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Lyapunov stability and weak attraction for control systems

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Abstract

In this paper we deal with Lyapunov stability and weak attraction for control systems. We give characterizations of the stability and asymptotical stability of a compact set by means of its components. We also study the asymptotical stability of the prolongation of a compact weak attractor.

Keywords: *Lyapunov stability, asymptotical stability, connected components, prolongations, control systems.*

AMS Subject Classification (2010): *93D20, 93D05, 34D20, 37C75.*

1. Introduction

The subject matter of this paper is to study Lyapunov stable sets and weak attractors for control systems.

The problem of characterizing the stability and the asymptotical stability of a set by means of its components has been extensively studied in the contexts of dynamical and semi-dynamical systems (see [2], [3], [4], [11] and references therein). In [3, Chapter V], in the context of dynamical systems on metric spaces, Bhatia and Szegö show that a compact set is stable if and only if its components are stable ([3, Chapter V, Theorem 1.19]). They also show that a compact asymptotically stable set has a finite number of components, which are asymptotically stable (see [3, Chapter V, Theorem 1.22]). Another classical result for dynamical systems on metric spaces states that the first prolongation of a compact weak attractor is the smallest asymptotically stable set containing it (see [4, Chapter V, Theorem 1.25]). This result depends on the compactness of the first prolongation of a compact weak attractor (see [4, Chapter V, Lemma 1.26]).

The theory of Lyapunov stability for semigroup actions on topological spaces was introduced by Braga Barros, Souza and Rocha in [8]. Several results on Lyapunov stability and attraction have been generalized from the settings of dynamical systems and dynamical polysystems on metric spaces to the setting of semigroup actions on topological spaces (see [1], [3], [4], [9], [10], [17]). The results obtained for semigroup actions can be applied to the especial context of control systems by considering systems driven by piecewise constant functions. The dynamics of such a system is determined by the action of its system semigroup and the control system itself can be viewed as a generalized dynamical system (see [10, Section 4], [12, Sections 4.2, 4.3 and 4.8] and [15, Section 2]). Some results have been obtained specifically in this context (see [5], [10], [14], [15], [16]).

In this paper, we generalize the results presented by Bhatia and Szegö in [3, Chapter V] on the components of compact stable and asymptotically stable sets for control systems. We show that, for a certain class of control systems, the stability and the asymptotical stability of a compact set can be characterized by means of its components. For the same class of control systems, we show that the first prolongation of a compact weak attractor is the smallest asymptotically stable set containing it. This result also follows from the compactness of the first prolongation of a weak attractor.

In the Section 2, we recall the basic concepts and results of the theories of Lyapunov stability and attraction for control systems.

In the Section 3, we show that a compact set is stable if and only if its components are stable (Theorem 3.1). Under certain hypotheses, we also show that a component of a compact asymptotically stable set is also asymptotically stable if it is isolated (Theorem 3.2) and we show that a compact set is asymptotically stable if and only if it has a finite number of components, each of which is asymptotically stable (Theorem 3.3). After, we show that the first prolongation of a compact weak attractor is a compact set (Proposition 3.1). This fact entails that the first prolongation of a compact weak attractor is the smallest asymptotically stable set containing it (Theorem 3.4).

2. Preliminaries

In this section, we recall some definitions and results on control systems that are used throughout this paper. We refer to [5], [6], [7], [12], [14], [15], [16] for the theory of control systems.

Let M be a finite-dimensional C^{∞}-manifold and let Σ be an *affine control system* on M given by

$$\begin{aligned} x'(t) &= X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)) \\ u &= (u_1, \ldots, u_m) \in \mathcal{U}_{pc} \end{aligned} \tag{2.1}$$

where $X_0, \ldots, X_m$ are C^{∞}-complete vector fields in M and $\mathcal{U}_{pc} = \{u : \mathbf{R} \to U : u \mathrm{piecewiseconstant}\}$, with $U \subset \mathbf{R}^m$. We assume that for each $u \in \mathcal{U}_{pc}$ and $x \in M$, the system Σ admits a unique solution $\varphi(t, x, u)$, $t \in \mathbf{R}$, with $\varphi(0, x, u) = x$. We use the notation

$$\mathbf{X}(x, u) = X_0(x) + \sum_{i=1}^{m} u_i X_i(x) \tag{2.2}$$

and assume that $\mathbf{X}_u = \mathbf{X}(\cdot, u)$ is a C^{∞}-complete vector field in M, for every $u \in U$. For each $t \in \mathbf{R}$ and $u \in U$, we have the diffeomorphism $\varphi_t^u : M \to M$ defined by $\varphi_t^u(x) = \varphi(t, x, u)$. The *system semigroup* of the control system Σ is defined as

$$\mathcal{S}_{\Sigma} = \left\{\varphi_{t_n}^{u_n} \circ \cdots \circ \varphi_{t_1}^{u_1} : u_i \in U, t_i \geq 0, n \in \mathbf{N}\right\}. \tag{2.3}$$

