Proyecciones Journal of Mathematics
Vol. 41, N^o 3, pp. 553-568, June 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-3933

On a subclass of meromorphic functions with positive coefficients defined by rapid operator

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Received : January 2020. Accepted : January 2021

Abstract

In this paper, we introduce and study a new subclass of meromorphic univalent functions defined by Rapid operator. We obtain coefficient inequalities, extreme points, radius of starlikeness and convexity. Finally we obtain partial sums and neighborhood properties for the class $\sum_p^*(\gamma, k, \mu, \theta)$.

Keywords and phrases: *Meromorphic, extreme point, partial sums, neighborhood.*

2010 Mathematics Subject Classification: *30C45.*

1. Introduction

Let S be denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in $U = \{z : z \in C \text{and} |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Denote by $S^*(\gamma)$ and $K(\gamma), 0 \leq \gamma < 1$ the subclasses of functions in S that are starlike and convex functions of order α respectively. Analytically $f \in S^*(\gamma)$ if and only if f is of the form 1.1 and satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, z \in U.$$

Similarly, $f \in K(\gamma)$ if and only if f is of the form 1.1 and satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma, z \in U.$$

Also denote by T the subclasses of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \tag{1.2}$$

introduced and studied by Silverman [21], let $T^*(\gamma) = T \cap S^*(\gamma), CV(\gamma) = T \cap K^*(\gamma)$. The classes $T^*(\gamma)$ and $K^*(\gamma)$ posses some interesting properties and have been extensively studied by Silverman [21] and others. In 1991, Goodman [10, 11] introduced an interesting subclass uniformly convex (uniformly starlike) of the class CV of convex functions (ST starlike functions) denoted by UCV (UST). A function $f(z)$ is uniformly convex (uniformly starlike) in U if $f(z)$ in CV (ST) has the property that for every circular arc γ contained in U with center ξ also in U, the arc $f(\gamma)$ is a convex arc (starlike arc) with respect to $f(\xi)$.

Motivated by Goodman [10, 11], Ronning [17, 18] introduced and studied the following subclasses of S. A function $f \in S$ is said to be in the class $S_p(\gamma, k)$ uniformly $k-$starlike functions if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, 0 \leq \gamma < 1, k \geq 0 \text{and} z \in U \tag{1.3}$$

and is said to be in the class UCV (γ, k), uniformly $k-$convex functions if it satisfies the condition

$$\Re\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)>k\left|\frac{zf''(z)}{f'(z)}\right|, 0\leq\gamma<1, k\geq 0 \text{and} z\in U. \tag{1.4}$$

Indeed it follows from 1.3 and 1.4 that

$$f\in UCV(\gamma,k)\Leftrightarrow zf'\in S_p(\gamma,k). \tag{1.5}$$

Further Ahuja et al. [1], Bharathi et al. [7], Murugusundaramoorthy et al. [12] and others have studied and investigated interesting properties for the classes $S_p(\gamma,k)$ and $UCV(\gamma,k)$.

Let $\sum$ denote the class of functions of the form

$$f(z)=z^{-1}+\sum_{n=1}^{\infty}a_nz^n, a_n\geq 0 \tag{1.6}$$

which are analytic in the punctured open disk
$U^*=\{z:z\in\mathbf{C}, 0<|z|<1\}=U\setminus\{0\}$.

Let $\sum_s, \sum^*(\gamma)$ and $\sum_k(\gamma)(0\leq\gamma<1)$ denote the subclasses of $\sum$ that are meromorphic univalent, meromorphically starlike functions of order γ and meromorphically convex functions of order γ respectively. Analytically, $f\in\sum^*(\gamma)$ if and only if f is of the form 1.6 and satisfies

$$-\Re\left(\frac{zf'(z)}{f(z)}\right)>\gamma, z\in U.$$

Similarly, $f\in\sum_k(\gamma)$ if and only if f is of the form 1.6 and satisfies

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\gamma, z\in U$$

and similar other classes of meromorphically univalent functions have been extensively studied by (for example) Altintas et al. [2], Aouf [3], Mogra et al. [13], Undegadi et al [24, 25, 26] and others (see [8, 14, 15]).

