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Two-parameter generalization of bihyperbolic Jacobsthal numbers

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Abstract

In this paper, we define a two-parameter generalization of bihyperbolic Jacobsthal numbers. We give Binet formula, the generating functions and some identities for these numbers.

Keywords: *Jacobsthal numbers, bihyperbolic numbers, bihyperbolic Jacobsthal numbers, recurrence relations, generating functions.*

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1. Introduction

Let $\mathbf{h}$ be the unipotent element such that $\mathbf{h} \neq \pm 1$ and $\mathbf{h}^2 = 1$. A hyperbolic number z is defined as $z = x + y\mathbf{h}$, where $x, y \in \mathbf{R}$. We will denote by $\mathbf{H}$ the set of hyperbolic numbers.

The addition and the subtraction of hyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients. The hyperbolic numbers multiplication can be made analogously as multiplication of algebraic expressions using the rule $\mathbf{h}^2 = 1$. The real numbers x and y are called the real and unipotent parts of the hyperbolic number z, respectively. For others details concerning hyperbolic numbers see for example [10, 11, 12].

Extension of complex numbers to higher dimension has an interest not only in mathematics also in modern physics and engineering. Quaternions are one of the well-known sets, however they form a non-commutative algebra.

In [9], Olariu introduced commutative hypercomplex numbers in different dimensions. One of 4-dimensional commutative hypercomplex number is called *hyperbolic fourcomplex number*. In [10], the authors used the name *bihyperbolic numbers*.

Note that bihyperbolic numbers are a special case of generalized Segre's quaternions, being a 4-dimensional commutative number system, and they are also named as *canonical hyperbolic quaternions* (see [5]). In this paper, we use the name bihyperbolic numbers. Analogously as bicomplex numbers are an extension of complex numbers, bihyperbolic numbers are a natural extension of hyperbolic numbers to 4-dimension.

Let $\mathbf{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + j_1 x_1 + j_2 x_2 + j_3 x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbf{R}$ and $j_1,\ j_2,\ j_3 \notin \mathbf{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1.$$

From the above rules the multiplication of bihyperbolic numbers can be made analogously as multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients.

The addition and multiplication on $\mathbf{H}_2$ are commutative and associative. Moreover, $(\mathbf{H}_2, +, \cdot)$ is a commutative ring.

For the algebraic properties of bihyperbolic numbers, see [1].

2. The (s,p)-Jacobsthal numbers

Let $n \geq 0$ be an integer. The Jacobsthal sequence $\{J_n\}$ is defined by the second order linear recurrence

$$J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2$$

with initial terms $J_0 = 0$, $J_1 = 1$. So the Jacobsthal sequence has the form 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, ... and its terms are named as Jacobsthal numbers. The direct formula for the nth Jacobsthal number has the form $J_n = \frac{2^n-(-1)^n}{3}$, named as the Binet formula for the Jacobsthal numbers.

There are many generalizations of this sequence – for example see [6, 7, 8, 17]. In [2] a two-parameter generalization of the Jacobsthal sequence was investigated. We recall it.

Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. The sequence $\{J_n(s,p)\}$ was defined by the following recurrence

$$J_n(s,p) = 2^{s+p}J_{n-1}(s,p) + (2^{2s+p} + 2^{s+2p})J_{n-2}(s,p) \text{ for } n \geq 2 \tag{2.1}$$

with initial conditions $J_0(s,p) = 1$, $J_1(s,p) = 2^s + 2^p + 2^{s+p}$.

It is easily seen that for $s = p = 0$ we have $J_n(0,0) = J_{n+2}$.

The sequence $\{J_n(s,p)\}$ is named as (s,p)-Jacobsthal sequence and its terms as (s,p)-Jacobsthal numbers.