It is easily seen that $\mathcal{S}_{\Sigma}$ acts on M as a semigroup of diffeomorphisms of M. For an element $x \in M$ and a subset $A \subset \mathcal{S}_{\Sigma}$, we define

(2.4) $$Ax = \{y \in M : \text{thereexists}\phi \in A\text{suchthat}\phi(x) = y\}.$$

The set $\mathcal{S}_\Sigma x$ is called the *positive orbit of* Σ *through* $x \in M$. Since the set of control functions of Σ is $\mathcal{U}_{pc}$, we have that

(2.5) $$\mathcal{S}_\Sigma x = \{\ y \in M :\ \text{there exist } t \geq 0 \text{ and } u \in \mathcal{U}_{pc} \text{ such that } \varphi(t, x, u) = y\ \}.$$

For subsets $X \subset M$ and $A \subset \mathcal{S}_\Sigma$, we define

(2.6) $$AX = \bigcup_{x \in X} Ax.$$

We say that X is *positively invariant* for the system Σ if $\mathcal{S}_\Sigma X \subset X$.

For $t \geq 0$, we consider the sets

(2.7) $$(\mathcal{S}_\Sigma)_{\geq t} = \left\{\varphi_{t_n}^{u_n} \circ \cdots \circ \varphi_{t_1}^{u_1} : u_i \in U, t_i \geq 0, \sum_{i=1}^{n} t_i \geq t, n \in \mathbf{N}\right\}$$

and

(2.8) $$(\mathcal{S}_\Sigma)_{\leq t} = \left\{\varphi_{t_n}^{u_n} \circ \cdots \circ \varphi_{t_1}^{u_1} : u_i \in U, t_i \geq 0, \sum_{i=1}^{n} t_i \leq t, n \in \mathbf{N}\right\}.$$

The family

(2.9) $$\mathcal{F}_{\text{ctr}} = \left\{(\mathcal{S}_\Sigma)_{\geq t} : t \geq 0\right\}$$

is a directed set when ordered by the reverse inclusion or, in other words, it is a time-depending filter basis on the subsets of $\mathcal{S}_\Sigma$ (that is, $\emptyset \notin \mathcal{F}_{\text{ctr}}$ and given $t, s \geq 0$, $(\mathcal{S}_\Sigma)_{\geq t+s} \subset (\mathcal{S}_\Sigma)_{\geq t} \cap (\mathcal{S}_\Sigma)_{\geq s}$).

Throughout this paper, we assume that the control range U of Σ is a compact and convex subset of $\mathbf{R}^m$. Thus, the closure of $\mathcal{U}_{pc}$ with respect to the weak* topology of $\mathcal{L}^\infty(\mathbf{R}, \mathbf{R}^n)$, denoted by $\mathcal{U} = \text{cl}(\mathcal{U}_{pc})$, is a compact Hausdorff space and the *solution map*

(2.10) $$\varphi : \mathbf{R} \times M \times \mathcal{U} \to M, \quad (t, x, u) \mapsto \varphi(t, x, u)$$

is continuous, where $\varphi(t, x, u)$ is the unique solution of the system Σ with respect to the initial condition $x(0) = x$ and the function $u \in \mathcal{U}$ at the time

t. Moreover, the cocycle property holds: $\varphi(t+s,x,u)=\varphi(t,\varphi(s,x,u),u')$, where u' is the shift $u'(s)=u\cdot t(s)=u(t+s)$ (see [10, Section 4], [12, Sections 4.2 and 4.3] and [15, Section 2] for the details).

In the theories of semigroup actions and control systems, some results need additional hypotheses on the family $\mathcal{F}_{\mathrm{ctr}}$. The following translation hypotheses were considered in [5], [6], [7], [8], [9], [10], [14], [15], [16].

Definition 2.1. *We say that the system* Σ *satisfies*

1. *the hypothesis* H_1 *if for all* $\phi\in\mathcal{S}_\Sigma$ *and* $t\geq 0$ *there exists* $s\geq 0$ *such that* $\phi\circ(\mathcal{S}_\Sigma)_{\geq s}\subset(\mathcal{S}_\Sigma)_{\geq t}$.
2. *the hypothesis* H_2 *if for all* $\phi\in\mathcal{S}_\Sigma$ *and* $t\geq 0$ *there exists* $s\geq 0$ *such that* $(\mathcal{S}_\Sigma)_{\geq s}\circ\phi\subset(\mathcal{S}_\Sigma)_{\geq t}$.
3. *the hypothesis* H_3 *if for all* $\phi\in\mathcal{S}_\Sigma$ *and* $t\geq 0$ *there exists* $s\geq 0$ *such that* $(\mathcal{S}_\Sigma)_{\geq s}\subset(\mathcal{S}_\Sigma)_{\geq t}\circ\phi$.