In [6], Athsan and Kulkarni introduced Rapid - operator for analytic functions and Rosy and Sunil Varma [19] modified their operator to meromorphic functions as follows.

Lemma 1.1. *For $f\in\sum$ given by 1.1, $0\leq\mu\leq 1$ and $0\leq\theta\leq 1$, if the operator $S_\mu^\theta:\sum\to\sum$ is defined by*

$$S_\mu^\theta f(z)=\frac{1}{(1-\mu)^\theta\Gamma(\theta+1)}\int_0^\infty t^{\theta+1}e^{\frac{-t}{1-\mu}}f(tz)dt \tag{1.7}$$

then

$$S_\mu^\theta f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} L(n,\theta,\mu) a_n z^n \tag{1.8}$$

where $L(n,\theta,\mu) = (1-\mu)^{n+1}\frac{\Gamma(n+\theta+2)}{\Gamma(\theta+1)}$ *and* Γ *is the familiar Gamma function.*

In order to prove our results wee need the following lemmas.

Lemma 1.2. *If* γ *is a real number and* $\omega = -(u+iv)$ *is a complex number then*

$$\Re(\omega) \geq \gamma \Leftrightarrow |\omega + (1-\gamma)| - |\omega - (1-\gamma)| \geq 0.$$

Lemma 1.3. *If* $\omega = u + iv$ *is a complex number and* γ *is a real number then*

$$-\Re(\omega) \geq k|\omega + 1| + \gamma \Leftrightarrow -\Re\left(\omega(1+ke^{i\theta}) + ke^{i\theta}\right) \geq \gamma, -\pi \leq \theta \leq \pi.$$

Motivated by Sivaprasad Kumar et al. [16] and Atshan et al . [5], now we define a new subclass $\sum^*(\gamma,k,\mu,\theta)$ of $\sum$.

Definition 1.4. *For* $0 \leq \gamma < 1, k \geq 0, 0 \leq \mu \leq 1$ *and* $0 \leq \theta \leq 1$, *we let* $\sum^*(\gamma,k,\mu,\theta)$ *be the subclass of* $\sum_s$ *consisting of functions of the form 1.6 and satisfying the analytic criterion*

$$-\Re\left(\frac{z(S_\mu^\theta f(z))'}{S_\mu^\theta f(z)} + \gamma\right) > k\left|\frac{z(S_\mu^\theta f(z))'}{S_\mu^\theta f(z)} + 1\right|. \tag{1.9}$$

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bounds, extreme points, radii of meromorphic starlikeness and convexity for the class $\sum^*(\gamma,k,\mu,\theta)$. Further, we obtain partial sums and neighborhood properties for the class also.

2. Coefficient estimates

In this section we obtain necessary and sufficient condition for a function f to be in the class $\sum^*(\gamma,k,\mu,\theta)$.

Theorem 2.1. *Let $f \in \sum$ be given by 1.6. Then $f \in \sum^*(\gamma, k, \mu, \theta)$ if and only if*

$$\sum_{n=1}^{\infty}[n(k+1)+(k+\gamma)]L(n,\theta,\mu)a_n \leq (1-\gamma). \tag{2.1}$$

Proof. Let $f \in \sum^*(\gamma, k, \mu, \theta)$. Then by definition and using Lemma 1.2, it is enough to show that

$$-\Re\left\{\frac{z(S_\mu^\theta f(z))'}{S_\mu^\theta f(z)}+(1+ke^{i\theta})+ke^{i\theta}\right\} > \gamma, -\pi \leq \theta \leq \pi. \tag{2.2}$$