Theorem 1. [2] *(Binet formula for (s,p)-Jacobsthal numbers)*

Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then the nth (s,p)-Jacobsthal number is given by

$$J_n(s,p) = c_1r_1^n + c_2r_2^n,$$

where

$$\begin{aligned} r_1 &= 2^{s+p-1} + \tfrac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}, \\ r_2 &= 2^{s+p-1} - \tfrac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}, \\ c_1 &= \frac{2^s + 2^p + 2^{s+p} - 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}, \\ c_2 &= \frac{-2^s - 2^p - 2^{s+p} + 2^{s+p-1} + \frac{1}{2}\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}{\sqrt{4^{s+p} + 2^{s+p+2}(2^s + 2^p)}}. \end{aligned} \tag{2.2}$$

Theorem 2. [2] *Let $n \geq 1$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\sum_{l=0}^{n-1} J_l(s,p) = \frac{J_n(s,p) + (2^{2s+p} + 2^{s+2p})J_{n-1}(s,p) - 1 - 2^s - 2^p}{2^{s+p}(1 + 2^s + 2^p) - 1}.$$

Jacobsthal numbers are well-known in the theory of recurrence equations and they have applications in distinct areas of mathematics. Recently they are investigated also in the context of hypercomplex numbers, see for example Jacobsthal quaternions, Jacobsthal hybrid numbers and their generalizations. Details can be found in [3, 4, 13, 14, 15, 16].

In this paper, we introduce and study bihyperbolic (s,p)-Jacobsthal numbers which are a generalization of bihyperbolic Jacobsthal numbers.

3. Bihyperbolic (s,p)-Jacobsthal numbers

Let $n \geq 0$ be an integer. We define the nth bihyperbolic (s,p)-Jacobsthal number $BhJ_n^{s,p}$ by the following relation

$$BhJ_n^{s,p} = J_n(s,p) + j_1 J_{n+1}(s,p) + j_2 J_{n+2}(s,p) + j_3 J_{n+3}(s,p),$$

where $J_n(s,p)$ is given by (2.1).

Note that for $s = p = 0$ we obtain $BhJ_n^{0,0} = BhJ_{n+2}$, where BhJ_n denotes nth bihyperbolic Jacobsthal number.

By some elementary calculations we find the following recurrence relation for the bihyperbolic (s,p)-Jacobsthal numbers.

Theorem 1. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$2^{s+p} JH_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p}) JH_n^{s,p} = JH_{n+2}^{s,p}.$$

Proof.

$$\begin{aligned}
&2^{s+p} JH_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p}) JH_n^{s,p} \\
&= 2^{s+p}(J_{n+1}(s,p) + j_1 J_{n+2}(s,p) + j_2 J_{n+3}(s,p) + j_3 J_{n+4}(s,p)) \\
&\ +(2^{2s+p} + 2^{s+2p})(J_n(s,p) + j_1 J_{n+1}(s,p) + j_2 J_{n+2}(s,p) + j_3 J_{n+3}(s,p)) \\
&= J_{n+2}(s,p) + j_1 J_{n+3}(s,p) + j_2 J_{n+4}(s,p) + j_3 J_{n+5}(s,p) \\
&= JH_{n+2}^{s,p}.
\end{aligned}$$

□

Theorem 2. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\begin{aligned}
&JH_n^{s,p} - j_1 JH_{n+1}^{s,p} - j_2 JH_{n+2}^{s,p} + j_3 JH_{n+3}^{s,p} \\
&= J_n(s,p) - J_{n+2}(s,p) - J_{n+4}(s,p) + J_{n+6}(s,p).
\end{aligned}$$

Proof.

