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On the resolution of the heat equation in unbounded non-regular domains of $\mathbf{R}^3$

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Abstract

We will prove well posedness and regularity results for the bi-dimensional heat equation, subject to mixed Dirichlet-Neumann type boundary conditions on the parabolic boundary of an unbounded (in one space variable direction) time-dependent domain. Our results are proved in anisotropic Hilbertian Sobolev spaces by using the domain decomposition method. This work complements the results obtained in [13] in the one-space variable case.

Key words. *Heat equation, Unbounded non-regular domains, Dirichlet-Neumann condition, Anisotropic Sobolev spaces.*

2010 Mathematics Subject Classification. *35K05, 35K20.*

1. Introduction

This work is devoted to the analysis of the following two-dimensional second order parabolic problem

$$(1.1)\qquad \begin{cases} \partial_t u - \partial_x^2 u - \partial_y^2 u = f \in L^2(\Theta), \\ \partial_x u|_{\Gamma_2} = 0, \\ u|_{\partial\Theta\setminus\Gamma_2} = 0, \end{cases}$$

where $L^2(\Theta)$ stands for the space of square-integrable functions on Θ with the measure $dtdxdy$. Here, Θ (see, Fig. 1) is an open set of $\mathbf{R}^3$ defined by

$$\Theta := \left\{(t,x) \in \mathbf{R}^2 :\ a < t < b,\ \psi(t) < x < +\infty\right\} \times]0,c[$$

where a, b, and c are real numbers such that $-\infty < a < 0 < b < +\infty$, and $c > 0$, while ψ is a Lipschitz continuous real-valued function on (a,b), and such that

$$\psi(t) := \begin{cases} \varphi_1(t) \text{ on } (a,0], \\ \varphi_2(t) \text{ on } [0,b). \end{cases}$$

The function φ_1 (respectively, φ_2) is positive and decreasing (respectively, increasing) on $(a,0]$ (respectively, on $[0,b)$) and verifies the hypothesis $\varphi_1(0) = \varphi_2(0) = 0$. The boundary $\partial\Theta$ of the domain Θ is defined by $\partial\Theta := \bigcup_{k=1}^{5} \Gamma_k$ where

$$\Gamma_1 = \left\{(t, \varphi_1(t), y) \in \mathbf{R}^3 : a < t < 0,\ 0 < y < c\right\},$$

$$\Gamma_2 = \left\{(t, \varphi_2(t), y) \in \mathbf{R}^3 : 0 < t < b,\ 0 < y < c\right\},$$

$$\Gamma_3 = \left\{(t,x,0) \in \mathbf{R}^3 :\ a < t < b,\ \psi(t) < x < +\infty\right\},$$

$$\Gamma_4 = \left\{(t,x,c) \in \mathbf{R}^3 :\ a < t < b,\ \psi(t) < x < +\infty\right\},$$

$$\Gamma_5 = \left\{(0,0,y) \in \mathbf{R}^3 :\ 0 < y < c\right\}.$$

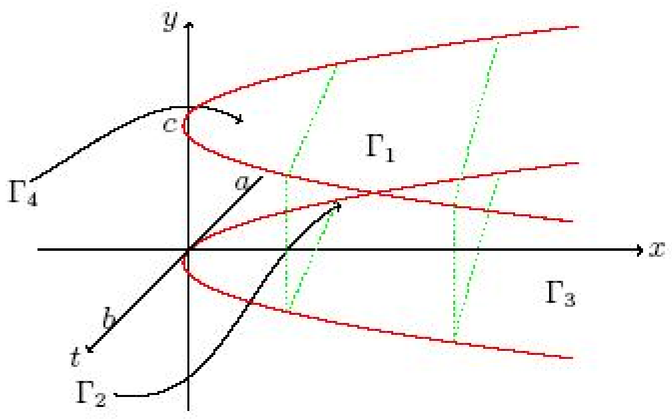

FIG.1 : The unbounded domain Θ.

Notice that the section of Θ in the t direction defined by

$$I_x := [\varphi_1^{-1}(x), \varphi_2^{-1}(x)]\times]0, c[$$

for x in $]0, +\infty[$, is such that the sections $I_n, n \in \mathbf{N}^*$ become bounded when n becomes large, i. e.,

$$\forall\, n \in \mathbf{N}^*,\ |\varphi_2^{-1}(n) - \varphi_1^{-1}(n)| \leq b - a. \tag{1.2}$$

The most interesting point of the parabolic problem studied here is the unboundedness of Θ with respect to the space variable x which prevents one using the methods in [19],[21] and [25]. It's the characteristic (1.2) of the x-sections of Θ which helps us to overcome this difficulty.

Besides being interesting in itself, Problem (1.1) governs for instance, the lateral diffusion of a pollutant in a flow of a river with variable width and constant depth (see [17] and [11]). Regarding the unboundedness in the $x-$direction, this models in our situation the fact where the length of the river is large (several kilometers, for example) relative to the $y-$direction (depth of five meters, for example). In fact, unboundedness in one space direction models a situation where this direction is large relative to the others directions. In practice, often domains are bounded in space directions, but to make mathematical analysis, we make the largest dimension tend towards infinity (see [5] and [7]).

Problem (1.1) is also of interest in combustion theory, where the non-cylindrical space-time of the boundary

$$\partial_{\varphi_1}\Theta := \{(t, \varphi_1(t), y) \in \mathbf{R}^3 : a < t < 0,\ 0 < y < c\}$$

and

$$\partial_{\varphi_2}\Theta := \{(t, \varphi_2(t), y) \in \mathbf{R}^3 : 0 < t < b,\ 0 < y < c\}$$

can be considered as an approximation of a flame front (see [18] and [20]). For similar problems and more applications (see [2], [6] and [8]).

In the case of bounded non-cylindrical domains, studies related to Problem (1.1) can be found in [10], [14] and [11] both in one-dimensional and bi-dimensional cases. Whereas second-order parabolic equations in bounded non-cylindrical domains are well studied (see for instance [1], [9], [12], [15], [17], [18], [22], [23], [24], [27] and the references therein), the literature concerning unbounded non-cylindrical domains does not seem to be extensive. The regularity of the heat equation solution in a non-smooth and unbounded domain (in the t direction) is obtained in [25] and [26] by using two different approaches. In [16], uniqueness classes of solutions of non-divergent second order parabolic equations were obtained. The heat equation in unbounded non-cylindrical domains with respect to the space variable x were considered in [7] and [3]. In [7], the analysis is done in the framework of evolution function spaces, and in [3], properties of solutions of the heat equation were obtained in the space of functions with $t-$and$-x$ derivatives are in weighted L^2-spaces. The class of domains used in [3] corresponds here to

$$\psi(t) := \begin{cases} -\alpha\sqrt{-t} \text{ on } [a, 0] \\ \delta\sqrt{t} \text{ on } [0, b] \end{cases}$$

for any positive constants α and δ. For spaces of low smoothness there is still hope to stay within analysis in the usual anisotropic Sobolev-Slobodetskii spaces under additional assumptions on the type of the domain singularity (here, in the neighborhood of infinity). Let us consider the anisotropic Sobolev space

$$\mathcal{H}^{1,2}_{\gamma}(\Theta) := \left\{u \in \mathcal{H}^{1,2}(\Theta) : \partial_x u|_{\Gamma_2} = u|_{\partial\Theta\setminus\Gamma_2} = 0\right\}$$

with

$$\mathcal{H}^{1,2}(\Theta) := \left\{u \in L^2(\Theta) : \partial_t u,\ \partial_x^j u,\ \partial_y^j u,\ \partial_{xy} u \in L^2(\Theta),\ j = 1, 2\right\}.$$

The space $\mathcal{H}^{1,2}(\Theta)$ is equipped with the natural norm, that is

$$\|u\|_{\mathcal{H}^{1,2}(\Theta)} = \Big(\|u\|^2_{L^2(\Theta)} + \|\partial_t u\|^2_{L^2(\Theta)} + \|\partial_{xy} u\|^2_{L^2(\Theta)}$$

$$+\sum_{j=1}^{2}\left[\left\|\partial_x^j u\right\|_{L^2(\Theta)}^2+\left\|\partial_y^j u\right\|_{L^2(\Theta)}^2\right]^{\frac{1}{2}}.$$

The main result of this paper is given in the following theorem, which complements similar results obtained in the one-space variable case (see, [13]).

