, \alpha \in \Lambda\}$ such that $\alpha \in \Lambda \bigcap P_\alpha = (0)$. Using the proof of Theorem 1, by expression (1.4) we get $[d(x), x]I[d(x), x] \subseteq P_\alpha$ for all $\alpha \in \Lambda$ and for all $x \in I$. Therefore, $[d(x), x]I[d(x), x] = 0$ and the semiprimeness of R forces that $[d(x), x] = 0$ for all $x \in I$. Accordingly, by [6, Theorem 3], we conclude that R contains a nonzero central ideal.
(2) Using the same technics as in the preceding proof, we can prove the same conclusion holds for $F(x \circ y) \pm x \circ y = 0$ for all $x, y \in I$. □

The authors in [3, Theorem 2.8] established that, if a 2-torsion free semiprime ring R admits a generalized derivation F associated with a nonzero derivation d such that $F[x, y] = [F(x), y] + [d(y), x]$ for all $x, y \in I$, where I is a nonzero ideal of R, then R contains a nonzero central ideal. Moreover, Ashraf and Almas Khan [2] considered the same identity, but for Lie ideals in $*$-prime rings. More specifically, they proved that, if R is a 2-torsion free $*$-prime ring, $F : R \to R$ is a generalized derivation with a nonzero derivation d which commutes with $*$ and U is a $*$-Lie ideal of R such that $F[u, v] = [F(u), v] + [d(v), u]$ for all $u, v \in U$, then $U \subseteq Z(R)$.

Motivated by the above results, the aim of the next theorem is to study the more general case when the same relation contained on center of R/P. More precisely we will prove the following result.

Theorem 2. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \nsubseteq I$. If R admits a generalized derivation F associated with a derivation d satisfying the condition $\overline{F[x, y] - [F(x), y] - [d(y), x]} \in Z(R/P)$ for all $x, y \in I$, then one of the following assertions holds:*
1) *$char(R/P) = 2$;*
2) *$d(R) \subseteq P$;*
3) *R/P is a commutative integral domain.*

Proof. Suppose that $char(R/P) \neq 2$. We are given that

$$\overline{F[x, y] - [F(x), y] - [d(y), x]} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{1.12}$$

If $Z(R/P) = \{\overline{0}\}$, then the relation (1.12) reduces to

$$F[x, y] - [F(x), y] - [d(y), x] \in P \quad \text{for all } x, y \in I. \tag{1.13}$$

Replacing y by yx in (1.13), we get

$$[x, y]d(x) - y[F(x), x] - [yd(x), x] \in P \quad \text{for all } x, y \in I.$$

Substituting ry for y in the above relation and using it, we can verify that

$$2[x,r]Id(x) \subseteq P \quad \text{for all } x, r \in I.$$

Whence, using 2-torsion freeness with Fact 1 we get $d(R) \subseteq P$ or R/P is commutative, a contradiction. Therefore, we obviously obtain $d(R) \subseteq P$. Now if $Z(R/P) \neq \{\overline{0}\}$, then there exists $\overline{z} \in Z(R/P)$ such that $\overline{z} \neq \overline{0}$. Substituting yz for y in our hypothesis, we get

$$\overline{[x,y]d(z) - y[F(x),z] - [yd(z),x]} \in Z(R/P) \quad \text{for all } x, y \in I$$

which proves that

$$\Big[[x,r]yd(z), r\Big] \in P \quad \text{for all } x, y, r \in I.$$

Since, this relation is exactly (1.6), then arguing as before, we find that either R/P is a commutative integral domain or $\overline{d(z)} = \overline{0}$. On the other hand, if we replace y by $-x$ in (1.12), then it is obvious to see that

$$\overline{[F(x),x] + [d(x),x]} \in Z(R/P) \quad \text{for all } x \in I. \tag{1.14}$$