$$\begin{aligned}
&JH_n^{s,p} - j_1 JH_{n+1}^{s,p} - j_2 JH_{n+2}^{s,p} + j_3 JH_{n+3}^{s,p}\\
&= J_n(s,p) + j_1 J_{n+1}(s,p) + j_2 J_{n+2}(s,p) + j_3 J_{n+3}(s,p)\\
&\ -j_1(J_{n+1}(s,p) + j_1 J_{n+2}(s,p) + j_2 J_{n+3}(s,p) + j_3 J_{n+4}(s,p))\\
&\ -j_2(J_{n+2}(s,p) + j_1 J_{n+3}(s,p) + j_2 J_{n+4}(s,p) + j_3 J_{n+5}(s,p))\\
&\ +j_3(J_{n+3}(s,p) + j_1 J_{n+4}(s,p) + j_2 J_{n+5}(s,p) + j_3 J_{n+6}(s,p))\\
&= J_n(s,p) + j_1 J_{n+1}(s,p) + j_2 J_{n+2}(s,p) + j_3 J_{n+3}(s,p)\\
&\ -j_1 J_{n+1}(s,p) - J_{n+2}(s,p) - j_3 J_{n+3}(s,p) - j_2 J_{n+4}(s,p)\\
&\ -j_2 J_{n+2}(s,p) - j_3 J_{n+3}(s,p) - J_{n+4}(s,p) - j_1 J_{n+5}(s,p)\\
&\ +j_3 J_{n+3}(s,p) + j_2 J_{n+4}(s,p) + j_1 J_{n+5}(s,p) + J_{n+6}(s,p)\\
&= J_n(s,p) - J_{n+2}(s,p) - J_{n+4}(s,p) + J_{n+6}(s,p).
\end{aligned}$$

□

Theorem 3. (*Binet formula*) *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$JH_n^{s,p} = c_1 \hat{r_1} r_1^n + c_2 \hat{r_2} r_2^n,$$

where r_1, r_2, c_1, c_2 are given by (2.2), respectively, and

$$\hat{r_1} = 1 + j_1 r_1 + j_2 r_1^2 + j_3 r_1^3, \quad \hat{r_2} = 1 + j_1 r_2 + j_2 r_2^2 + j_3 r_2^3.$$

Proof. By Theorem 1 (section 2) we get

$$\begin{aligned}
BhJ_n^{s,p} &= J_n(s,p) + j_1 J_{n+1}(s,p) + j_2 J_{n+2}(s,p) + j_3 J_{n+3}(s,p)\\
&= c_1 r_1^n + c_2 r_2^n + j_1 \left(c_1 r_1^{n+1} + c_2 r_2^{n+1}\right)\\
&\ +j_2 \left(c_1 r_1^{n+2} + c_2 r_2^{n+2}\right) + j_3 \left(c_1 r_1^{n+3} + c_2 r_2^{n+3}\right)\\
&= c_1 r_1^n (1 + j_1 r_1 + j_2 r_1^2 + j_3 r_1^3) + c_2 r_2^n (1 + j_1 r_2 + j_2 r_2^2 + j_3 r_2^3)\\
&= c_1 \hat{r_1} r_1^n + c_2 \hat{r_2} r_2^n,
\end{aligned}$$

which ends the proof. □

The next theorem presents a summation formula for the bihyperbolic (s,p)-Jacobsthal numbers.

Theorem 4. *Let $n \geq 0$, $s \geq 0$, $p \geq 0$ be integers. Then*

$$\begin{aligned}
\sum_{l=0}^{n} JH_l^{s,p} &= \frac{JH_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p}) JH_n^{s,p} - (1 + 2^s + 2^p)(1 + j_1 + j_2 + j_3)}{2^{s+p}(1 + 2^s + 2^p) - 1}\\
&\ -j_1 - j_2(1 + 2^s + 2^p + 2^{s+p})\\
&\ -j_3(1 + 2^s + 2^p + 2^{s+p} + 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p}).
\end{aligned}$$