The system Σ satisfies the hypotheses H_1 and H_2 (see [6, Section 5] and [7, Section 4]). In the following, we present examples of classes of systems satisfying the hypothesis H_3.

Example 2.1. *Let* G *be a connected Lie group with nontrivial center and* g *its Lie algebra. Let* $a\subset\mathrm{z}(\mathrm{g})$ *be a vector subspace in the center* $\mathrm{z}(\mathrm{g})$ *of* g. *Take a nonzero vector field* $X\in\mathrm{g}$. *Consider the control system* Σ *on* G *defined by concatenations of vector fields in the set* $F=\{X+Y:Y\in a\}$. *The system semigroup is the set*

$$\mathcal{S}_\Sigma=\{\exp t(X+Y):t\geq 0\text{ and }Y\in a\}. \tag{2.11}$$

The system Σ *satisfies the hypothesis* H_3 *(see [7, Example 4] and [14, Example 2.3]).*

Example 2.2. *Let* $M=\mathbf{R}^d$. *Consider the bilinear control system* Σ *on* M *given by*

$$x'(t)=\sum_{i=1}^{n}u_i(t)\,A_i(x(t))\,, \tag{2.12}$$

where $U=\{u\in\mathbf{R}^n:a\leq\|u\|\leq b\}$, *with* $a>0$, *and* $A_1,\ldots,A_n\in\mathbf{R}^{d\times d}$ *are pairwise commutative matrices. Then, the system* Σ *satisfies the hypothesis* H_3 *(see [5, Example 2.1] and [14, Example 2.4]).*

For more examples of systems satisfying the hypothesis H_3, see [5], [7], [14], [15], [16].

The next concept of limit set for control systems was studied in [6, Section 5].

Definition 2.2. *The ω-limit set of a subset $X \subset M$ for the system Σ is defined as*

$$\omega(X) = \left\{ y \in M : \begin{array}{c} \text{there exist sequences } t_n \to +\infty\ ,\ (x_n) \text{ in} X \\ \text{and } (u_n) \text{ in } \mathcal{U}_{pc} \text{ such that } \varphi\ (t_n, x_n, u_n) \to y \end{array} \right\}. \tag{2.13}$$

For an element $x \in M$, we write $\omega(\{x\}) = \omega(x)$. If $\mathrm{cl}(\mathcal{S}_\Sigma x)$ is compact, then $\omega(x)$ is nonempty, connected, compact and positively invariant. Moreover,

$$\mathrm{cl}(\mathcal{S}_\Sigma x) = \mathcal{S}_\Sigma x \cup \omega(x) \tag{2.14}$$

(see [15] for the details).

We also have the following result.

Proposition 2.1. *Let $x \in M$. For all $t > 0$ and $u \in \mathcal{U}_{pc}$, we have*

$$\omega(\varphi(t, x, u)) \subset \omega(x). \tag{2.15}$$

Proof. See [15, Proposition 2.1] (see also [8, Proposition 2.1]). □

We refer to [5], [7], [8], [10], [14], [15], [16] for the theory of limit sets for control systems.

From now on, we assume that M has a metric d. We denote by $B(x, \varepsilon)$ the ε-ball centered in $x \in M$ with respect to the metric d. We also denote by $B(X, \varepsilon)$ the ε-neighborhood of a subset X of M, that is, for $\varepsilon > 0$,

$$B(X, \varepsilon) = \{y \in M : d(y, X) < \varepsilon\}. \tag{2.16}$$

The next definition of prolongation for control systems was considered in [15].

Definition 2.3. *Let $x \in M$ and $t \geq 0$. The first positive t-prolongation of x is defined as*

$$D^{+}(x,t)=\left\{ y\in M:\ \begin{array}{c}\text{there exist sequences } (t_n) \text{ in } [t,+\infty),\\ (x_n) \text{ in } M \text{ and } (u_n) \text{ in } \mathcal{U}_{pc} \text{ such that}\\ x_n\to x \text{ and } \varphi(t_n,x_n,u_n)\to y\end{array}\right\}. \tag{2.17}$$

For a subset X of M, the first positive t-prolongation of X is defined as

$$D^{+}(X,t)=\bigcup_{x\in X} D^{+}(x,t). \tag{2.18}$$

For an element $x\in M$, we write $D^{+}(x)=D^{+}(x,0)$. It is easily seen that $\mathrm{cl}(\mathcal{S}_{\Sigma}x)\subset D^{+}(x)$, for every $x\in M$. In particular, $x\in D^{+}(x)$, for every $x\in M$. For a subset X of M, we also write $D^{+}(X)=D^{+}(X,0)$. The set $D^{+}(X)$ is positively invariant (see [8, Proposition 2.7]).