A linearization of relation (1.14), leads to

$$\overline{[F(z),x]} \in Z(R/P) \quad \text{for all } x \in I. \tag{1.15}$$

Writing $xF(z)$ instead of x in (1.15), we get $\overline{F(z)} \in Z(R/P)$. Now substituting zx for x in (1.14), we obtain

$$\overline{[F(z)x, zx]} + 2\overline{[zd(x), zx]} \in Z(R/P) \quad \text{for all } x \in I$$

and therefore

$$2\overline{[d(x),x]}\,\overline{z}^2 \in Z(R/P) \quad \text{for all } x \in I.$$

Hence from the last relation, we get $\overline{[d(x),x]} \in Z(R/P)$ for all $x \in I$. Invoking Lemma 1, we conclude that $d(R) \subseteq P$ or R/P is commutative. Consequently, in both cases, we have either $d(R) \subseteq P$ or R/P is a commutative integral domain. □

As an application of our theorem, the following corollary improves the result of [2] for the case when the underlying identity belongs to the center of a prime ring.

Corollary 3. *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d satisfying $F[x,y] - [F(x),y] - [d(y),x] \in Z(R)$ for all $x, y \in I$, then R is a commutative integral domain.*

Proposition 2 ([3], Theorem 2.8). *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d satisfying $F[x,y] = [F(x),y] + [d(y),x]$ for all $x,y \in I$, then R contains a nonzero central ideal.*

In [17] Dhara and al. showed that, a prime ring R must be commutative if it admits two generalized derivations F and G associated with derivations d and g respectively and satisfies the properties $F(x)F(y) \pm G(xy) \pm yx \in Z(R)$ for all $x,y \in I$, where I is a nonzero two sided ideal of R.

Motivated by the above results, our aim in the following theorem is to investigate a more general context of differential identities with generalized derivations acting in a center of quotient ring R/P by omitting the primeness assumption imposed on the ring.

Theorem 3. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \nsubseteq I$. If (F,d) and (G,g) two generalized derivations of R associated with derivations d and g satisfying the condition $\overline{F(x)F(y) + G(xy) \pm yx} \in Z(R/P)$ for all $x,y \in I$, then R/P is a commutative integral domain.*

Proof. Assume that

$$\overline{F(x)F(y) + G(xy) - yx} \in Z(R/P) \quad \text{for all } x,y \in I. \tag{1.16}$$

Replacing y by yr in (1.16), we have

$$\Big[F(x)yd(r) + xyg(r) + y[x,r],r\Big] \in P \quad \text{for all } x,y,r \in I. \tag{1.17}$$

Substituting xr for x in (1.17), we get

$$\Big[F(x)ryd(r) + xd(r)yd(r) + xryg(r) + y[x,r]r,r\Big] \in P \quad \text{for all } x,y,r \in I. \tag{1.18}$$

Using (1.17) together with (1.18), we find that

$$\Big[xd(r)yd(r) + [y[x,r],r],r\Big] \in P \quad \text{for all } x,y,r \in I. \tag{1.19}$$

Right multiplying (1.19) by r and combining it with the last relation, it follows that

$$\Big[x[d(r)yd(r),r],r\Big] \in P \quad \text{for all } x,y,r \in I. \tag{1.20}$$

Writing $d(r)yd(r)x$ instead of x in (1.20), we obtain

$$[d(r)yd(r),r]x[d(r)yd(r),r] \in P \quad \text{for all} \;\; x,y,r \in I.$$

Applying Fact 1, we get $[d(r)yd(r),r] \in P$ for all $y,r \in I$, that is

$$d(r)yd(r)r - rd(r)yd(r) \in P \quad \text{for all} \;\; y,r \in I. \tag{1.21}$$

Replacing y by $yd(r)t$ in (1.21), we get

$$d(r)yd(r)td(r)r - rd(r)yd(r)td(r) \in P \quad \text{for all} \;\; y,r,t \in I. \tag{1.22}$$

Putting t instead of y in (1.21) and left multiplying it by $d(r)y$, we arrive at

$$d(r)yd(r)td(r)r - d(r)yrd(r)td(r) \in P \quad \text{for all} \;\; y,r,t \in I. \tag{1.23}$$