Theorem 1.1. *Problem (1.1) admits a unique solution* $u \in \mathcal{H}^{1,2}(\Theta)$.

It is not difficult to prove the uniqueness of the solution. Indeed, let us consider $u \in \mathcal{H}_\gamma^{1,2}(\Theta)$ a solution of the problem (1.1) with a null right-hand side term, that is, u satisfies

$$\begin{cases} \partial_t u - \partial_x^2 u - \partial_y^2 u = 0 \ \text{ in } \Theta, \\ \partial_x u|_{\Gamma_2} = u|_{\partial\Theta\setminus\Gamma_2} = 0. \end{cases} \tag{1.3}$$

Using the Green's formula, we have

$$\int_\Theta \left(\partial_t u - \partial_x^2 u - \partial_y^2 u\right) u dt dx dy = \int_{\partial\Theta} [\frac{1}{2}|u|^2 \nu_t - u(\partial_x u \nu_x + \partial_y u \nu_y)] d\sigma$$

$$+\int_\Theta [(\partial_x u)^2 + (\partial_y u)^2] dt dx dy \tag{1.4}$$

where ν_t, ν_x, ν_y are the components of the unit outward normal vector at the boundary of Θ. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Θ where $t = 0, y = 0$ and $y = c$ we have $u = 0$ and consequently the corresponding boundary integral vanishes. On the parts of the boundary of Θ where $x = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_x = \frac{-1}{\sqrt{1+(\varphi_i')^2(t)}}, \nu_y = 0, \nu_t = \frac{\varphi_i'(t)}{\sqrt{1+(\varphi_i')^2(t)}} \text{ and } \partial_x u(t,\varphi_2(t), y)$$

$$= u(t, \varphi_1(t), y) = 0.$$

Thus, we get

$$\int_{\partial\Theta}\left(\frac{1}{2}|u|^2 \nu_t - u\ \partial_x u\ \nu_x - u\ \partial_y u\ \nu_y\right) d\sigma = \int_0^c \int_0^b \frac{\varphi_2'(t)}{2} u^2(t, \varphi_2(t), y) dt\ dy. \tag{1.5}$$

Then, from (1.4) and (1.5), we obtain

$$\int_{\Theta}\left(\partial_t u-\partial_x^2 u-\partial_y^2 u\right) u\ dt\ dx\ dy = \int_0^c\int_0^b \frac{\varphi_2'(t)}{2}u^2(t,\varphi_2(t),y)dt\ dy + \int_{\Theta}\left[(\partial_x u)^2+(\partial_y u)^2\right]dt\ dx\ dy.$$

Consequently, since u is solution of (1.3), from the last equality we deduce

$$\int_{\Theta}\left[(\partial_x u)^2+(\partial_y u)^2\right]dt\ dx\ dy=0,$$

because

$$\int_0^c\int_0^b \frac{\varphi_2'(t)}{2}u^2(t,\varphi_2(t),y)dt\ dy\geq 0$$

thanks to the fact that φ_2 is an increasing function on $[0,b)$. This implies that $(\partial_x u)^2=(\partial_y u)^2=0$ and consequently $\partial_x^2 u=\partial_y^2 u=0$. Then, the hypothesis $\partial_t u-\partial_x^2 u-\partial_y^2 u=0$ gives $\partial_t u=0$. Thus, u is constant. The boundary conditions imply that $u=0$.

The organization of this paper is as follows. In section 2, we study an auxiliary problem related to Problem (1.1) in a bounded domain. In section 3, we prove a uniform estimate of the type

$$\|u_n\|_{\mathcal{H}^{1,2}(\Theta_n)}\leq C\,\|f\|_{L^2(\Theta)},$$

where C is a constant independent of n and for each $n\in\mathbf{N}$, $u_n\in\mathcal{H}^{1,2}(\Theta_n)$ is the solution in a truncated bounded domain Θ_n approximating Θ. The previous estimate will allow us to pass to the limit and complete the proof of Theorem 1.1.

2. An auxiliary problem

In this section, we replace the unbounded domain Θ by the bounded domain $\Theta_r, r>0$ defined by

$$\Theta_r:=\left\{(t,x,y)\in\mathbf{R}^3:\ a<t<b,\ \psi(t)<x<r,\ 0<y<c\right\},$$

with boundary $\partial\Theta_r:=\bigcup_{k=0}^{5}\Gamma_{k,r}$ (see, Fig. 2) where

$$\Gamma_{0,r}=\left\{(t,r,y)\in\mathbf{R}^3:\ d_1<t<d_2,\ 0<y<c\right\},$$

$$\Gamma_{1,r} = \left\{ (t, \varphi_1(t), y) \in \mathbf{R}^3 : \, d_1 < t < 0, \, 0 < y < c \right\},$$

$$\Gamma_{2,r} = \left\{ (t, \varphi_2(t), y) \in \mathbf{R}^3 : \, 0 < t < d_2, \, 0 < y < c \right\},$$

$$\Gamma_{3,r} = \left\{ (t, x, 0) \in \mathbf{R}^2 : \, a < t < b, \, \psi(t) < x < r \right\},$$

$$\Gamma_{4,r} = \left\{ (t, x, c) \in \mathbf{R}^3 : \, a < t < b, \, \psi(t) < x < r \right\},$$

$$\Gamma_{5,r} = \left\{ (0, 0, y) \in \mathbf{R}^3 : \, 0 < y < c \right\},$$

and $d_1 = \varphi_1^{-1}(r)$, $d_2 = \varphi_2^{-1}(r)$.

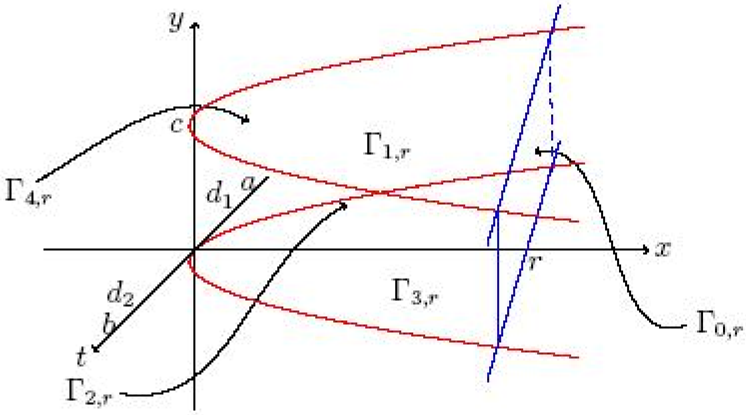

Fig.2 : The bounded domain Θ_r.

Taking into account the previous definitions, and denoting $f_r = f|_{\Theta_r}$, we consider the boundary value problem

$$\text{(2.1)} \qquad \begin{cases} \partial_t u_r - \partial_x^2 u_r - \partial_y^2 u_r = f_r \in L^2(\Theta_r), \\ u_r|_{\partial\Theta_r \setminus \Gamma_{2,r}} = 0, \\ \partial_x u_r|_{\Gamma_{2,r}} = 0. \end{cases}$$

2.1. Existence of solutions to problem (2.1)

Here, we replace Θ_r by

$$\Theta_r^{(n)} = \left\{ (t, x, y) \in \Theta_r : \, \alpha_n < t < \beta_n \right\},$$

where the sequence $(\alpha_n)_{n \in \mathbf{N}}$ (respectively, $(\beta_n)_{n \in \mathbf{N}}$) is strictly negative (respectively, strictly positive) and converges to d_1 (respectively, to d_2) (see, Fig.3). Thus $\varphi_1(\alpha_n) < r$ and $\varphi_2(\beta_n) < r$.