For convenience

$$\begin{aligned} C(z) &= -\left[z(S_\mu^\theta f(z))'\right](1+ke^{i\theta}) - ke^{i\theta}S_\mu^\theta f(z) \\ D(z) &= S_\mu^\theta f(z) \end{aligned}$$

That is, the equation 2.2 is equivalent to

$$-\Re\left(\frac{C(z)}{D(z)}\right) \geq \gamma.$$

In view of Lemma 1.2, we only need to prove that

$$|C(z)+(1-\gamma)D(z)| - |C(z)-(1-\gamma)D(z)| \geq 0.$$

Therefore

$$\begin{aligned} |C(z)+(1-\gamma)D(z)| \geq \quad & (2-\gamma)\tfrac{1}{|z|} \\ & -\sum_{n=1}^{\infty}[n(k+1)+(k+\gamma-1)]L(n,\theta,\mu)a_n|z|^n \\ \text{and } |C(z)-(1-\gamma)D(z)| \leq \quad & (\gamma)\tfrac{1}{|z|} \\ & +\sum_{n=1}^{\infty}[n(k+1)+(k+\gamma+1)]L(n,\theta,\mu)a_n|z|^n. \end{aligned}$$

It is to show that

$$\begin{aligned} & |C(z)+(1-\gamma)D(z)| - |C(z)-(1+\gamma)D(z)| \\ \geq \; & 2(1-\gamma)\tfrac{1}{|z|} - 2\sum_{n=1}^{\infty}[n(k+1)+(k+\gamma)]L(n,\theta,\mu)a_n|z|^n \\ \geq \; & 0, \text{by the given condition 2.1.} \end{aligned}$$

Conversely suppose $f \in \sum^*(\gamma, k, \mu, \theta)$. Then by Lemma 1.3, we have 2.2.

Choosing the values of z on the positive real axis the inequality 2.2 reduces to

$$\Re\left\{\frac{[1-\gamma-2\lambda(1+ke^{i\theta})]\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}[n(1+ke^{i\theta})+(\gamma+ke^{i\theta})]L(n,\theta,\mu)z^{n-1}}{\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}L(n,\theta,\mu)a_nz^{n-1}}\right\}\geq 0.$$

Since $\Re(-e^{i\theta})\geq -|e^{i\theta}|=-1$, the above inequality reduces to

$$\Re\left\{\frac{[1-\gamma]\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}[n(1+k)+(\gamma+k)]L(n,\theta,\mu)a_nr^{n-1}}{\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}L(n,\theta,\mu)r^{n-1}}\right\}\geq 0.$$

Letting $r\to 1^-$ and by the mean value theorem, we have obtained the inequality 2.1. □

Corollary 2.2. *If* $f\in\sum^*(\gamma,k,\mu,\theta)$ *then*

$$a_n\leq\frac{(1-\gamma)}{[n(1+k)+(\gamma+k)]L(n,\theta,\mu)}. \tag{2.3}$$

Theorem 2.3. *If* $f\in\sum^*(\gamma,k,\mu,\theta)$ *then for* $0<|z|=r<1$,

$$\frac{1}{r}-\frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)}r\leq|f(z)|$$
$$\leq\frac{1}{r}+\frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)}r. \tag{2.4}$$

This result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)}z, at z=r, ir. \tag{2.5}$$

Proof. Since $f(z)=\frac{1}{z}+\sum\limits_{n=1}^{\infty}a_nz^n$, we have

$$|f(z)|=\frac{1}{r}+\sum_{n=1}^{\infty}a_nr^n\leq\frac{1}{r}+r\sum_{n=2}^{\infty}a_n. \tag{2.6}$$

Since $n\geq 1,(2k+\gamma+1)\leq n(k+1)(k+\gamma)L(n,\theta,\mu)$, using Theorem 2.1, we have

$$\begin{aligned}(2k+\gamma+1)\sum_{n=1}^{\infty} a_n &\leq \sum_{n=1}^{\infty} n(k+1)(k+\gamma)L(n,\theta,\mu)\\ &\leq (1-\gamma)\\ \Rightarrow \sum_{n=1}^{\infty} a_n &\leq \frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)}.\end{aligned}$$

Using the above inequality in 2.7, we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)} r$$

and $|f(z)| \geq \frac{1}{r} - \frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)} r$.