Combining (1.22) with (1.23), one can verify that

$$d(r)yrd(r)td(r) - rd(r)yd(r)td(r) \in P \quad \text{for all} \;\; y,r,t \in I. \tag{1.24}$$

On the other hand, right multiplying (1.21) by $td(r)$ and then subtracting it from (1.24), it is obvious to see that

$$d(r)y[d(r),r]td(r) \in P \quad \text{for all} \;\; y,r,t \in I$$

which forces that

$$[d(r),r]I[d(r),r]I[d(r),r] \subseteq P \quad \text{for all} \;\; r \in I.$$

According to Fact 1, it follows that $[d(r),r] \in P$ for all $r \in I$, which proves that $\Big[[d(r),r],t\Big] \in P$ for all $r,t \in I$. By virtue of Lemma 1, the last equation implies that either $d(R) \subseteq P$ or R/P is an integral domain. Now if we take $d(R) \subseteq P$, then the expression (1.17) becomes

$$x[yg(r),r] + [x,r]yg(r) + \Big[y[x,r],r\Big] \in P \quad \text{for all} \;\; x,y,r \in I. \tag{1.25}$$

Putting xy instead of y in (1.25), one can see that

$$[x,r]xyg(r) + [x,r]y[x,r] \in P \quad \text{for all} \;\; x,y,r \in I. \tag{1.26}$$

Substituting $r+x$ for r in (1.26), we obtain

$$[x,r]xyg(x) \in P \quad \text{for all} \;\; x,y,r \in I.$$

Accordingly,

$$[g(x), x]t[g(x), x]y[g(x), x] \in P \quad \text{for all } \; x, y, t \in I.$$

Whence, using again Fact 1, we conclude that either $g(R) \subseteq P$ or R/P is commutative. Thus, in the first case the expression (1.26) reduces to $[x, r]y[x, r] \in P$ for all $x, y, r \in I$. Since P is prime, the last equation implies that $[R, R] \subseteq P$ and therefore R/P is a commutative integral domain.

By similar manner, the same conclusion holds for $\overline{F(x)F(y) + G(xy) + yx} \in Z(R/P)$ for all $x, y \in I$. This completes the proof of our theorem. □

As an application of Theorem 3, the following corollary extended the results of Dhara [17, Theorem 1] for semiprime ring.

Corollary 4. *Let R be a semiprime ring and I a nonzero ideal of R. If (F, d) and (G, g) two generalized derivations of R associated with derivations d and g. Then the following assertions are equivalent:*
1) $F(x)F(y) \pm G(xy) \pm yx \in Z(R)$ for all $x, y \in I$
2) R is commutative.

Proof. We need only prove that (1) $\Longrightarrow$ (2). Assume that

$$F(x)F(y) + G(xy) \pm yx \in Z(R) \quad \text{for all } \; x, y \in I. \tag{1.27}$$

By view of the semiprimeness of the ring R, there exists a family of prime ideals $\mathcal{P} = \{P_\alpha \,/\, \alpha \in \Lambda\}$ such that $\alpha \in \Lambda \bigcap P_\alpha = (0)$, thereby obtaining $[F(x)F(y) + G(xy) \pm yx, r] \in P_\alpha$ for all $x, y, r \in I$ and for all $\alpha \in \Lambda$. Hence, it follows that $\overline{F(x)F(y) + G(xy) \pm yx} \in Z(R/P_\alpha)$ for all $\alpha \in \Lambda$. Invoking Theorem 3, we conclude that R/P_α is a commutative integral domain which, because of $\alpha \in \Lambda \bigcap P_\alpha = (0)$, assures that R is commutative. We notice that, if (G, g) is a generalized derivation on R, then $(-G, -g)$ is also a generalized derivation on R. Thus by putting $(-G, -g)$ instead of (G, g) in the expression (1.27), we get the required result. □

If we replace G by $G \pm id_R$ in the Corollary 4, then one can obviously obtain the following result.

