P-adic discrete semigroup of contractions

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Abstract

Let $A \in B(X)$ be a spectral operator on a non-archimedean Banach space over $\mathbb{C}_p$. In this paper, we give a necessary and sufficient condition on the resolvent of $A$ so that the discrete semigroup consisting of powers of $A$ is contractions.

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1. Introduction and Preliminaries

In the archimedean operators theory, necessary and sufficient conditions on the resolvent of a densely defined closed linear operator for it to be the infinitesimal generator of a strongly continuous semigroup \((T(s))_{s \in \mathbb{R}^+}\) such that there is \(M \geq 1\), \(\|T(s)\| \leq M\). In particular, we have the following theorem and its corollary.

**Theorem 1.1.** [7] A necessary and sufficient condition that a closed linear operator \(A\) with dense domain be the infinitesimal generator of a strongly continuous semigroup \((T(s))_{s \in \Omega}\) such that for all \(s \in \mathbb{R}^+\), \(\|T(s)\| \leq M\) is that

\[
\|R_{\lambda}(A)^n\| \leq \frac{M}{\lambda^n}
\]

for \(\lambda > 0\) and \(n \in \mathbb{N}\), where \(R_{\lambda}(A) = (\lambda I - A)^{-1}\).

**Corollary 1.2.** [7] A necessary and sufficient condition that a closed linear operator \(A\) with dense domain be the infinitesimal generator of a strongly continuous semigroup \((T(s))_{s \in \Omega}\) such that for all \(s \in \mathbb{R}^+\), \(\|T(s)\| \leq 1\) is that

\[
\|R_{\lambda}(A)\| \leq \frac{M}{\lambda}
\]

for \(\lambda > 0\).

Throughout this paper, \(X\) is a non-archimedean (n.a) Banach space over a (n.a) non trivially complete valued field \(K\) which is also algebraically closed with valuation \(|\cdot|\), \(B(X)\) denote the set of all bounded linear operators on \(X\) into \(X\), \(\mathbb{Q}_p\) is the field of \(p\)-adic numbers (\(p \geq 2\) being a prime) equipped with \(p\)-adic valuation \(|\cdot|_p\), \(\mathbb{Z}_p\) denotes the ring of \(p\)-adic integers of \(\mathbb{Z}_p\) is the unit ball of \(\mathbb{Q}_p\). For more details and related issues, we refer to [4] and [6]. We denote the completion of algebraic closure of \(\mathbb{Q}_p\) under the \(p\)-adic absolute value \(|\cdot|_p\) by \(\mathbb{C}_p\) (see [4]). Let \(r > 0\), \(\Omega_r\) be the clopen ball of \(K\) centred at 0 with radius \(r > 0\), that is \(\Omega_r = \{t \in K : |t| < r\}\). Recall that a free non-archimedean Banach space \(X\) is a non-archimedean Banach space for which there exists a family \((e_i)_{i \in I}\) in \(X \setminus \{0\}\) such that any element \(x \in X\) can be written in the form of a convergent sum \(x = \sum_{i \in I} x_i e_i\), \(x_i \in K\), i.e., \(\lim_{i \in I} x_i e_i = 0\) (the limit is with respect to the Fréchet filter on \(I\)) and \(\|x\| = \sup_{i \in I} |x_i||e_i|\). Let \(X\) be a free non-archimedean Banach

space, recall that every bounded linear operator $A$ on $X$ can be written in a unique fashion as a pointwise convergent series, that is, there exists an infinite matrix $(a_{i,j})_{i,j\in\mathbb{N} \times \mathbb{N}}$ with coefficients in $K$ such that

$$A = \sum_{i,j\in\mathbb{N}} a_{i,j} e_j' \otimes e_i,$$

and $\forall j \in \mathbb{N}$, $\lim_{i \to \infty} |a_{i,j}|\|e_i\| = 0$,

where $(\forall i \geq 1) e_j' \langle u \rangle = u_i \left( e_j' \text{ is the linear form associated with } e_i \right)$.