The boundary of the truncated domain $\Theta_r^{(n)}$ is $\partial\Theta_r^{(n)} := \bigcup_{k=0}^{6} \Gamma_{k,r}^{(n)}$ (see Fig.3) where

$$\Gamma_{0,r}^{(n)} = \left\{(t,r,y) \in \mathbf{R}^3 : \alpha_n < t < \beta_n,\, 0 < y < c\right\},$$

$$\Gamma_{1,r}^{(n)} = \left\{(t,\varphi_1(t),y) \in \mathbf{R}^3 : \alpha_n < t < 0,\, 0 < y < c\right\},$$

$$\Gamma_{2,r}^{(n)} = \left\{(t,\varphi_2(t),y) \in \mathbf{R}^3 : 0 < t < \beta_n,\, 0 < y < c\right\},$$

$$\Gamma_{3,r}^{(n)} = \left\{(t,x,0) \in \mathbf{R}^2 : \alpha_n < t < \beta_n,\, \psi(t) < x < r\right\},$$

$$\Gamma_{4,r}^{(n)} = \left\{(t,x,c) \in \mathbf{R}^2 : \alpha_n < t < \beta_n,\, \psi(t) < x < r\right\},$$

$$\Gamma_{5,r}^{(n)} = \left\{(\alpha_n,x,y) \in \mathbf{R}^3 : \varphi_1(\alpha_n) < x < r,\, 0 < y < c\right\},$$

$$\Gamma_{6,r}^{(n)} = \left\{(\beta_n,x,y) \in \mathbf{R}^3 : \varphi_2(\beta_n) < x < r,\, 0 < y < c\right\}.$$

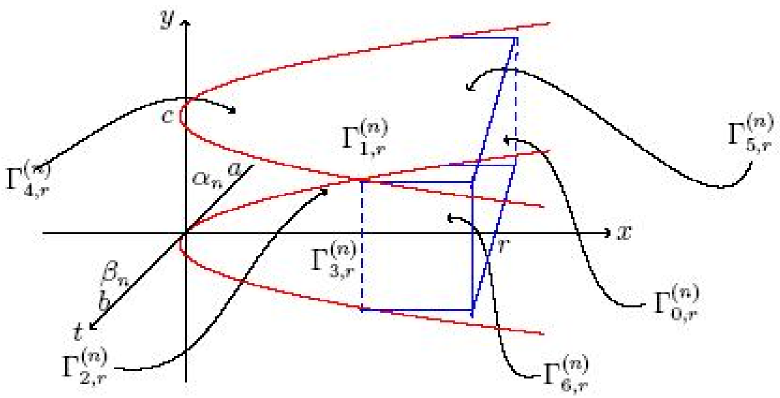

Fig.3 : The truncated domain $\Theta_r^{(n)}$.

The change of variable

$$\Phi : (t,x,y) \quad \mapsto \quad (t,z,y) = \left(t, \frac{x - \psi_r^{(n)}(t)}{r - \psi_r^{(n)}(t)}, y\right)$$

where

$$\psi_r^{(n)}(t) := \begin{cases} \varphi_1(t) \text{ on } [\alpha_n, 0], \\ \\ \varphi_2(t) \text{ on } [0, \beta_n], \end{cases}$$

transforms $\Theta_r^{(n)}$ into the parallelepiped $Q^{(n)} =]\alpha_n, \beta_n[\times]0,1[\times]0,c[$. Also, we define

$$\Gamma_2^{(n)} = \left\{(t,0,y) \in \mathbf{R}^3 : 0 < t < \beta_n,\ 0 < y < c\right\},$$

$$\Gamma_6^{(n)} = \left\{(\beta_n, x, y) \in \mathbf{R}^3 : 0 < z < 1,\ 0 < y < c\right\}.$$

Lemma 2.1. *For every $g^{(n)} \in L^2\left(Q^{(n)}\right)$, there exists a unique $v^{(n)} \in \mathcal{H}^{1,2}\left(Q^{(n)}\right)$ solution of Problem (2.4).*

Proof. *Since the coefficient $b(t)$ is continuous in $\overline{Q^{(n)}}$, the optimal regularity result is given by Ladyzhenskaya-Solonnikov-Ural'tseva [19].* □

Lemma 2.2. *The following operator is compact:*

$$B : \mathcal{H}_\gamma^{1,2}\left(Q^{(n)}\right) \to L^2\left(Q^{(n)}\right),\ \ v^{(n)} \mapsto Bv^{(n)} = a(t,z)\,\partial_z v^{(n)}.$$

Here,

$$\mathcal{H}_\gamma^{1,2}\left(Q^{(n)}\right) = \left\{v^{(n)} \in \mathcal{H}^{1,2}\left(Q^{(n)}\right) : v^{(n)}\Big|_{\partial Q^{(n)}\setminus(\Gamma_2^{(n)}\cup\Gamma_6^{(n)})} = \partial_z v^{(n)}\Big|_{\Gamma_2^{(n)}} = 0\right\}.$$

Proof. $Q^{(n)}$ has the "horn property" of Besov [4], so

$$\partial_z : \mathcal{H}_\gamma^{1,2}\left(Q^{(n)}\right) \to \mathcal{H}^{\frac{1}{2},1}\left(Q^{(n)}\right),\ \ v^{(n)} \mapsto \partial_z v^{(n)}$$

is continuous. Since $Q^{(n)}$ is bounded, the canonical injection is compact from $\mathcal{H}^{\frac{1}{2},1}\left(Q^{(n)}\right)$ into $L^2\left(Q^{(n)}\right)$, see for instance [4]. Here

$$\mathcal{H}^{\frac{1}{2},1}\left(Q^{(n)}\right) = L^2\left(\alpha_n, \beta_n; H^1\,]0,1[\times]0,c[\right) \cap H^{\frac{1}{2}}\left(\alpha_n, \beta_n; L^2\,]0,1[\times]0,c[\right),$$

see [21] for the complete definitions of the $\mathcal{H}^{r,s}$ Hilbertian Sobolev spaces. Then, ∂_z is a compact operator from $\mathcal{H}_\gamma^{1,2}\left(Q^{(n)}\right)$ into $L^2\left(Q^{(n)}\right)$. Since $a(.,.)$ is a bounded function for $t \in]\alpha_n, \beta_n[$, the operator $B = a\partial_z$ is also compact from $\mathcal{H}_\gamma^{1,2}\left(Q^{(n)}\right)$ into $L^2\left(Q^{(n)}\right)$. □

With the previous notations and Lemmas 2.1 and 2.2, we are in position to prove the following the following theorem.

Theorem 2.1. *Let the function* $f_r^{(n)} = f|_{\Theta_r^{(n)}} \in L^2\left(\Theta_r^{(n)}\right)$*. The problem*

$$
(2.2)\qquad \begin{cases} \partial_t u_r^{(n)} - \partial_x^2 u_r^{(n)} - \partial_y^2 u_r^{(n)} = f_r^{(n)} \ \ \text{in } \Theta_r^{(n)}, \\ u_r^{(n)}\Big|_{\partial\Theta_r^{(n)}\setminus(\Gamma_{2,r}^{(n)}\cup\Gamma_{6,r}^{(n)})} = 0, \\ \partial_x u_r^{(n)}\Big|_{\Gamma_{2,r}^{(n)}} = 0, \end{cases}
$$

admits a unique solution $u_r^{(n)} \in \mathcal{H}^{1,2}\left(\Theta_r^{(n)}\right)$.

Proof. The uniqueness of the solution is easy to check. Let us prove its existence. Denoting

$$
u_r^{(n)}(t,x,y) = v^{(n)}(t,z,y) \quad \text{and } f_r^{(n)}(t,x,y) = g^{(n)}(t,z,y),
$$

then Problem (2.2) becomes

$$
(2.3)\ \begin{cases} \partial_t v^{(n)} + a(t,z)\,\partial_z v^{(n)} - b(t)\,\partial_z^2 v^{(n)} - \partial_y^2 v^{(n)} = g^{(n)}, \text{ in } Q^{(n)} \\ v^{(n)}\Big|_{\partial Q^{(n)}\setminus(\Gamma_2^{(n)}\cup\Gamma_6^{(n)})} = 0, \\ \partial_z v^{(n)}\Big|_{\Gamma_2^{(n)}} = 0, \end{cases}
$$

where $b(t) := \left(\frac{1}{r-\psi_r^{(n)}(t)}\right)^2$, $a(t,z) := \frac{(z-1)\psi_r^{(n)'}(t)}{r-\psi_r^{(n)}(t)}$. The aforementioned change of variable conserves the spaces L^2 and $\mathcal{H}^{1,2}$ because $-b(t)$ and $a(t,z)$ are bounded functions when $t \in]\alpha_n, \beta_n[$. In other words

$$
f_r^{(n)} \in L^2(\Theta_r^{(n)}), \quad u_r^{(n)} \in \mathcal{H}^{1,2}(\Theta_r^{(n)}).
$$