The prolongations of compact sets are closed sets.

Proposition 2.2. *Let K be a compact subset of M. Then, $D^{+}(K)$ is closed.*

Proof. See [8, Proposition 2.9]. □

For every $x\in M$, the first positive prolongation $D^{+}(x)$ is connected if it is compact (see [15, Theorem 2.3]). Concerning the first positive prolongations of subsets of M, we have the following result.

Proposition 2.3. *Let X be a connected subset of M. Assume that $D^{+}(x)$ is compact, for every $x\in X$. Then $D^{+}(X)$ is connected.*

Proof. It is well-known that if $\{X_i\}_{i\in I}$ is a family of connected sets and X is a connected set such that $X\cap X_i\neq\emptyset$, for every $i\in I$, then $X\cup(\cup_{i\in I}X_i)$ is connected (for instance, see [13, Section 23]). Consider the family $\{D^{+}(x)\}_{x\in X}$. Each element of this family is a connected set and $x\in X\cap D^{+}(x)$, for every $x\in X$. Since X is connected and $X\subset D^{+}(X)$, it follows that

$$D^{+}(X)=X\cup D^{+}(X)=X\cup\left(\bigcup_{x\in X} D^{+}(x)\right) \tag{2.19}$$

is connected. □

The next definition of prolongational limit set for control systems was considered in [15].

Definition 2.4. *Let $x \in M$. The first positive prolongational limit set of x for the system Σ is defined as*

$$(2.20)\quad J^{+}(x) = \left\{ y \in M : \begin{array}{c} \text{there exist sequences } (t_n) \text{ in } \mathbf{R}^{+}, (x_n) \text{ in } M \\ \text{and } (u_n) \text{ in } \mathcal{U}_{pc} \text{ such that } t_n \to +\infty , x_n \to x \\ \text{and } \varphi(t_n, x_n, u_n) \to y \end{array} \right\}.$$

For every $x \in M$, the set $J^{+}(x)$ is closed and positively invariant (see [8, Proposition 2.10]). Moreover, $\omega(x) \subset J^{+}(x)$ and

$$(2.21)\qquad D^{+}(x) = \mathcal{S}_{\Sigma}x \cup J^{+}(x)$$

(see [15, Theorem 2.2]).

The next proposition gives a useful property of the prolongational limit sets of elements of ω-limit sets.

Proposition 2.4. *Assume that the system Σ satisfies the hypothesis* H_3. *Take $x \in M$ and $z \in \omega(x)$. Then $J^{+}(x) \subset J^{+}(z)$.*

Proof. See [8, Proposition 2.12]. □

We refer to [8], [10], [15], [16] for the theory of prolongational limit sets.

The concepts of domains of attraction and attractors for control systems were studied in [8], [9].

Definition 2.5. *Let X be a subset of M. The domain of weak attraction and the domain of attraction of X for the system Σ are respectively defined by*

$$(2.22)\qquad \mathcal{A}_w(X) = \left\{ y \in M : \begin{array}{l} \text{for every } \varepsilon > 0 \text{ and } t \geq 0, \\ (\mathcal{S}_{\Sigma})_{\geq t}\, y \cap B(X, \varepsilon) \neq \emptyset \end{array} \right\}$$

and

$$(2.23)\quad \mathcal{A}(X) = \left\{ y \in M : \begin{array}{l} \text{for every } \varepsilon > 0 \text{ , there exists } t \geq 0 \text{ such that} \\ (\mathcal{S}_{\Sigma})_{\geq t}\ y \subset B(X, \varepsilon) \end{array} \right\}.$$

We say that X is a weak attractor (respectively an attractor) for the system Σ if there exists $\varepsilon > 0$ such that $B(X,\varepsilon) \subset \mathcal{A}_w(X)$ (respectively $B(X,\varepsilon) \subset \mathcal{A}(X)$).

Since $\mathcal{F}_{\mathrm{ctr}}$ is a filter basis, we have that $\mathcal{A}(X) \subset \mathcal{A}_w(X)$. Thus, every attractor is a weak attractor (see [9, Section 3]).

For a compact subset K of M, it was shown in [9, Theorems 3.5 and 3.6] that

$$\mathcal{A}_w(K) = \{x \in M : \omega(x) \cap K \neq \emptyset\} \tag{2.24}$$

and

$$A(K) = \{x \in M : \omega(x) \neq \emptyset \text{and} \omega(x) \subset K\}. \tag{2.25}$$

It is immediate to verify that, in the definition of weak attractor and attractor, the ε-neighborhoods can be replaced by open neighborhoods for compact sets.

The next result will be used afterwards in this paper.