Moreover, for each $j \in \mathbb{N}$, $A e_j = \sum_{i\in\mathbb{N}} a_{i,j} e_i$ and its norm is defined by

$$\| A \| = \sup_{i,j} |a_{i,j}|\|e_i\|.$$

For more details see [1], Proposition 3.7.

**Definition 1.3.** [1] Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of $K$. We define $E_\omega$ by

$$E_\omega = \{ x = (x_i)_i : \forall i \in \mathbb{N}, x_i \in K, \text{ and } \lim_{i \to \infty} |\omega_i|^\frac{1}{2} |x_i| = 0 \} ,$$

it is equipped with the norm

$$\forall x \in E_\omega : x = (x_i)_i, \quad \|x\| = \sup_{i\in\mathbb{N}}(|\omega_i|^\frac{1}{2} |x_i|).$$

**Remark 1.4.** (1) [1], Exemple 2.21. The space $(E_\omega, \| \cdot \|)$ is a non archimedean Banach space.

(2) If

$$\langle \cdot, \cdot \rangle : E_\omega \times E_\omega \longrightarrow K$$

$$(x, y) \mapsto \sum_{i=0}^{\infty} x_i y_i \omega_i,$$

where $x = (x_i)_i$ and $y = (y_i)_i$. Then, the space $(E_\omega, \| \cdot \|, \langle \cdot, \cdot \rangle)$ is called a $p$-adic (or non archimedean) Hilbert space.

(2) The orthogonal basis $\{ e_i, i \in \mathbb{N} \}$ is called the canonical basis of $E_\omega$, where for all $i \in \mathbb{N}, \|e_i\| = |\omega_i|^\frac{1}{2}$. 
Definition 1.5. [6] Let $A \in B(X)$, set $\nu(A) = \inf_n \|A^n\|^\frac{1}{n} = \lim_n \|A^n\|^\frac{1}{n}$, $A$ is said to be a spectral operator if $\sup\{\|\lambda\| : \lambda \in \sigma(A)\} = \nu(A)$. For $A \in B(X)$, set $U_A = \{\lambda \in K : (I - \lambda A)^{-1} \text{ exists in } B(X)\}$. ($U_A$ is open and $0 \in U_A$) and $C_A = \{\alpha \in K : B(0,|\beta|) \subset U_A \text{ for some } \beta \in K, |\beta| > |\alpha|\}$.

Proposition 1.6. [6] Let $A \in B(X)$, the following are equivalent.

(i) $A$ is a spectral operator.

(ii) For all $\lambda \in C_A$, $(I - \lambda A)^{-1} = \sum_{n=0}^{\infty} \lambda^n A^n$.

(iii) For each $\alpha \in C_A^*$, the function $\lambda \mapsto (I - \lambda A)^{-1}$ is analytic on $B(0,|\alpha|)$.

2. Discrete semigroups of bounded linear operators on non-archimedean Banach space

We begin with the following definition.

Definition 2.1. Let $X$ be a non-archimedean Banach space over $K$. A family $(T(n))_{n} \in N$ of bounded linear operators from $X$ into $X$ is said to be a discrete semigroup of bounded linear operators on $X$ if

(i) $T(0) = I$, where $I$ is the unit operator of $X$;

(ii) For all $m, n \in N$, $T(m + n) = T(m)T(n)$.

Remark 2.2. Let $A \in B(X)$, $T(n) = A^n$ is a discrete semigroup of bounded linear operators on $X$, its generator $A$.

Definition 2.3. Let $X$ be a non-archimedean Banach space over $K$. A discrete semigroup $(T(n))_{n} \in N$ is said to be uniformly bounded if $\sup_{n \in N} \|T(n)\|$ is finite.

In contrast with the classical setting, we have the following example.
Example 2.4. Let $K = \mathbb{Q}_p$, if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then $A$ generate a discrete semigroup of bounded linear operators $(T(n))_{n \in \mathbb{N}}$ given by:

$$\forall n \in \mathbb{N}, \ T(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$  

In fact, it is easy to check that:

(i) $T(0) = I$ where $I$ is the unit operator on $\mathbb{Q}_p^2$.