Consider the simplified problem

$$
(2.4)\qquad \begin{cases} \partial_t v^{(n)} - b(t)\,\partial_z^2 v^{(n)} - \partial_y^2 v^{(n)} = g^{(n)}, \text{ in } Q^{(n)} \\ v^{(n)}\Big|_{\partial Q^{(n)}\setminus(\Gamma_2^{(n)}\cup\Gamma_6^{(n)})} = 0, \\ \partial_z v^{(n)}\Big|_{\Gamma_2^{(n)}} = 0. \end{cases}
$$

From Lemma 2.1 we deduce that the operator

$$
\partial_t - b(.)\partial_z^2 - \partial_y^2
$$

is an isomorphism from $\mathcal{H}_{\gamma}^{1,2}\left(Q^{(n)}\right)$ into $L^2\left(Q^{(n)}\right)$. On the other hand, by Lemma 2.2 the operator $a\partial_z$ is compact. Consequently, the operator

$$\partial_t + a\,(.,.)\,\partial_z - b(.)\partial_z^2 - \partial_y^2$$

is an isomorphism from $\mathcal{H}_{\gamma}^{1,2}\left(Q^{(n)}\right)$ into $L^2\left(Q^{(n)}\right)$. This ends the proof of the theorem. □

We shall need the following result in order to justify the calculus of the next section.

Lemma 2.3. *The space*

$$\left\{v^{(n)} \in H^4\left(]\alpha_n, 0[\times]0,1[\times]0,c[\right) : v^{(n)}\Big|_{\partial Q^{(n)}\setminus(\Gamma_2^{(n)}\cup\Gamma_6^{(n)})} = \partial_z v^{(n)}\Big|_{\Gamma_2^{(n)}} = 0\right\}$$

is dense in the space

$$\left\{v^{(n)} \in \mathcal{H}^{1,2}\left(]\alpha_n, 0[\times]0,1[\times]0,c[\right) : v^{(n)}\Big|_{\partial Q^{(n)}\setminus(\Gamma_2^{(n)}\cup\Gamma_6^{(n)})} = \partial_z v^{(n)}\Big|_{\Gamma_2^{(n)}} = 0\right\}.$$

Proof. It is a consequence of [21, Vol. 1, Theorem 2.1]. □

Remark 2.1. *We can replace in Lemma 2.3,* $]\alpha_n, 0[\times]0,1[\times]0,c[$ *by* $\Theta_r^{(n)}\Big|_{t<0}$ *with the help of the change of variable defined above.*

2.2. Case of a non-regular bounded domain

Now, we return to the bounded domain Θ_r. For $\alpha_n < 0 < \beta_n$, we denote $f_r^{(n)} = f|_{\Theta_r^{(n)}}$ and $u_r^{(n)} \in \mathcal{H}^{1,2}\left(\Theta_r^{(n)}\right)$ the solution of Problem (2.2) in $\Theta_r^{(n)}$. Such a solution exists by Theorem 2.1.

2.2.1. A uniform estimate

First, let us denote

$$Q_1 = \Theta_r^{(n)}\Big|_{t<0}, \quad Q_2 = \Theta_r^{(n)}\Big|_{t>0} \quad \text{and } f_i = f|_{Q_i}, i = 1, 2.$$

Then, consider the following problems:

$$\text{(2.5)} \qquad \begin{cases} \partial_t u_1 - \partial_x^2 u_1 - \partial_y^2 u_1 = f_1 & \text{in } Q_1, \\ u_1|_{\partial Q_1\setminus\Gamma} = 0, \end{cases}$$

$$(2.6)\qquad \begin{cases} \partial_t v - \partial_x^2 v - \partial_y^2 v = f_2 \text{ in } Q_2, \\ \partial_x v|_{\Gamma_{2,r}^{(n)}} = 0, \\ v|_{\partial Q_2 \setminus (\Gamma_{2,r}^{(n)} \cup \Gamma_{6,r}^{(n)})} = 0, \end{cases}$$

where

$$\Gamma = \left\{ (0,x,y) \in \mathbf{R}^2 : x \in]0,r[\, , y \in]0,c[\right\}.$$

By a similar argument like that used in Subsection 2.1, Problems (2.5) and (2.6) admit (unique) solutions $u_1 \in \mathcal{H}^{1,2}(Q_1)$ and $v \in \mathcal{H}^{1,2}(Q_2)$.

The following Lemmas will be needed in order to establish the uniform estimate of Proposition 2.1.

Lemma 2.4. *The solutions u_1 and v of Problems (2.5) and (2.6) respectively verify the following equalities:*

$$\|f_1\|^2 = \|\partial_t u_1\|^2_{L^2(Q_1)} + \left\|\partial_x^2 u_1\right\|^2_{L^2(Q_1)} + \left\|\partial_y^2 u_1\right\|^2_{L^2(Q_1)} + 2\|\partial_{xy} u_1\|^2_{L^2(Q_1)} + \|\nabla u_1\|^2_{L^2(\Gamma)} + I_n, \tag{2.7}$$

$$\|f_2\|^2 = \|\partial_t v\|^2_{L^2(Q_2)} + \left\|\partial_x^2 v\right\|^2_{L^2(Q_2)} + \left\|\partial_y^2 v\right\|^2_{L^2(Q_2)} + 2\|\partial_{xy} v\|^2_{L^2(Q_2)} + \|\nabla v\|^2_{L^2(\Gamma')} + J_n, \tag{2.8}$$

where

$$I_n = -\int_0^c \int_{\alpha_n}^0 \varphi_1'(t)\left[\partial_x u_1(t, \varphi_1(t), y)\right]^2 dt\, dy, \quad J_n$$

$$= \int_0^c \int_0^{\beta_n} \varphi_2'(t)\left[\partial_y v(t, \varphi_2(t), y)\right]^2 dt\, dy,$$

$$\Gamma = \left\{ (0,x,y) \in \mathbf{R}^3 : x \in]0,r[\, , y \in]0,c[\right\}, \Gamma'$$

$$= \left\{ (\beta_n, x, y) \in \mathbf{R}^3 : x \in]\varphi_2(\beta_n), r[\, , y \in]0,c[\right\}.$$

Proof. Let us denote the inner product in $L^2(Q_1)$ by $\langle .,. \rangle$, then we have

$$\begin{aligned} \|f_1\|^2_{L^2(Q_1)} &= \langle \partial_t u_1 - \partial_x^2 u_1 - \partial_y^2 u_1, \partial_t u_1 - \partial_x^2 u_1 - \partial_y^2 u_1 \rangle \\ &= \|\partial_t u_1\|^2_{L^2(Q_1)} + \|\partial_x^2 u_1\|^2_{L^2(Q_1)} + \left\|\partial_y^2 u_1\right\|^2_{L^2(Q_1)} \\ &\quad -2\langle \partial_t u_1, \partial_x^2 u_1 \rangle - 2\langle \partial_t u_1, \partial_y^2 u_1 \rangle + 2\langle \partial_x^2 u_1, \partial_y^2 u_1 \rangle \end{aligned}$$

Estimation of $-2\langle \partial_t u_1, \partial_x^2 u_1 \rangle$: We have

$$\begin{aligned}\langle \partial_t u_1, \partial_x^2 u_1 \rangle &= \int_{Q_1} \partial_t u_1 \partial_x^2 u_1 dt\, dx\, dy \\ &= -\int_{Q_1} \partial_x \partial_t u_1 . \partial_x u_1 dt\, dx\, dy + \int_{\partial Q_1} \partial_t u_1 . \partial_x u_1 \nu_x d\sigma.\end{aligned}$$