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)} z$.
□

Corollary 2.4. *If $f \in \sum^*(\gamma, k, \mu, \theta)$ then*

$$\frac{1}{r^2} - \frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)} \leq |f'(z)|$$

$$\leq \frac{1}{r^2} + \frac{(1-\gamma)}{(2k+\gamma+1)(1-\mu)^2(\theta+1)(\theta+2)}.$$

The result is sharp for the function given by 2.6

3. Extreme points

Theorem 3.1. *Let $f_0(z) = \frac{1}{z}$ and*

$$f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(1-\gamma)}{[n(1+k)+k]L(n,\theta,\mu)} z^n, n \geq 1. \tag{3.1}$$

Then $f \in \sum^*(\gamma, k, \mu, \theta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} u_n f_n(z), u_n \geq 0 \text{and} \sum_{n=1}^{\infty} u_n = 1. \tag{3.2}$$

Proof. Suppose $f(z)$ can be expressed as in 3.2. Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} u_n f_n(z) = u_0 f_0(z) + \sum_{n=1}^{\infty} u_n f_n(z) \\ &= \tfrac{1}{z} + \sum_{n=1}^{\infty} u_n \tfrac{(1-\gamma)}{[n(1+k)+k]L(n,\theta,\mu)} z^n. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{n=1}^{\infty} u_n \tfrac{(1-\gamma)}{[n(1+k)+k]L(n,\theta,\mu)} \tfrac{[n(1+k)+k)]L(n,\theta,\mu)}{(1-\gamma)} z^n \\ = \quad &\sum_{n=1}^{\infty} u_n = 1 - u_0 \leq 1. \end{aligned}$$

So by Theorem 2.1, $f \in \sum^*(\gamma, k, \mu, \theta)$.

Conversely suppose that $f \in \sum^*(\gamma, k, \mu, \theta)$. Since

$$a_n \leq \frac{(1-\gamma)}{[n(1+k)+k]L(n,\theta,\mu)} n \geq 1.$$

We set $u_n = \frac{[n(1+k)+(\gamma+k)]L(n,\theta,\mu)}{(1-\gamma)} a_n, n \geq 1$ and $u_0 = 1 - \sum_{n=1}^{\infty} u_n$.

Then we have $f(z) = \sum_{n=0}^{\infty} u_n f_n(z) = u_0 f_0(z) + \sum_{n=1}^{\infty} u_n f_n(z)$.

Hence the results follows. □

4. Radii of meromorphically starlike and meromorphically convexity

Theorem 4.1. *Let $f \in \sum^*(\gamma, k, \mu, \theta)$. Then f is meromorphically starlike of order $\delta, (0 \leq \delta \leq 1)$ in the unit disc $|z| < r_1$, where*

$$r_1 = \inf_n \left[\frac{(1-\delta)}{(n+2-\delta)} \frac{[n(1+k)+k]L(n,\theta,\mu)}{(1-\gamma)} \right]^{\frac{1}{n+1}}, n \geq 1.$$

The result is sharp for the extremal function $f(z)$ given by 3.1.

Proof. The function $f \in \sum^*(\gamma, k, \mu, \theta)$ of the form 1.6 is meromorphically starlike of order δ is the disc $|z| < r_1$ if and only if it satisfies the condition

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| < (1-\delta). \tag{4.1}$$

Since

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq \left|\frac{\sum\limits_{n=1}^{\infty}(n+1)a_n z^{n+1}}{1+\sum\limits_{n=1}^{\infty}a_n z^{n+1}}\right| \leq \frac{\sum\limits_{n=1}^{\infty}(n+1)|a_n||z|^{n+1}}{1-\sum\limits_{n=1}^{\infty}|a_n||z|^{n+1}}.$$

The above expression is less than $(1-\delta)$ if $\sum\limits_{n=1}^{\infty}\frac{(n+2-\delta)}{(1-\delta)}a_n|z|^{n+1} < 1$.