Proof. By the definition of the bihyperbolic (s,p)-Jacobsthal numbers we have

$$\begin{aligned}\sum_{l=0}^{n} JH_l^{s,p} &= JH_0^{s,p} + JH_1^{s,p} + \ldots + JH_n^{s,p}\\ &= J_0(s,p) + j_1 J_1(s,p) + j_2 J_2(s,p) + j_3 J_3(s,p)\\ &\quad + J_1(s,p) + j_1 J_2(s,p) + j_2 J_3(s,p) + j_3 J_4(s,p) + \ldots\\ &\quad + J_n(s,p) + j_1 J_{n+1}(s,p) + j_2 J_{n+2}(s,p) + j_3 J_{n+3}(s,p)\\ &= J_0(s,p) + J_1(s,p) + \ldots + J_n(s,p)\\ &\quad + j_1 \left(J_1(s,p) + J_2(s,p) + \ldots + J_{n+1}(s,p) + J_0(s,p) - J_0(s,p)\right)\\ &\quad + j_2 (J_2(s,p) + J_3(s,p) + \ldots + J_{n+2}(s,p) + J_0(s,p) + J_1(s,p)\\ &\quad - J_0(s,p) - J_1(s,p))\\ &\quad + j_3 (J_3(s,p) + J_4(s,p) + \ldots + J_{n+3}(s,p) + J_0(s,p)\\ &\quad + J_1(s,p) + J_2(s,p) - J_0(s,p) - J_1(s,p) - J_2(s,p)).\end{aligned}$$

Using Theorem 2 (section 2), we obtain

$$\begin{aligned}&\sum_{l=0}^{n} JH_l^{s,p}\\ &= \tfrac{1}{2^{s+p}(1+2^s+2^p)-1} [J_{n+1}(s,p) + (2^{2s+p} + 2^{s+2p}) J_n(s,p) - 1 - 2^s - 2^p\\ &+ j_1 (J_{n+2}(s,p) + (2^{2s+p} + 2^{s+2p}) J_{n+1}(s,p) - 1 - 2^s - 2^p)\\ &+ j_2 (J_{n+3}(s,p) + (2^{2s+p} + 2^{s+2p}) J_{n+2}(s,p) - 1 - 2^s - 2^p)\\ &+ j_3 (J_{n+4}(s,p) + (2^{2s+p} + 2^{s+2p}) J_{n+3}(s,p) - 1 - 2^s - 2^p))]\\ &- (j_1 J_0(s,p) + j_2 (J_0(s,p) + J_1(s,p)) + j_3 (J_0(s,p) + J_1(s,p) + J_2(s,p)))\end{aligned}$$

$$\begin{aligned}&= \tfrac{1}{2^{s+p}(1+2^s+2^p)-1} [J_{n+1}(s,p) + j_1 J_{n+2}(s,p) + j_2 J_{n+3}(s,p) + j_3 J_{n+4}(s,p)\\ &+ (2^{2s+p} + 2^{s+2p}) (J_n(s,p) + j_1 J_{n+1}(s,p) + j_2 J_{n+2}(s,p) + j_3 J_{n+3}(s,p))\\ &- (1 + 2^s + 2^p)(1 + j_1 + j_2 + j_3)]\\ &- j_1 - j_2 (1 + 2^s + 2^p + 2^{s+p})\\ &- j_3 (1 + 2^s + 2^p + 2^{s+p} + 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p})\\ &= \frac{JH_{n+1}^{s,p} + (2^{2s+p} + 2^{s+2p}) JH_n^{s,p} - (1 + 2^s + 2^p)(1 + j_1 + j_2 + j_3)}{2^{s+p}(1 + 2^s + 2^p) - 1}\\ &- j_1 - j_2 (1 + 2^s + 2^p + 2^{s+p})\\ &- j_3 (1 + 2^s + 2^p + 2^{s+p} + 2^{2s+p+1} + 2^{s+2p+1} + 2^{2s+2p}).\end{aligned}$$

□

In particular, we obtain the following formula for the bihyperbolic Jacobsthal numbers.