Hence,

$$\begin{aligned}-2 \langle \partial_t u_1, \partial_x^2 u_1 \rangle &= \int_{Q_1} \partial_t \left(\partial_x u_1\right)^2 dt\, dx\, dy - 2\int_{\partial Q_1} \partial_t u_1 . \partial_x u_1 \nu_x d\sigma \\ &= \int_{\partial Q_1} \left[\left(\partial_x u_1\right)^2 \nu_t - 2 \partial_t u_1 . \partial_x u_1 \nu_x\right] d\sigma\end{aligned}$$

where ν_t, ν_x, ν_y are the components of the unit outward normal vector at $\partial Q_1 = \Gamma_{1,r}^{(n)} \cup \Gamma_{5,r}^{(n)} \cup \Gamma_{0^-,r}^{(n)} \cup \Gamma_{4^-,r}^{(n)} \cup \Gamma_{3^-,r}^{(n)}$ where

$$\Gamma_{0^-,r}^{(n)} = \left\{(t, r, y) \in \mathbf{R}^3 : \alpha_n < t < 0,\ 0 < y < c\right\},$$

$$\Gamma_{3^-,r}^{(n)} = \left\{(t, x, 0) \in \mathbf{R}^2 : \alpha_n < t < 0,\ \psi(t) < x < r\right\},$$

$$\Gamma_{4^-,r}^{(n)} = \left\{(t, x, c) \in \mathbf{R}^2 : \alpha_n < t < 0,\ \psi(t) < x < r\right\}.$$

We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_1 where $t = \alpha_n, x = r, y = 0$ and $y = c$, we have $u_1 = 0$ and consequently $\partial_x u_1 = 0$. The corresponding boundary integral vanishes. On the part of the boundary of Q_1 where $t = 0$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_{\partial Q_1} \left(\partial_x u_1\right)^2 \nu_t \, d\sigma = \int_0^c \int_0^r \left(\partial_x u_1\right)^2 dx\, dy,$$

is nonnegative. On the part of the boundary where $x = \varphi_1(t)$, we have

$$\nu_x = \frac{-1}{\sqrt{1 + \left(\varphi_1'\right)^2 (t)}}, \ \nu_t = \frac{\varphi_1'(t)}{\sqrt{1 + \left(\varphi_1'\right)^2 (t)}} \text{ and } u_1\left(t, \varphi_1(t), y\right) = 0. \tag{2.9}$$

Differentiating u_1 with respect to t, we obtain

$$\partial_t u_1\left(t, \varphi_1(t), y\right) + \varphi_1'(t)\, \partial_x u_1\left(t, \varphi_1(t), y\right) = 0. \tag{2.10}$$

Consequently, in view of (2.9) and (2.10) we infer that

$$I_n = -2 \int_{\partial Q_1} \partial_t u_1 . \partial_x u_1 \nu_x \, d\sigma = -2 \int_0^c \int_{\alpha_n}^0 \varphi_1'(t) \left[\partial_x u_1\left(t, \varphi_1(t), y\right)\right]^2 dt\, dy.$$

Therefore,

$$-2\langle \partial_t u_1, \partial_x^2 u_1\rangle = \|\partial_x u_1\|_{L^2(\Gamma)}^2 + I_n. \tag{2.11}$$

Estimation of $-2\langle \partial_t u_1, \partial_y^2 u_1\rangle$: We have

$$-2\left\langle \partial_t u_1, \partial_y^2 u_1\right\rangle = \int_{\partial Q_1} \left[(\partial_y u_1)^2 \nu_t - 2\partial_t u_1.\partial_y u_1 \nu_y\right] d\sigma.$$

Using the Dirichlet boundary conditions, we obtain

$$-2\langle \partial_t u_1, \partial_y^2 u_1\rangle = \int_0^c \int_0^r (\partial_y u_1)^2 \, dx\, dy = \|\partial_y u_1\|_{L^2(\Gamma)}^2 . \tag{2.12}$$

Estimation of $2\langle \partial_x^2 u_1, \partial_y^2 u_1\rangle$: We have

$$\partial_x^2 u_1.\partial_y^2 u_1 = \partial_x \left(\partial_x u_1.\partial_y^2 u_1\right) - \partial_y \left(\partial_x u_1.\partial_x\partial_y u_1\right) + \left(\partial_x\partial_y u_1\right)^2 .$$

Then

$$\begin{aligned} 2\langle \partial_x^2 u_1, \partial_y^2 u_1\rangle &= 2\textstyle\int_{Q_1} \partial_x \left(\partial_x u_1.\partial_y^2 u_1\right) dt\, dx\, dy \\ &\quad -2\textstyle\int_{Q_1} \partial_y \left(\partial_x u_1.\partial_x\partial_y u_1\right) dt\, dx\, dy \\ &\quad +2\textstyle\int_{Q_1} \left(\partial_x\partial_y u_1\right)^2 dt\, dx\, dy \\ &= 2\textstyle\int_{Q_1} \left(\partial_x\partial_y u_1\right)^2 dt\, dx\, dy \\ &\quad +2\textstyle\int_{\partial Q_1} \left[\partial_x u_1 \partial_y^2 u_1 \nu_x - \partial_x u_1.\partial_x\partial_y u_1 \nu_y\right] d\sigma. \end{aligned}$$

Thanks to the boundary conditions, the above boundary integral vanishes. Consequently

$$2\langle \partial_x^2 u_1, \partial_y^2 u_1\rangle = 2\left\|\partial_x\partial_y u_1\right\|_{L^2(Q_1)}^2 . \tag{2.13}$$

Summing up the estimates (2.11), (2.12) and (2.13) of the inner products, formula (2.7) follows. By using a similar argument, we can prove formula (2.8). □

Let us now, consider the following problem

$$\begin{cases} \partial_t w - \partial_x^2 w - \partial_y^2 w = 0 \text{ in } Q_2, \\ w|_{\Gamma} = u_1|_{\Gamma}, \\ w|_{\partial Q_2 \setminus (\Gamma \cup \Gamma' \cup \Gamma_{2,r}^{(n)})} = 0, \\ \partial_x w|_{\Gamma_{2,r}^{(n)}} = 0, \end{cases} \tag{2.14}$$

where u_1 is the solution of Problem (2.5). Thanks to [21, Theorem 4.3, Vol.2], Problem (2.14) admits a unique solution $w \in \mathcal{H}^{1,2}(Q_2)$. Note that we can approach $u_1|_\Gamma$ (which is in $H^1(\Gamma)$) by regular functions (for example, by functions in $H^2(\Gamma)$), then it is easy to prove the following lemma.

Lemma 2.5. *The solution w of Problem (2.14) verifies*

$$\|\nabla u_1\|^2_{L^2(\Gamma)} = \|\partial_t w\|^2_{L^2(Q_2)} + \left\|\partial_x^2 w\right\|^2_{L^2(Q_2)} + \left\|\partial_y^2 w\right\|^2_{L^2(Q_2)} + 2\,\|\partial_{xy} w\|^2_{L^2(Q_2)}$$
$$+ \|\nabla w\|^2_{L^2(\Gamma')} + K_n, \tag{2.15}$$

where
$$K_n = \int_0^c \int_0^{\beta_n} \varphi_2'(t)\,[\partial_y w\,(t, \varphi_2(t), y)]^2\,dt\,dy.$$

Now, considering u_1, v and w solutions of the problem (2.5), (2.6) and (2.14) respectively, we set

$$u_r^{(n)} := \begin{cases} u_1 \text{ in } Q_1, \\ u_2 \text{ in } Q_2, \end{cases}$$

where $u_2 = v + w$. Note that $u_r^{(n)} \in \mathcal{H}^{1,2}\left(\Theta_r^{(n)}\right)$ is then the solution of Problem (2.2) obtained in Theorem 2.1, and satisfies the following result.