Corollary 5. *Let R be a semiprime ring and I a nonzero ideal of R. If (F, d) and (G, g) two generalized derivations of R associated with derivations d and g. Then the following assertions are equivalent:*
1) $F(x)F(y) \pm G(xy) \pm [x, y] \in Z(R)$ for all $x, y \in I$
2) $F(x)F(y) \pm G(xy) \pm x \circ y \in Z(R)$ for all $x, y \in I$
3) R is commutative.

In [4] Ashraf and al. established that if R is a prime ring, I is a nonzero ideal of R and F is a generalized derivation of R associated with nonzero derivation d such that $F(xy) \pm xy \in Z(R)$ or $F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, then R is commutative.

Our fundamental aim is to generalize this result in two directions. First of all, we will treat a more general differential identity involving two generalized derivations. More specifically, we will study the more general case by considering the following situations:
(i) $\overline{F(x)F(y) \pm G(xy)} \in Z(R/P)$ for all $x, y \in I$, (ii) $\overline{[F(x), y] \pm G(xy)} \in Z(R/P)$ for all $x, y \in I$ and (iii) $\overline{F(x) \circ y \pm G(xy)} \in Z(R/P)$ for all $x, y \in I$. Secondly, we will assume that the above algebraic identities belong to $Z(R/P)$, where P is any prime ideal rather than the zero ideal.

Theorem 4. *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \not\subseteq I$. If F and G are generalized derivations of R associated with derivations d and g respectively, satisfying one of the following properties:*
1) $\overline{F(x)F(y) \pm G(xy)} \in Z(R/P)$ for all $x, y \in I$;
2) $\overline{[F(x), y] \pm G(xy)} \in Z(R/P)$ for all $x, y \in I$;
3) $\overline{F(x) \circ y \pm G(xy)} \in Z(R/P)$ for all $x, y \in I$;
then $\Big(d(R) \subseteq P$ and $g(R) \subseteq P\Big)$ or R/P is a commutative integral domain.

Proof. (1) We are given that

$$\overline{F(x)F(y) + G(xy)} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{1.28}$$

Replacing y by yr in (1.28), we find that

$$\overline{\Big(F(x)F(y) + G(xy)\Big)r + F(x)yd(r) + xyg(r)} \in Z(R/P) \quad \text{for all } x, y, r \in I$$

and therefore

$$[F(x)yd(r) + xyg(r), r] \in P \quad \text{for all } x, y, r \in I. \tag{1.29}$$

Substituting xt for x in (1.29) and subtracting it with (1.29), we arrive at

$$[xd(t)yd(r), r] \in P \quad \text{for all } x, y, r, t \in I. \tag{1.30}$$

Putting ux instead of x in (1.30), we obtain

$$[u,r]xd(t)yd(r) + u[xd(t)yd(r),r] \in P \quad \text{for all } x,y,r,t,u \in I$$

in such a way that

$$[u,r]Id(t)yd(r) \subseteq P \quad \text{for all } y,r,t,u \in I \tag{1.31}$$

which because of primeness, gives that either $[I,r] \subseteq P$ or $d(t)yd(r) \in P$ for all $y,r,t \in I$. The sets of r for which theses conditions holds are additive subgroups of I with union equal to I; so that by Brauer's trick, we have R/P is commutative or $d(R) \subseteq P$. In the later case the relation (1.29) yields

$$[xyg(r),r] \in P \quad \text{for all } x,y,r \in I. \tag{1.32}$$

Substituting wx for x in (1.32) where $w \in R$, we get $[w,r]xyg(r) \in P$ for all $x,y,r \in I$. As a special case of the last equation, we may write

$$[w,r]xg(r)yg(r) \in P \quad \text{for all } x,y,r \in I \text{ and } w \in R. \tag{1.33}$$

On the other hand, taking $w = g(r)$ in the above relation and combining it with (1.33), it is obvious to see that

$$[g(r),r]x[g(r),r]y[g(r),r] \in P \quad \text{for all } x,y,r \in I.$$

Since P is prime, the last equation assures that $[g(r),r] \in P$ which leads to $\Big[[g(r),r],t\Big] \in P$ for all $r,t \in I$. Applying Lemma 1, it follows that either $g(R) \subseteq P$ or R/P is commutative. Now assume that $\overline{F(x)F(y) - G(xy)} \in Z(R/P)$. Thus by putting $(-G,-g)$ instead of (G,g) in the relation (1.28), we get the required result.