Corollary 5. *Let $n \geq 1$ be an integer. Then*

$$\sum_{l=0}^{n} BhJ_l = \frac{BhJ_{n+2} - BhJ_1}{2}.$$

Proof. By Theorem 4 for $s = p = 0$ we have

$$\begin{aligned}\sum_{l=0}^{n} JH_l^{0,0} &= \frac{JH_{n+1}^{0,0} + 2JH_n^{0,0} - 3(1 + j_1 + j_2 + j_3)}{2} - (j_1 + 4j_2 + 9j_3) \\ &= \frac{JH_{n+2}^{0,0} - (3 + 5j_1 + 11j_2 + 21j_3)}{2}.\end{aligned}$$

Using fact that $J_n(0,0) = J_{n+2}$ and $BhJ_0 = j_1 + j_2 + 3j_3$, $BhJ_1 = 1 + j_1 + j_2 + 5j_3$, we get

$$\begin{aligned}\sum_{l=0}^{n} BhJ_l &= \frac{BhJ_{n+2} - (3 + 5j_1 + 11j_2 + 21j_3)}{2} + BhJ_0 + BhJ_1 \\ &= \frac{BhJ_{n+2} - (3 + 5j_1 + 11j_2 + 21j_3) + 2(1 + 2j_1 + 4j_2 + 8j_3)}{2} \\ &= \frac{BhJ_{n+2} - (1 + j_1 + 3j_2 + 5j_3)}{2} = \frac{BhJ_{n+2} - BhJ_1}{2},\end{aligned}$$

which ends the proof. □

Now, we give the ordinary generating functions for the bihyperbolic (s,p)-Jacobsthal numbers.

Theorem 6. *The generating function for the bihyperbolic (s,p)-Jacobsthal sequence $\{BhJ_n^{s,p}\}$ has the following form*

$$G(x) = \frac{BhJ_0^{s,p} + (BhJ_1^{s,p} - 2^{s+p}BhJ_0^{s,p})x}{1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2}.$$

Proof. Assuming that the generating function of the bihyperbolic (s,p)-Jacobsthal sequence $\{BhJ_n^{s,p}\}$ has the form $G(x) = \sum_{n=0}^{\infty} BhJ_n^{s,p}x^n$, we obtain that

$$\begin{aligned}&(1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2)G(x) \\ &= (1 - 2^{s+p}x - (2^{2s+p} + 2^{s+2p})x^2) \cdot (BhJ_0^{s,p} + BhJ_1^{s,p}x + BhJ_2^{s,p}x^2 + \ldots) \\ &= BhJ_0^{s,p} + BhJ_1^{s,p}x + BhJ_2^{s,p}x^2 + \ldots \\ &\quad -2^{s+p}BhJ_0^{s,p}x - 2^{s+p}BhJ_1^{s,p}x^2 - 2^{s+p}BhJ_2^{s,p}x^3 - \ldots \\ &\quad -(2^{2s+p} + 2^{s+2p})BhJ_0^{s,p}x^2 - (2^{2s+p} + 2^{s+2p})BhJ_1^{s,p}x^3 \\ &\quad -(2^{2s+p} + 2^{s+2p})BhJ_2^{s,p}x^4 - \ldots \\ &= BhJ_0^{s,p} + (BhJ_1^{s,p} - 2^{s+p}BhJ_0^{s,p})x,\end{aligned}$$

since $BhJ_n^{s,p} = 2^{s+p}BhJ_{n-1}^{s,p} + (2^{2s+p} + 2^{s+2p})BhJ_{n-2}^{s,p}$ and the coefficients of x^n for $n \geq 2$ are equal to zero. □

In particular, we obtain the generating function for bihyperbolic Jacobsthal numbers

$$g(x) = \frac{BhJ_0 + (BhJ_1 - BhJ_0)x}{1 - x - 2x^2}.$$

Recall that

$$\begin{aligned} BhJ_0 &= j_1 + j_2 + 3j_3, \\ BhJ_1 &= 1 + j_1 + 3j_2 + 5j_3 \end{aligned}$$

and

$$BhJ_1 - BhJ_0 = 1 + 2j_2 + 2j_3.$$

4. Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflict of interest.

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