Proposition 2.5. *Let K be a compact subset of M. Then*

1. *$\mathcal{A}_w(K)$ is an open set containing K if it is a weak attractor.*
2. *$\mathcal{A}(K)$ is an open set containing K if it is an attractor and the system Σ satisfies the hypothesis* H_3.
3. *$\mathcal{A}_w(K)$ is positively invariant if the system Σ satisfies the hypothesis* H_3.
4. *$\mathcal{A}(K)$ is positively invariant.*

Proof. See [9, Propositions 3.2 and 3.3]. □

The following result concerns the prolongational limit sets of elements in the domain of weak attraction of a compact set.

Proposition 2.6. *Assume that the system Σ satisfies the hypothesis* H_3. *Let K be a compact subset of M and take $x \in \mathcal{A}_w(K)$. Then $J^+(x) \subset D^+(K)$.*

Proof. Take $x \in \mathcal{A}_w(K)$. Then $\omega(x) \cap K \neq \emptyset$. Thus, given $z \in \omega(x) \cap K$, we have from Proposition 2.4 that

$$J^+(x) \subset J^+(z) \subset D^+(z) \subset D^+(K). \tag{2.26}$$

□

The next definition of Lyapunov stable set was considered in [8].

Definition 2.6. *We say that a subset X of M is*

1. *stable for the system Σ if for every $\varepsilon > 0$ and $x \in X$, there exists $\delta > 0$ such that $\mathcal{S}_\Sigma B(x, \delta) \subset B(X, \varepsilon)$.*
2. *uniformly stable for the system Σ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{S}_\Sigma B(X, \delta) \subset B(X, \varepsilon)$.*
3. *orbitally stable for the system Σ if every neighborhood U of X contains a positively invariant neighborhood V of X.*
4. *asymptotically stable for the system Σ if it is a uniformly stable attractor.*

It follows immediately from Definition 2.6 that a subset X of M is stable if it is uniformly stable. The converse holds if X is compact (see [8, Theorem 3.2]). It also follows immediately from Definition 2.6 that a compact subset K of M is stable if and only if it is orbitally stable.

Concerning the positive invariance of stable sets, we have the following result.

Proposition 2.7. *Let X be a closed and stable subset of M. Then X is positively invariant.*

Proof. See [8, Proposition 3.1]. □

The next result provides a connexion between prolongations and stable sets.

Proposition 2.8. *Let K be a compact subset of M. Then K is stable if and only if $D^+(K) = K$.*

Proof. See [8, Proposition 3.3]. □

Finally, we give a sufficient condition for a compact weak attractor to be asymptotically stable.

Proposition 2.9. *Assume that the system Σ satisfies the hypothesis* H_3. *Let K be a compact weak attractor. Assume that K is stable. Then K is asymptotically stable.*

Proof. See [8, Theorem 3.5]. □

We refer to [5], [6], [7], [8], [9], [10] for the theories of attraction and Lyapunov stability.

3. Main results

In this section, we present results concerning the components of stable and asymptotically stable sets. We characterize the stability and the asymptotically stability of a compact set by means of its components. We also show that the first prolongation of a compact weak attractor is the smallest asymptotically stable set containing it.

Throughout this section, we consider a system Σ as in the previous section and assume that K is a compact subset of M.

We start by characterizing the stability of K by means of the stability of its components.

Theorem 3.1. *The set K is stable if and only if every component of K is stable.*

Proof. Assume that K is stable and let K_1 be a component of K. Notice that K_1 is connected and compact. Since K is stable, we have from Proposition 2.8 that $D^+(K) = K$. Since $D^+(x) \subset D^+(K_1) \subset D^+(K) = K$, for all $x \in K_1$, it follows that $D^+(x)$ is compact, for all $x \in K_1$. Thus we have from Proposition 2.3 that $D^+(K_1)$ is a connected subset of K containing K_1. Then $D^+(K_1) = K_1$ and we conclude from Proposition 2.8 that K_1 is stable. Conversely, let $\{K_i\}_{i\in I}$ be the collection of the components of K and assume that K_i is stable, for all $i \in I$. Hence, $D^+(K_i) = K_i$, for all $i \in I$. Thus,

$$\begin{aligned} D^+(K) &= \bigcup_{x\in K} D^+(x) = \bigcup_{i\in I} \bigcup_{x\in K_i} D^+(x) \\ &= \bigcup_{i\in I} D^+(K_i) = \bigcup_{i\in I} K_i = K. \end{aligned}$$

Therefore, K is stable. □

We need the following lemma.

Lemma 3.1. *Suppose that K is positively invariant. Let K_1 be a component of K. Then,*

1. *K_1 is also compact and positively invariant.*
2. *$\mathcal{A}(K_1) \cap (K - K_1) = \emptyset$.*

Proof.

1. It is immediate that K_1 is compact. Now notice that the orbits of the system Σ are connected. Thus, if $x \in K_1$, we have that $\mathcal{S}_\Sigma x$ is a connected subset of K that intersects K_1. Hence, $\mathcal{S}_\Sigma x \subset K_1$. Therefore,

$$\mathcal{S}_\Sigma K_1 = \bigcup_{x \in K_1} \mathcal{S}_\Sigma x \subset K_1 \tag{3.1}$$

and K_1 is positively invariant.