Proposition 2.1. *There exists a constant $C > 0$ independent of n such that*

$$\left\|\partial_t u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_x^2 u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_y^2 u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_{xy} u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)}$$
$$\le C\,\|f_r\|^2_{L^2(\Theta_r)}.$$

Proof. Summing up the estimates (2.7), (2.8) and (2.15), we then obtain

$$\begin{aligned}
\left\|f_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} &= \|f_1\|^2_{L^2(Q_1)} + \|f_2\|^2_{L^2(Q_2)} \\
&\ge \|\partial_t u_1\|^2_{L^2(Q_1)} + \|\partial_t v\|^2_{L^2(Q_2)} + \|\partial_t w\|^2_{L^2(Q_2)} \\
&\quad + \left\|\partial_x^2 u_1\right\|^2_{L^2(Q_1)} + \left\|\partial_x^2 v\right\|^2_{L^2(Q_2)} + \left\|\partial_x^2 w\right\|^2_{L^2(Q_2)} \\
&\quad + \left\|\partial_y^2 u_1\right\|^2_{L^2(Q_1)} + \left\|\partial_y^2 v\right\|^2_{L^2(Q_2)} + \left\|\partial_y^2 w\right\|^2_{L^2(Q_2)} \\
&\quad + \|\partial_{xy} u_1\|^2_{L^2(Q_1)} + \|\partial_{xy} v\|^2_{L^2(Q_2)} + \|\partial_{xy} w\|^2_{L^2(Q_2)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left\|f_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} \\
&\geq \|\partial_t u_1\|^2_{L^2(Q_1)} + \tfrac{1}{2}\|\partial_t u_2\|^2_{L^2(Q_2)} + \left\|\partial_x^2 u_1\right\|^2_{L^2(Q_1)} + \tfrac{1}{2}\left\|\partial_x^2 u_2\right\|^2_{L^2(Q_2)} \\
&+ \left\|\partial_y^2 u_1\right\|^2_{L^2(Q_1)} + \tfrac{1}{2}\left\|\partial_y^2 u_2\right\|^2_{L^2(Q_2)} + \|\partial_{xy} u_1\|^2_{L^2(Q_1)} + \tfrac{1}{2}\|\partial_{xy} u_2\|^2_{L^2(Q_2)} \\
&\geq \tfrac{1}{2}\left(\left\|\partial_t u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_x^2 u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_y^2 u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_{xy} u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)}\right).
\end{aligned}$$

But

$$\left\|f_r^{(n)}\right\|_{L^2\left(\Theta_r^{(n)}\right)} \leq \|f_r\|_{L^2(\Theta_r)},$$

then,

$$\begin{aligned}
&\left\|\partial_t u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_x^2 u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_y^2 u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_{xy} u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} \\
&\leq 2\left\|f_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} \\
&\leq 2\|f_r\|^2_{L^2(\Theta_r)}.
\end{aligned}$$

This ends the proof of the proposition. □

Theorem 2.2. *There exists a constant $K > 0$ independent of n such that*

$$\left\|u_r^{(n)}\right\|^2_{\mathcal{H}^{1,2}\left(\Theta_r^{(n)}\right)} \leq K\left\|f_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} \leq K\|f\|^2_{L^2(\Theta_r)}.$$

Proof. The majoration of $\left\|\partial_t u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_x^2 u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)}$ $+\left\|\partial_y^2 u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_{xy} u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)}$ is given by Proposition 2.1. Thus, it remains to show the following claim.

Claim 2.1. *There exists $C > 0$ independent of n such that*

$$\left\|\partial_x u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|\partial_y u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} + \left\|u_r^{(n)}\right\|^2_{L^2\left(\Theta_r^{(n)}\right)} \leq C.$$

Indeed, through direct calculations we have

$$\int_{\Theta_r^{(n)}} f_r^{(n)} u_r^{(n)} e^{-t}\, dtdxdy = \int_{\Theta_r^{(n)}} \partial_t \left(\frac{[u_r^{(n)}]^2}{2} e^{-t} \right) - \partial_x (u_r^{(n)} \partial_x u_r^{(n)}) e^{-t}$$
$$-\partial_y (u_r^{(n)} \partial_y u_r^{(n)}) e^{-t}\, dtdxdy + \int_{\Theta_r^{(n)}} |\nabla u_r^{(n)}|^2 e^{-t}\, dtdxdy$$
$$+\tfrac{1}{2} \int_{\Theta_r^{(n)}} |u_r^{(n)}|^2 e^{-t}\, dtdxdy,$$

where $|\nabla u_r^{(n)}|^2 = (\partial_x u_r^{(n)})^2 + (\partial_y u_r^{(n)})^2$. Since $u_r^{(n)}\Big|_{\partial\Theta_r^{(n)} \setminus (\Gamma_{2,r}^{(n)} \cup \Gamma_{6,r}^{(n)})} =$

$\partial_x u_r^{(n)}\Big|_{\Gamma_{2,r}^{(n)}} = 0$, it follows that

$$\int_{\Theta_r^{(n)}} \partial_t \left(\frac{[u_r^{(n)}]^2}{2} e^{-t} \right) - \partial_x (u_r^{(n)} \partial_x u_r^{(n)}) e^{-t} - \partial_y (u_r^{(n)} \partial_y u_r^{(n)}) e^{-t}\, dtdxdy$$
$$= \int_{\partial\Theta_r^{(n)}} \frac{[u_r^{(n)}]^2}{2} e^{-t} \nu_t - u_r^{(n)} \partial_x u_r^{(n)} \nu_x e^{-t} - u_r^{(n)} \partial_y u_r^{(n)} \nu_y e^{-t}\, d\sigma$$
$$= \int_{\Gamma_{2,r}^{(n)} \cup \Gamma_{6,r}^{(n)}} \frac{[u_r^{(n)}]^2}{2} \nu_t e^{-t}\, d\sigma - \int_{\Gamma_{6,r}^{(n)}} u_r^{(n)} \partial_x u_r^{(n)} \nu_x e^{-t}\, d\sigma$$
$$- \int_{\Gamma_{2,r}^{(n)} \cup \Gamma_{6,r}^{(n)}} u_r^{(n)} \partial_y u_r^{(n)} \nu_y e^{-t}\, d\sigma$$
$$= \int_{\varphi_2(\beta_n)}^{r} \int_0^c \frac{[u_r^{(n)}(\beta_n, x, y)]^2}{2}\, dxdy + \int_0^{\beta_n} \int_0^c \varphi_2'(t) \frac{[u_r^{(n)}(t, \varphi_2(t), y)]^2}{2}\, dtdy.$$

From this we obtain the boundary integral is nonnegative and therefore by (2.2.1), we get

$$\int_{\Theta_r^{(n)}} f_r^{(n)} u_r^{(n)} e^{-t}\, dtdxdy \geq \int_{\Theta_r^{(n)}} |\nabla u_r^{(n)}|^2 e^{-t}\, dtdxdy + \frac{1}{2} \int_{\Theta_r^{(n)}} |u_r^{(n)}|^2 e^{-t}\, dtdxdy$$
$$\geq e^{-d_2} \left(\int_{\Theta_r^{(n)}} |\nabla u_r^{(n)}|^2\, dtdxdy + \frac{1}{2} \int_{\Theta_r^{(n)}} |u_r^{(n)}|^2\, dtdxdy \right).$$

On the other hand, using the Young's inequality yields

$$\int_{\Theta_r^{(n)}} f_r^{(n)} u_r^{(n)} e^{-t}\, dtdxdy \leq \frac{1}{\epsilon} e^{-2d_1} \|f_r\|^2_{L^2(\Theta_r)} + \epsilon \left(\int_{\Theta_r^{(n)}} |\nabla u_r^{(n)}|^2\, dtdxdy \right.$$
$$\left. + \frac{1}{2} \int_{\Theta_r^{(n)}} |u_r^{(n)}|^2\, dtdxdy, \quad \forall \epsilon > 0. \right.$$

This combined with (**??**) gives

$$\left(e^{-d_2} - \epsilon \right) \left(\int_{\Theta_r^{(n)}} |\nabla u_r^{(n)}|^2\, dtdxdy + \frac{1}{2} \int_{\Theta_r^{(n)}} |u_r^{(n)}|^2\, dtdxdy \right)$$
$$\leq \frac{1}{\epsilon} e^{-2d_1} \|f_r\|^2_{L^2(\Theta_r)}, \quad \forall \epsilon > 0.$$

Now, taking $\epsilon = \frac{e^{-d_2}}{2}$, we obtain the desired result and hence this shows Claim 2.1. □

2.2.2. Passing to the limit

We are now in position to prove the first main result of this work.

Theorem 2.3. *Problem (2.1) admits a (unique) solution u_r belonging to*

$$\mathcal{H}_{\gamma}^{1,2}(\Theta_r) = \left\{ u_r \in \mathcal{H}^{1,2}(\Theta_r) : u_r|_{\partial\Theta_r \setminus \Gamma_{2,r}} = \partial_x u_r|_{\Gamma_{2,r}} = 0 \right\}.$$

Proof. Choose a sequence $\left(\Theta_r^{(n)}\right)_{n\in\mathbf{N}}$ of the domains defined above. In what follows, we show that

Claim 2.2. $\Theta_r^{(n)} \to \Theta_r$, *as* $n \to +\infty$.