Using the fact that $f(z) \in \sum^*(\gamma, k, \mu, \theta)$ if and only if

$$\sum_{n=1}^{\infty}\frac{[n(1+k)+k]L(n,\theta,\mu)}{(1-\gamma)}a_n \leq 1.$$

Thus, 4.1 will be true if $\frac{(n+2-\delta)}{(1-\delta)}|z|^{n+1} < \frac{[n(1+k)+(\gamma+k)]L(n,\theta,\mu)}{(1-\gamma)}$ or equivalently $|z|^{n+1} < \frac{(1-\delta)}{(n+2-\delta)}\frac{[n(1+k)+k]L(n,\theta,\mu)}{(1-\gamma)}$ which yields the starlikeness of the family. □

The proof of the following theorem is analogous to that of Theorem 4.1, and so we omit the proof.

Theorem 4.2. *Let $f \in \sum^*(\gamma, k, \mu, \theta)$. Then f is meromorphically convex of order $\delta, (0 \leq \delta \leq 1)$ in the unit disc $|z| < r_2$, where*

$$r_2 = \inf_n \left[\frac{(1-\delta)}{n(n+2-\delta)}\frac{[n(1+k)+(\gamma+k)]L(n,\theta,\mu)}{(1-\gamma)}\right]^{\frac{1}{n+1}}, n \geq 1.$$

The result is sharp for the extremal function $f(z)$ given by 3.1.

5. Partial Sums

Let $f \in \sum$ be a function of the form 1.6. Motivated by Silverman [22] and Silvia [23] and also see [4], we define the partial sums f_m defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n, (m \in N). \tag{5.1}$$

In this section we consider partial sums of function from the class $\sum^*(\gamma, k, \mu, \theta)$ and obtain sharp lower bounds for the real part of the ratios of f to f_m and f' to f'_m.

Theorem 5.1. *Let $f \in \sum^*(\gamma, k, \mu, \theta)$ be given by 1.6 and define the partial sums $f_1(z)$ and $f_m(z)$ by*

$$f_1(z) = \frac{1}{z} \text{and} f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} |a_n| z^n, (m \in N \setminus \{1\}). \tag{5.2}$$

Suppose also that $\sum\limits_{n=1}^{\infty} d_n |a_n| \leq 1$, where

$$d_n \geq \begin{cases} 1, & \text{if } n = 1, 2, \cdots, m \\ \frac{[n(1+k)+(\gamma+k)]L(n,\theta,\mu)}{(1-\gamma)}, & \text{if } n = m+1, m+2, \cdots \end{cases}. \tag{5.3}$$

Then $f \in \sum^*(\gamma, k, \mu, \theta)$. Furthermore

$$\Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}}$$

and $\Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1+d_{m+1}}$.

Proof. For the coefficient d_n given by 5.3 it is not difficult to verify that

$$d_{m+1} > d_m > 1. \tag{5.4}$$

Therefore we have

$$\sum_{n=1}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |a_n| d_m \leq 1 \tag{5.5}$$

by using the hypothesis 5.3. By setting

$$g_1(z) = d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right) = 1 + \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=1}^{\infty} |a_n| z^{n-1}}$$

then it sufficient to show that

$$\Re\Big(g_1(z)\Big) \geq 0 \text{ or } \left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq 1, (z \in U)$$

and applying 5.7, we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{m+1}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n|-d_{m+1}\sum\limits_{n=m+1}^{\infty}|a_n|} \leq 1, (z \in U)$$

which ready yields the assertion 5.4 of Theorem 5.1. In order to see that

$$f(z) = \frac{1}{z} + \frac{z^{m+1}}{d_{m+1}} \tag{5.6}$$

gives sharp result, we observe that for

$$z = re^{\frac{i\pi}{m}} \text{that} \frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+2}}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}} \text{as} r \to 1^-.$$