2. Suppose by contradiction that K_0 is another component of K such that there exists $x \in \mathcal{A}(K_1) \cap K_0$. Since $x \in \mathcal{A}(K_1)$, we have that $\omega(x) \neq \emptyset$ and $\omega(x) \subset K_1$. On the other hand, it follows from item 1 that K_0 is compact and positively invariant. Thus $\omega(x) \subset K_0$, which contradicts $K_1 \cap K_0 = \emptyset$.

□

The next result gives us a relation between asymptotical statibility and isolated components.

Theorem 3.2. *Assume that the system Σ satisfies the hypothesis* H_3. *Suppose that K is asymptotically stable. Let K_1 be a component of K. Then K_1 is asymptotically stable if and only if it is an isolated component of K.*

Proof. Assume that K_1 is asymptotically stable. Then K_1 is stable and, by Proposition 2.7, it is positively invariant. Thus, it follows from Lemma 3.1 that $\mathcal{A}(K_1) \cap (K - K_1) = \emptyset$. Since the system Σ satisfies the hypothesis H_3 and K_1 is also an attractor, we have from Proposition 2.5 that $\mathcal{A}(K_1)$ is an open neighborhood of K_1. Therefore, K_1 is isolated.

Conversely, suppose that K_1 is isolated. Then there exists an open neighborhood W of K_1 in M such that $W \cap (K - K_1) = \emptyset$. Since K is asymptotically stable, it follows from Proposition 2.5 that $W \cap \mathcal{A}(K)$ is an open neighborhood of K_1 in M. Since M is locally compact, there exists a closed neighborhood U of K_1 in M such that $U \subset (W \cap \mathcal{A}(K)) \subset \mathcal{A}(K)$ and $U \cap (K - K_1) = \emptyset$. We have from Theorem 3.1 that K_1 is stable. Then there exists a neighborhood V of K_1 in M such that $\mathcal{S}_\Sigma V \subset V \subset U \subset \mathcal{A}(K)$. Now, take $x \in V$. Then $\omega(x) \neq \emptyset$ and $\omega(x) \subset K$. On the other hand,

$$\omega(x) \subset \mathrm{cl}(\mathcal{S}_\Sigma x) \subset \mathrm{cl}(\mathcal{S}_\Sigma V) \subset \mathrm{cl}(V) \subset \mathrm{cl}(U) = U. \tag{3.2}$$

Hence $\omega(x) \subset K \cap U$. Since $U \cap (K - K_1) = \emptyset$, it follows that $\omega(x) \subset K_1$. Thus, $V \subset \mathcal{A}(K_1)$ and K_1 is an attractor. Since K_1 is a compact stable attractor, we conclude that K is asymptotically stable. □

As a consequence of Theorem 3.2, we have the following theorem for asymptotically stable sets.

Theorem 3.3. *Assume that the system Σ satisfies the hypothesis* H_3. *The set K is asymptotically stable if and only if K has a finite number of components, each of which is asymptotically stable.*

Proof. Assume that K is asymptotically stable. We have from Proposition 2.5 that $\mathcal{A}(K)$ is an open neighborhood of K. Since M is locally connected, the components of $\mathcal{A}(K)$ are open sets. Then the components of $\mathcal{A}(K)$ form an open covering of K. Since K is compact, there exist a finite number of components that cover K, say $C_1, \ldots, C_n$. Notice that each component of K is contained in one of the components $C_1, \ldots, C_n$. Moreover, each component $C_1, \ldots, C_n$ contains precisely one component of K. In fact, let $\{K_l\}_{l \in L}$ be the collection of the components of K. Fix $i \in \{1, \ldots, n\}$ and $x \in C_i$. Since $\mathcal{A}(K)$ is positively invariant, it follows that C_i is positively invariant. Thus, $\omega(x) \subset C_i$, because C_i is also a closed set. On the other hand, we have that $\omega(x) \neq \emptyset$ and $\omega(x) \subset K$. It follows from the connectedness of $\omega(x)$ that $\omega(x) \subset K_{l_1}$, for some $l_1 \in L$ such that $K_{l_1} \subset C_i$. Hence, $C_i \subset \mathcal{A}(K_{l_1})$. If there would exist $l_2 \neq l_1$ in L such that $K_{l_2} \subset C_i$, then $K_{l_2} \subset \mathcal{A}(K_{l_1})$, which would contradict the item 2 of Lemma 3.1. Thus, each component $C_1, \ldots, C_n$ contains precisely one component of K. That entails that the number of the components of K is finite and each one of them is isolated. From the Theorem 3.2, we conclude that the components of K are asymptotically stable.