Indeed, arguing by contradiction by supposing that

$$\limsup_{n\to\infty} \Theta_r^{(n)} \neq \liminf_{n\to\infty} \Theta_r^{(n)}.$$

To prove this end, we recall the following definitions:

$$\limsup_{n\to\infty} \Theta_r^{(n)} = \bigcap_{n\geq 1} \bigcup_{n\geq j} \Theta_r^{(j)} \quad \text{and} \quad \liminf_{n\to\infty} \Theta_r^{(n)} = \bigcup_{n\geq 1} \bigcap_{j\geq n} \Theta_r^{(j)}.$$

From this, it turns out that it is enough to prove

$$\bigcap_{n\geq 1} \bigcup_{n\geq j} \Theta_r^{(j)} \neq \bigcup_{n\geq 1} \bigcap_{j\geq n} \Theta_r^{(j)}.$$

Thus, there exists $z \in \bigcap_{n\geq 1} \bigcup_{n\geq j} \Theta_r^{(j)}$ such that $z \notin \bigcup_{n\geq 1} \bigcap_{j\geq n} \Theta_r^{(j)}$. Since $z \in \bigcap_{n\geq 1} \bigcup_{n\geq j} \Theta_r^{(j)}$, we have

$$\forall n \geq 1,\ \exists n_0 \in \mathbf{N}^* : z \in \Theta_r^{(n_0)}.$$

Notice that, since $\beta_n < d_2$ and $d_1 < \alpha$ for any $n \geq 1$, we derive that $z \in \Theta_r$. On the other hand, since $\beta_j \to d_2$ and $\alpha_j \to d_1$ as $j \to \infty$, we may assume that there exists $n_0 \in \mathbf{N}^*$ such that for any $j \geq n_0$, $z \in \Theta_r^{(j)}$, that is $z \in \bigcup_{n\geq 1} \bigcap_{j\geq n} \Theta_r^{(j)}$. But this contradicts our assumptions and hence this shows Claim 2.2. As a byproduct, we obtain for any $(t, x, y) \in \Theta_r$, $\lim_{n\to\infty} \chi_{\Theta_r^{(n)}}(t, x, y) = 1$. Here χ_A is the characteristic function of a set A.

Consider the solution $u_r^{(n)} \in \mathcal{H}^{1,2}\left(\Theta_r^{(n)}\right)$ of the mixed boundary value problem

$$\begin{cases} \partial_t u_r^{(n)} - \partial_x^2 u_r^{(n)} - \partial_y^2 u_r^{(n)} = f_r^{(n)} \in L^2\left(\Theta_r^{(n)}\right), \\ u_r^{(n)}\Big|_{\partial\Theta_r^{(n)}\setminus(\Gamma_{2,r}^{(n)}\cup\Gamma_{6,r}^{(n)})} = 0, \\ \partial_x u_r^{(n)}\Big|_{\Gamma_{2,r}^{(n)}} = 0. \end{cases}$$

Such a solution $u_r^{(n)}$ exists by Theorem 2.1. Hereafter, we set

$$\pi_1^{(n)} := \{(t,x,y) \in \Theta_r : \varphi_1^{-1}(r) < t < \alpha_n\},$$

$$\pi_2^{(n)} := \{(t,x,y) \in \Theta_r : \beta_n < t < \varphi_2^{-1}(r)\},$$

$$\sigma := \{(t,x,y) \in \Theta_r : t = \beta_n\}.$$

Notice that without loss of generality we may assume $\frac{d_2}{2} < \beta_n < d_2$ for any $n \in \mathbf{N}^*$ $\Big($To guess how this is possible one can think of this sequence $\beta_n = \frac{nd_2}{2(n+1)} + \frac{d_2}{2}\Big)$. Let $\widetilde{u_r^{(n)}}$ be the extension of $u_r^{(n)}$ given by

$$\widetilde{u_r^{(n)}} := \begin{cases} u_r^{(n)} & \text{if} \quad (t,x,y) \in \Theta_r^{(n)}, \\ 0 & \text{if} \quad (t,x,y) \in \pi_1^{(n)}, \\ u_r^{(n)}(2\beta_n - t, x, y) & \text{if} \quad (t,x,y) \in \pi_2^{(n)}. \end{cases}$$

It is easy to observe that $\widetilde{u_r^{(n)}}$ is then in $\mathcal{H}^{1,2}(\Theta_r)$ and satisfies

$$\left\| \widetilde{u_r^{(n)}} \right\|_{\mathcal{H}^{1,2}(\Theta_r)} \leq C \left\| f_r \right\|_{L^2(\Theta_r)}^2.$$

Thus, for a suitable increasing sequence of integers n_k, $k = 1, 2, \ldots$, there exist functions

$$u_r, v_r \text{ and } v_{r,i,j}, \ 1 \leq i+j \leq 2$$

in $L^2(\Theta_r)$ such that

$$\begin{array}{lcll} \widetilde{u_r^{n_k}} & \rightharpoonup & u_r \quad \text{weakly in } L^2(\Theta_r), & k \to \infty, \\ \partial_t \widetilde{u_r^{n_k}} & \rightharpoonup & v_r \quad \text{weakly in } L^2(\Theta_r), & k \to \infty, \\ \partial_x^i \partial_y^j \widetilde{u_r^{n_k}} & \rightharpoonup & v_{r,i,j} \quad \text{weakly in } L^2(\Theta_r), \quad 1 \leq i+j \leq 2, & k \to \infty. \end{array}$$

From where it follows,

$$v_r = \partial_t u_r,\ v_{r,i,j} = \partial_x^i \partial_y^j u_r,\ 1 \le i + j \le 2$$

in the sense of distributions in Θ_r and so in $L^2(\Theta_r)$. Furthermore, we have

$$\widetilde{u_r^{n_k}} \to u_r \text{ in } C([0,T]; C^1(\Omega_{r,t})), \text{ as } k \to \infty, \tag{2.16}$$

where

$$\Omega_{r,t} = \{(x,y) : \psi_r(t) < x < r,\ 0 < y < c\}.$$

From these analysis we infer that

$$\partial_t u_r - \partial_x^2 u_r - \partial_y^2 u_r = f_r \text{ in } \Theta_r.$$

In addition, in view of

$$\left.\widetilde{u_r^{(n)}}\right|_{\partial\Theta_r \setminus \Gamma_{2,r}} = 0,\ \left.\partial_x \widetilde{u_r^{(n)}}\right|_{\Gamma_{2,r}} = 0,$$

and (2.16), we derive that the solution u_r satisfies the boundary conditions. Hence the proof now is complete. □

Hereafter, we need to prepare some lemmas to prove Theorem 1.1.

Lemma 2.6. *There exists a constant $K_1 > 0$ independent of m such that*

$$\|u_m\|^2_{L^2(\Theta_m)} \le K_1 \|f_m\|^2_{L^2(\Theta_m)}.$$

Proof. For a real number $\lambda \neq 0$, we have

$$\begin{aligned} & \int_{\Theta_m} f_m u_m e^{-2\lambda^2 t}\, dtdxdy \\ = & \int_{\Theta_m} (\partial_t u_m u_m - \partial_x^2 u_m u_m - \partial_y^2 u_m u_m) e^{-2\lambda^2 t} dtdxdy, \\ = & \int_{\Theta_m} [\partial_t \left(\tfrac{u_m^2}{2} e^{-2\lambda^2 t}\right) + \lambda^2 u_m^2 e^{-2\lambda^2 t}]\, dtdxdy \\ + & \int_{\Theta_m} [|\nabla u_m|^2 + \partial_x(-\partial_x u_m u_m) + \partial_y(-\partial_y u_m u_m)] e^{-2\lambda^2 t} dtdxdy \\ = & \lambda^2 \int_{\Theta_m} u_m^2 e^{-2\lambda^2 t}\, dtdxdy + \int_{\Theta_m} |\nabla u_m|^2 e^{-2\lambda^2 t}\, dtdxdy \\ + & \int_0^c \int_0^{\varphi_2^{-1}(m)} \tfrac{\varphi_2'(t)}{2} u_m^2(t, \varphi_2(t), y) e^{-2\lambda^2 t}\, dtdy \\ \ge & \lambda^2 e^{-2\lambda^2 b} \|u_m\|^2_{L^2(\Theta_m)}. \end{aligned}$$