(2) Suppose that

$$\overline{[F(x),y] + G(xy)} \in Z(R/P) \quad \text{for all } x,y \in I. \tag{1.34}$$

Replacing y by yr in (1.34), we get

$$\overline{\Big([F(x),y] + G(xy)\Big)r + y[F(x),r] + xyg(r)} \in Z(R/P) \quad \text{for all } x,y,r \in I$$

which leads to

$$[y[F(x),r] + xyg(r),r] \in P \quad \text{for all } x,y,r \in I. \tag{1.35}$$

Writing ry instead of y in (1.35) and subtracting it from (1.35), we arrive at

$$\Big[[x,r]yg(r),r\Big] \in P \quad \text{for all } x,y,r \in I. \tag{1.36}$$

Since the expression (1.36) is similar as relation (1.6), reasoning in the same manner as above, we find that

$$[x,r][t,r]yg(r) \in P \quad \text{for all } x,y,r,t \in I$$

in particular

$$[g(r),r]I[g(r),r]I[g(r),r] \subseteq P \quad \text{for all } r \in I.$$

Invoking Fact 1, we get either R/P is an integral domain or $g(R) \subseteq P$. By the second case the relation (1.35) reduces to

$$\Big[y[F(x),r],r\Big] \in P \quad \text{for all } x,y,r \in I. \tag{1.37}$$

Putting $F(x)y$ instead of y in the expression (1.37) and using it, we obtain

$$[F(x),r] \in P \quad \text{for all } x,r \in I. \tag{1.38}$$

Replacing x by xr in the expression (1.38), we find that $[xd(r),r] \in P$ for all $x,r \in I$. The substitution tx for x in the last equation gives

$$[t,r]Id(r) \subseteq P \quad \text{for all } r,t \in I.$$

Finally, we claim that either $d(R) \subseteq P$ or R/P is an integral domain.

Furthermore, if we have $\overline{[F(x),y]-G(xy)} \in Z(R/P)$, then arguing as above, we arrive at $\Big(d(R) \subseteq P$ and $g(R) \subseteq P\Big)$ or R/P is a commutative integral domain.

(3) Using the same techniques as in the second case with a slight modifications, one can see that the same conclusion holds for $\overline{F(x)\circ y \pm G(xy)} \in Z(R/P)$ for all $x,y \in I$. Whence, the proof of our theorem is complete. □

The following corollary is an immediate consequence of the above theorem.

Corollary 6. *Let R be a prime ring and I a nonzero ideal of R. If F and G are generalized derivations of R associated with derivations d and g respectively such that at least one is nonzero, then the following assertions are equivalent:*

1) $F(x)F(y) \pm G(xy) \in Z(R)$ *for all* $x, y \in I$
2) $[F(x), y] \pm G(xy) \in Z(R)$ *for all* $x, y \in I$
3) $F(x) \circ y \pm G(xy) \in Z(R)$ *for all* $x, y \in I$
4) $F(x)F(y) \pm G(xy) \pm xy \in Z(R)$ *for all* $x, y \in I$
5) $[F(x), y] \pm G(xy) \pm xy \in Z(R)$ *for all* $x, y \in I$
6) $F(x) \circ y \pm G(xy) \pm xy \in Z(R)$ *for all* $x, y \in I$
7) *R is a commutative integral domain.*

As an application of Corollary 6, we have the following result.