Conversely, assume that K has a finite number of components, namely $K_1, ..., K_n$, which are asymptotically stable. Take a neighborhhod W of K in M. For each $i \in \{1, \ldots, n\}$, there exists a neighborhood V_i of K_i in M such that $\mathcal{S}_\Sigma V_i \subset V_i \subset W$. Then, $V = \bigcup_{i=1}^{n} V_i$ is a positively invariant neighborhood of K contained in W. Thus, K is stable. Now, we show that K is also an attractor. For each $i \in \{1, \ldots, n\}$, the set $\mathcal{A}(K_i)$ is an open neighborhood of K_i. Then $A = \bigcup_{i=1}^{n} \mathcal{A}(K_i)$ is an open neighborhood of K in M and $A \subset \mathcal{A}(K)$. Hence K is an attractor and, therefore, it is asymptotically stable. □

Now we are focused on showing that the prolongation of a compact weak attractor is the smallest asymptotically stable set containing it. In order to prove that, we need the following result.

Proposition 3.1. *Assume that K is a weak attractor. Then $D^+(K)$ is compact.*

Proof. Since M is locally compact and $\mathcal{A}_w(K)$ is an open neighborhood of K, we can choose an $\varepsilon > 0$ such that $\mathrm{cl}(B(K,\varepsilon))$ is compact and $\mathrm{cl}(B(K,\varepsilon)) \subset \mathcal{A}_w(K)$. We show that there exists $T > 0$ such that

$$D^+(K) \subset (\mathcal{S}_\Sigma)_{\leq T}\mathrm{cl}(B(K,\varepsilon)), \tag{3.3}$$

where $(\mathcal{S}_\Sigma)_{\leq T}\mathrm{cl}(B(K,\varepsilon)) = \varphi([0,T]\times \mathrm{cl}(B(K,\varepsilon))\times\mathcal{U})$. Pick $x \in \partial\mathrm{cl}(B(K,\varepsilon))$ and define

$$\tau_x = \inf\{t > 0 : \varphi(t,x,u) \in B(K,\varepsilon),\ \text{with } u \in \mathcal{U}_{pc}\}. \tag{3.4}$$

Notice that τ_x is well-defined as $x \in \mathcal{A}_w(K)$. Now let

$$T = \sup\{\tau_x : x \in \partial\mathrm{cl}(B(K,\varepsilon))\}. \tag{3.5}$$

We affirm that $T < +\infty$. In fact, suppose by contradiction that there exists a sequence $(x_n) \subset \partial\mathrm{cl}(B(K,\varepsilon))$ such that $\tau_{x_n} \to +\infty$. From the compactness of $\partial\mathrm{cl}(B(K,\varepsilon))$ we can assume without loss of generality that $x_n \to x$ in $\partial\mathrm{cl}(B(K,\varepsilon)) \subset \mathcal{A}_w(K)$. Take $\tau > 0$ and $u \in \mathcal{U}_{pc}$ such that $\varphi(\tau,x,u) \in B(K,\varepsilon)$. Then $\varphi(\tau,x_n,u) \to \varphi(\tau,x,u)$. Take $\mu > 0$ such that $B(\varphi(\tau,x,u),\mu) \subset B(K,\varepsilon)$. There exists $n_0 \in \mathbf{N}$ such that $\varphi(\tau,x_n,u) \in B(\varphi(\tau,x,u),\mu)$ whenever $n \geq n_0$. Thus, $\tau_{x_n} \leq \tau$ whenever $n \geq n_0$, which is a contradiction.

Now, take $y \in D^+(K) - \mathrm{cl}(B(K,\varepsilon))$. We have that $y \in D^+(x)$, for some $x \in K$, and $y \notin \mathrm{cl}(B(K,\varepsilon))$. There exist sequences $(t_n) \subset (0,+\infty)$, $(x_n) \subset M$ and $(u_n) \subset \mathcal{U}_{pc}$ such that $x_n \to x$ and $\varphi(t_n,x_n,u_n) \to y$. Take $n_1, n_2 \in \mathbf{N}$ such that $x_n \in B(K,\varepsilon)$ whenever $n \geq n_1$ and $\varphi(t_n,x_n,u_n) \notin B(K,\varepsilon)$ whenever $n \geq n_2$ and let $n_3 = \max\{n_1,n_2\}$. For each $n \geq n_3$, there exists $s_n \in (0,t_n)$ such that $\varphi(s_n,x_n,u_n) \in \partial\mathrm{cl}(B(K,\varepsilon))$ and $\varphi(t,x_n,u_n) \notin \mathrm{cl}(B(K,\varepsilon))$ if $t \in (s_n,t_n]$. Since

$$\varphi(t_n,x_n,u_n) = \varphi((t_n-s_n)+s_n,x_n,u_n) = \varphi(t_n-s_n,\varphi(s_n,x_n,u_n),u_n'), \tag{3.6}$$