On the other hand, for all $\epsilon > 0$, we have

$$\int_{\Theta_m} f_m u_m e^{-2\lambda^2 t}\, dtdxdy \le \frac{e^{-4\lambda^2 a}}{\epsilon} \|f_m\|^2_{L^2(\Theta_m)} + \epsilon \|u_m\|^2_{L^2(\Theta_m)}.$$

Therefore,

$$\frac{\|f_m\|^2_{L^2(\Theta_m)} e^{-4\lambda^2 a}}{\epsilon} \geq \left(\lambda^2 e^{-2\lambda^2(b)} - \epsilon\right) \|u_m\|^2_{L^2(\Theta_m)}.$$

Hence, by choosing $\epsilon = \frac{\lambda^2 e^{-2\lambda^2 b}}{2}$, we obtain

$$\|u_m\|^2_{L^2(\Theta_m)} (\Theta_m)^2 \leq K_1 \|f_m\|^2_{L^2(\Theta_m)}.$$

□

Lemma 2.7. *There exists a constant $K_2 > 0$ independent of m such that*

$$\|\partial_x u_m\|^2_{L^2(\Theta_m)} + \|\partial_y u_m\|^2_{L^2(\Omega_m)} \leq K_2 \|f_m\|^2_{L^2(\Theta_m)}.$$

Proof. We have

$$\int_{\Theta_m} [\partial_x(u_m \partial_x u_m) + \partial_y(u_m \partial_y u_m)] dt dx dy = \int_{\partial\Theta_m} [u_m \partial_x u_m \nu_x + u_m \partial_y u_m \nu_y] d\sigma$$

where ν_t, ν_x, ν_y are the components of the unit outward normal vector at $\partial\Theta_m$. On the parts of the boundary of Θ_m where $x = m, y = 0, y = c, (t, x) = (0, 0)$ and $x = \varphi_1(t)$, we have $u_m = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $x = \varphi_2(t)$, we have $\partial_x u_m = 0$ and $\nu_y = 0$. Consequently, the corresponding boundary integral vanishes. Finally,

$$\int_{\Theta_m} [\partial_x(u_m \partial_x u_m) + \partial_y(u_m \partial_y u_m)] dt dx dy = 0.$$

On the other hand, we have

$$\begin{aligned} & \int_{\Theta_m} [\partial_x(u_m \partial_x u_m) + \partial_y(u_m \partial_y u_m)] dt dx dy \\ = & \int_{\Theta_m} u_m \partial_x^2 u_m dt dx dy + \int_{\Theta_m} (\partial_x u_m)^2 dt dx dy \\ & + \int_{\Theta_m} u_m \partial_y^2 u_m dt dx dy + \int_{\Theta_m} (\partial_y u_m)^2 dt dx dy. \end{aligned}$$

Then,

$$0 = \int_{\Theta_m} u_m \partial_x^2 u_m dt dx dy + \|\partial_x u_m\|^2_{L^2(\Theta_m)} + \int_{\Theta_m} u_m \partial_y^2 u_m dt dx dy + \|\partial_y u_m\|^2_{L^2(\Theta_m)}.$$

Consequently,

$$\begin{aligned}
&\|\partial_x u_m\|^2_{L^2(\Theta_m)} + \|\partial_y u_m\|^2_{L^2(\Theta_m)} \\
&= -\textstyle\int_{\Theta_m} u_m \partial_x^2 u_m dtdxdy - \int_{\Theta_m} u_m \partial_y^2 u_m dtdxdy \\
&\leq \textstyle\int_{\Theta_m} u_m^2 dtdxdy + \int_{\Theta_m} (\partial_x^2 u_m)^2 dtdxdy \\
&+ \textstyle\int_{\Theta_m} u_m^2 dtdxdy + \int_{\Theta_m} (\partial_y^2 u_m)^2 dtdxdy \\
&= 2\,\|u_m\|^2_{L^2(\Theta_m)} + \left\|\partial_x^2 u_m\right\|^2_{L^2(\Theta_m)} + \left\|\partial_y^2 u_m\right\|^2_{L^2(\Theta_m)}.
\end{aligned}$$

Lemma 2.6 and Proposition 2.1 which remains valid in Ω_m give

$$\|\partial_x u_m\|^2_{L^2(\Theta_m)} + \|\partial_y u_m\|^2_{L^2(\Theta_m)} \leq 2K_1 \|f_m\|^2_{L^2(\Theta_m)} + 2\|f_m\|^2_{L^2(\Theta_m)} \leq K_2 \|f_m\|^2_{L^2(\Theta_m)}.$$

□

3. Proof of Theorem 1.1

For a large enough positive integer m, we define Θ_m by

$$\Theta_m = \{(t, x, y) \in \Theta : 0 < x < m\}.$$

Let $u_m \in \mathcal{H}^{1,2}_\gamma(\Theta_m)$ the solution of the following problem in Θ_m :

$$\text{(3.1)} \qquad \begin{cases} \partial_t u_m - \partial_x^2 u_m - \partial_y^2 u_m = f_m \in L^2(\Theta_m), \\ u_m|_{\partial\Theta_m \setminus \Gamma_{2,m}} = 0, \\ \partial_x u_m|_{\Gamma_{2,m}} = 0, \end{cases}$$

where

$$f_m = f|_{\Theta_m}, \; \Gamma_{2,m} = \left\{(t, \varphi_2(t), y) \in \mathbf{R}^2 : 0 < t < \varphi_2^{-1}(m), 0 < y < c\right\}.$$

Such a solution u_m exists by Theorem 2.3. Now we are in position to conclude the proof of our main theorem 1.1. The rest of the proof is quite similar to that of Theorem 2.3, so here we just sketch it. Let $\widetilde{u_m}$ be the extension of u_m given by

$$\widetilde{u_m} := \begin{cases} u_m & \text{if} \quad (t, x, y) \in \Theta_m, \\ 0 & \text{if} \quad (t, x, y) \in \Theta \setminus \Theta_m. \end{cases}$$

It is easy to observe from Lemmas 2.6-2.7 that $\widetilde{u_m}$ belongs to $\mathcal{H}^{1,2}(\Theta)$ and satisfies

$$\|\widetilde{u_m}\|_{\mathcal{H}^{1,2}(\Theta)} \leq C\,\|f\|^2_{L^2(\Theta)}.$$

Since the rest of the proof is quite similar to that in the previous section, so we omit the details and this completes the proof.

Remark 3.1. *Let us consider the following problem:*

$$
\left\{
\begin{array}{l}
\partial_t v - \partial_x^2 v - \partial_y^2 v = f \in L^2(D), \\
\partial_x v|_{\Gamma_1} = 0, \\
v|_{\partial D \setminus \Gamma_1} = 0,
\end{array}
\right. \tag{3.2}
$$

where

$$
D := \left\{ (t,x) \in \mathbf{R}^2 : \ l_1 < t < l_2; -\infty < x < \Phi(t) \right\} \times]0, l_3[,
$$

where l_1, l_2 *and* l_3 *are real numbers such that* $-\infty < l_1 < 0 < l_2 < +\infty$, *and* $l_3 > 0$ *while* Φ *is a Lipschitz continuous real-valued function on* (l_1, l_2), *and such that*

$$
\Phi(t) := \left\{
\begin{array}{lll}
\psi_1(t) & \text{on} & (l_1, 0], \\
\psi_2(t) & \text{on} & [0, l_2).
\end{array}
\right.
$$

The function ψ_1 *(respectively,* ψ_2 *) is a negative and increasing (respectively, decreasing) on* $(l_1, 0]$ *(respectively, on* $[0, l_2)$*) and verifies the hypothesis* $\psi_1(0) = \psi_2(0) = 0$. *Here,* Γ_1 *is the part of the boundary of* D *where* $x = \psi_1(t)$.

By using the same arguments like those used in solving Problem (1.1), we can show that Problem (3.2) admits a (unique) solution v *belonging to* $\mathcal{H}^{1,2}(D)$.

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