Corollary 7. *Let R be a prime ring and I a nonzero ideal of R. If F and G are generalized derivations of R associated with derivations d and g respectively such that at least one is nonzero, then the following assertions are equivalent:*

1) $[F(x), y] \pm G(xy) \pm yx \in Z(R)$ *for all* $x, y \in I$
2) $[F(x), y] \pm G(xy) \pm [x, y] \in Z(R)$ *for all* $x, y \in I$
3) $[F(x), y] \pm G(xy) \pm x \circ y \in Z(R)$ *for all* $x, y \in I$
4) $F(x) \circ y \pm G(xy) \pm yx \in Z(R)$ *for all* $x, y \in I$
5) $F(x) \circ y \pm G(xy) \pm [x, y] \in Z(R)$ *for all* $x, y \in I$
6) $F(x) \circ y \pm G(xy) \pm x \circ y \in Z(R)$ *for all* $x, y \in I$
7) *R is a commutative integral domain.*

As a consequence of Theorem 4, the next proposition gives a commutativity criteria for semi-prime ring.

Proposition 3. *Let R be a semiprime ring and I a nonzero ideal of R. If R admits two generalized derivations F and G associated with nonzero derivations d and g respectively, satisfying one of the following conditions:*
1) $F(x)F(y) \pm G(xy) \in Z(R)$ *for all* $x, y \in I$
2) $[F(x), y] \pm G(xy) \in Z(R)$ *for all* $x, y \in I$
3) $F(x) \circ y \pm G(xy) \in Z(R)$ *for all* $x, y \in I$
then R contains a nonzero central ideal.

Proof. Assume that $F(x)F(y) \pm G(xy) \in Z(R)$ for all $x, y \in I$. The ring R is semiprime then there exists a family of prime ideals $\mathcal{P} = \{P_\alpha \,/\, \alpha \in \Lambda\}$ such that $\alpha \in \Lambda \bigcap P_\alpha = (0)$. Therefore $[F(x)F(y) \pm G(xy), r] \in P_\alpha$ for all $\alpha \in \Lambda$. Using the proof of Theorem 4, by equation (1.31) we get

$$[d(r), r]I[d(r), r]I[d(r), r] = 0 \quad \text{for all} \;\; r \in I.$$

In light of the semiprimeness of R, we easily obtain $[d(r), r] = 0$ for all $r \in I$. According to [6, Theorem 3], we conclude that R contains a nonzero central ideal.
Using the same technics as in the preceding proof, the same conclusion holds for the identities $[F(x), y] \pm G(xy) \in Z(R)$ and $F(x) \circ y \pm G(xy) \in Z(R)$ for all $x, y \in I$. □

In the following proposition we will extend [17, Corollary 6] for semi-prime ring.

Proposition 4. *Let R be semiprime ring and I a nonzero ideal of R. Suppose that R admits two generalized derivations F and G associated with derivations d and g respectively such that at least one is nonzero. If the condition $F(x)F(y) \pm G(xy) \pm xy \in Z(R)$ holds for all $x, y \in I$, then R contains a nonzero central ideal.*

The following example proves that the condition "`R/P is 2-torsion free`" is necessary in Theorem 1.

Example 1. Let us set $R = M_2(\mathbf{Z}_2)$ and $P = (0)$. It is straightforward to check that R is a prime ring with $char(R) = 2$ and P is a prime ideal of R. Define $F : R \to R$ by $F(X) = X \circ A$, where
$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,
then F is a nonzero generalized derivation of R associated with the inner derivation $d(X) = [X, A]$ satisfying

$$F(X \circ Y) = \begin{pmatrix} ca' + dc' + c'a + d'c & 0 \\ 0 & ca' + dc' + c'a + d'c \end{pmatrix} \in Z(R)$$

for all $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$
in R. However, R is non-commutative.

The following example proves that the condition of the "`primeness`" imposed on the ideal is crucial in our Theorems.

Example 2. Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Big/ a, b, c \in \mathbf{Z} \right\}$ and $P = (0)$. Let I the ideal of R defined by $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \Big/ a \in \mathbf{Z} \right\}$. Define the maps on R as follows $F(x) = 2e_{11}x - xe_{11}$ and $G(x) = e_{12}x + xe_{11}$. Then it is clearly to see that F and G are generalized derivations of R associated with nonzero derivations d and g respectively, where $d(x) = e_{11}x - xe_{11}$ and $g(x) = -e_{11}x + xe_{11}$. Moreover F and G satisfies the conditions of all Theorems, but R is not commutative.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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