we have that $0 < t_n - s_n \leq T$ otherwise if $t_n - s_n > T$ we would have $\varphi(t_n, x_n, u_n) \in \mathrm{cl}(B(K, \varepsilon))$. Hence,

$$(3.7)\ \varphi(t_n, x_n, u_n) = \varphi(t_n - s_n, \varphi(s_n, x_n, u_n), u_n') \in (\mathcal{S}_\Sigma)_{\leq T}\mathrm{cl}(B(K, \varepsilon))$$

whenever $n \geq n_3$. Since the solution map (2.10) is continuous, $(\mathcal{S}_\Sigma)_{\leq T}\mathrm{cl}(B(K, \varepsilon))$ is compact and therefore $y \in (\mathcal{S}_\Sigma)_{\leq T}\mathrm{cl}(B(K, \varepsilon))$. The inclusion (3.3) holds as

$$(3.8)\quad \mathrm{cl}(B(K, \varepsilon)) = \varphi(\{0\} \times \mathrm{cl}(B(K, \varepsilon) \times \mathcal{U}) \subset (\mathcal{S}_\Sigma)_{\leq T}\mathrm{cl}(B(K, \varepsilon)).$$

Finally, we have from Proposition 2.2 that $D^+(K)$ is closed. Therefore, we conclude from (3.3) that $D^+(K)$ is compact. □

We now have the following theorem.

Theorem 3.4. *Assume that the system* Σ *satisfies the hypothesis* H_3 *and that* K *is a compact weak attractor. Then,* $D^+(K)$ *is a compact asymptotically stable set and* $\mathcal{A}(D^+(K)) = \mathcal{A}_w(K)$. *Moreover,* $D^+(K)$ *is the smallest asymptotically stable set containing* K.

Proof. Let $\varepsilon, T > 0$ be as in the inclusion (3.3). We have from Proposition 2.5 that $\mathcal{A}_w(K)$ is positively invariant. Thus,

$$(3.9)\qquad D^+(K) \subset (\mathcal{S}_\Sigma)_{\leq T}\mathrm{cl}(B(K, \varepsilon)) \subset \mathcal{A}_w(K).$$

Then, $\mathcal{A}_w(K)$ is an open and positively invariant neighborhood of $D^+(K)$. For a given $x \in \mathcal{A}_w(K)$, it follows from Proposition 2.6 that $\omega(x) \neq \emptyset$ and

$$(3.10)\qquad \omega(x) \subset J^+(x) \subset D^+(K).$$

Thus, $\mathcal{A}_w(K) \subset \mathcal{A}(D^+(K))$ and $D^+(K)$ is an attractor. We now show the other inclusion. Take $x \in \mathcal{A}(D^+(K))$. We have that $\omega(x) \neq \emptyset$ and $\omega(x) \subset D^+(K) \subset \mathcal{A}_w(K)$. Then there exist $t > 0$ and $u \in \mathcal{U}_{pc}$ such that $\varphi(t, x, u) \in \mathcal{A}_w(K)$ so that $\omega(\varphi(t, x, u)) \cap K \neq \emptyset$. It follows from Proposition 2.1 that $\omega(x) \cap K \neq \emptyset$. Thus $x \in \mathcal{A}_w(K)$ and the equality $\mathcal{A}_w(K) = \mathcal{A}(D^+(K))$ holds.

In order to show that $D^+(K)$ is stable, take $x \in D^+(K) \subset \mathcal{A}_w(K)$. We have from Proposition 2.6 that $J^+(x) \subset D^+(K)$. Since $D^+(x) = \mathcal{S}_\Sigma x \cup$

$J^+(x)$ and $D^+(K)$ is positively invariant, we have that $D^+(x) \subset D^+(K)$. Thus

$$D^+(D^+(K)) = \left(\bigcup_{x \in D^+(K)} D^+(x) \right) \subset D^+(K). \tag{3.11}$$

Since the other inclusion always holds, we have that $D^+(D^+(K)) = D^+(K)$. We conclude by Proposition 2.8 that $D^+(K)$ is stable. Therefore, $D^+(K)$ is asymptotically stable as it is also an attractor.

Finally, let K' be a compact asymptotically stable set such that

$$K \subset K' \subset D^+(K). \tag{3.12}$$

Then

$$D^+(K) \subset D^+(K') \subset D^+(D^+(K)) = D^+(K) \tag{3.13}$$

and we conclude that $D^+(K) = D^+(K')$. Therefore, we have from the stability of K' and Proposition 2.8 that

$$D^+(K) = D^+(K') = K'. \tag{3.14}$$

□

As a final remark, we point out that Theorem 3.4 is an extension of Proposition 2.9. In fact, we obtain Proposition 2.9 from Theorem 3.4 by assuming that K is stable.

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