Similarly, if we takes $g_2(z) = (1 + d_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1+d_{m+1}}\right)$
and making use of 5.7, we denote that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| < \frac{(1+d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n| - (1-d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}$$

which leads us immediately to the assertion 5.5 of Theorem 5.1.
The bound in 5.5 is sharp for each $m \in N$ with extremal function $f(z)$ given by 5.8 . □

The proof of the following theorem is analogous to that of Theorem 5.1, so we omit the proof.

Theorem 5.2. *If $f \in \sum^*(\gamma, k, \mu, \theta)$ be given by 1.6 and satisfies the condition 2.1 then*

$$\Re\left(\frac{f'(z)}{f'_m(z)}\right) > 1 - \frac{m+1}{d_{m+1}}$$

and $\Re\left(\frac{f'_m(z)}{f'(z)}\right) > \frac{d_{m+1}}{m+1+d_{m+1}}$,

where

$$d_n \geq \begin{cases} n, & \text{if } n = 2,3,\cdots,m \\ \frac{[n(1+k)+(\gamma+k)]L(n,\theta,\mu)}{(1-\gamma))}, & \text{if } n = m+1, m+2, \cdots \end{cases}.$$

The bounds are sharp with the extremal function $f(z)$ of the form 2.3.

6. Neighbourhoods for the class $\sum^{*\xi}(\gamma, k, \mu, \theta)$

In this section, we determine the neighborhood for the class $\sum^{*\xi}(\gamma, k, \mu, \theta)$ which we define as follows

Definition 6.1. *A function $f \in \sum$ is said to be in the class $\sum^{*\xi}(\gamma, k, \mu, \theta)$ if there exits a function $g \in \sum^{*}(\gamma, k, \mu, \theta)$ such that*

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \xi, (z \in E, 0 \leq \xi < 1). \tag{6.1}$$

Following the earlier works on neighbourhoods of analytic functions by Goodman [9] and Ruscheweyh [20], we define the $\delta-$neighbourhoods of function $f \in \sum$ by

$$N_\delta(f) = \left\{g \in \sum : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n and \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta\right\}. \tag{6.2}$$

Theorem 6.2. *If $g \in \sum^{*}(\gamma, k, \mu, \theta)$ and*

$$\xi = 1 - \frac{\delta(2k + \gamma + 1)L(1, \theta, \mu)}{(2k + \gamma + 1)L(1, \theta, \mu) - (1 - \gamma)} \tag{6.3}$$

then $N_\delta(g) \subset \sum^{*\xi}(\gamma, k, \mu, \theta)$.

Proof. Let $f \in N_\delta(g)$. Then we find from 7.2 that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \tag{6.4}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, n \in N. \tag{6.5}$$

Since $g \in \sum^{*}(\gamma, k, \mu, \theta)$, we have

$$\sum_{n=1}^{\infty} b_n \leq \frac{(1 - \gamma)}{(2k + \gamma + 1)L(1, \theta, \mu)}. \tag{6.6}$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum\limits_{n=1}^{\infty}|a_n-b_n|}{1-\sum\limits_{n=1}^{\infty}b_n} = \frac{\delta(2k+\gamma+1)L(1,\theta,\mu)}{(2k+\gamma+1)L(1,\theta,\mu)-(1-\gamma)} = 1-\xi$$

provided ξ is given by 7.3. Hence by definition, $f \in \sum^{*\xi}(\gamma, k, \mu, \theta)$ for ξ given by which completes the proof. □

Acknowledgement

The authors are thankful to the editor and referee(s) for their valuable comments and suggestions which helped very much in improving the paper.

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