(ii) For all $m, n \in \mathbb{N}$, $T(m + n) = T(m)T(n)$.

(iii) For all $z = (x, y) \in \mathbb{Q}_p^2$, $n \in \mathbb{N}$, we have

$$\|T(n)z\| = \left\| \begin{pmatrix} x + ny \\ y \end{pmatrix} \right\|,$$

$$= \max\{\|x + ny\|_p, |y|_p\},$$

$$\leq \max\{\|x\|_p, |ny|_p, |y|_p\},$$

$$\leq \max\{\|x\|_p, |y|\} \text{ with } |n|_p \leq 1,$$

$$\leq \|z\|.$$  

Then $(T(n))_{n \in \mathbb{N}}$ is an uniformly bounded discrete semigroup of bounded linear operators on $\mathbb{Q}_p^2$.

We have the following definition.

Definition 2.5. Let $X$ be a non-archimedean Banach space over $K$, let $(T(n))_{n \in \mathbb{N}}$ be a discrete semigroup of bounded linear operators on $X$, $(T(n))_{n \in \mathbb{N}}$ is said to be semigroup of contractions if for all $n \in \mathbb{N}$, $\|T(n)\| \leq 1$.

Example 2.6. Let $X = E_\omega$ where for all $i \in \mathbb{N}$, $\omega_i = p^i$. Let $A$ be unilateral shift given by

for all $i \in \mathbb{N}$, $Ae_i = e_{i+1}$.

Then, for all $n \in \mathbb{N}$, $A^n e_i = e_{n+i}$, hence, for all $n \in \mathbb{N}$, $\frac{\|A^n e_i\|}{\|e_i\|} = p^{-n} \leq 1$. Consequently, for all $n \in \mathbb{N}$, $\|A^n\| \leq 1$. Moreover, $(A^n)_{n \in \mathbb{N}}$ is a discrete semigroup of contractions on $E_\omega$.  

We start with the following statements.

**Lemma 2.7.** Let \((T(n))_{n \in \mathbb{N}}\) be a discrete semigroup of bounded linear operators on \(X\) such that \(\sup_{n \in \mathbb{N}} \|T(n)\| \leq M\). Then there exists an equivalent norm on \(X\) such that \((T(n))_{n \in \mathbb{N}}\) becomes a contraction.

**Proof.** Set: \(|x|_1 = \sup_{n \in \mathbb{N}} \|T(n)x\|\). We have \(\|T(0)\| = 1\) and \((\forall n \in \mathbb{N},) \|T(n)\| \leq M\), then \((\forall x \in X) \|x\| \leq |x|_1 \leq M\|x\|\). Hence \(|\cdot|_1\) is a norm on \(X\) which is equivalent to the original norm \(\|\cdot\|\) on \(X\). Furthermore, for all \(x \in X\), all \(n \in \mathbb{N}\), \(\|T(n)x\|_1 = \sup_{m \in \mathbb{N}} \|T(n)T(m)x\| = \sup_{m \in \mathbb{N}} \|T(n + m)x\| = \sup_{m \geq n} \|T(m)x\| \leq \sup_{m \in \mathbb{N}} \|T(m)x\| = |x|_1\).

In the next Proposition, we assume that \(Q_p \subset K\).

**Proposition 2.8.** The set of all discrete semigroup of contractions form a \(\mathbb{Z}_p\)-subspace of \(B(X)\).

**Proof.** Set \(C\) denote the set of all discrete semigroup of contractions on \(X\) into \(X\).

1. \(\|I_X\| \leq 1\), Hence \(C \neq \emptyset\).

2. Let \((T(n))_{n \in \mathbb{N}}\) and \((S(n))_{n \in \mathbb{N}}\) in \(C\) and \(\lambda \in \mathbb{Z}_p\), we have

\[
\|T(n) + \lambda S(n)\| \leq \max \left\{\|T(n)\|; |\lambda|\|S(n)\|\right\};
\]

\[
\leq 1.
\]

Hence, for all \(n \in \mathbb{N}\) and for all \(\lambda \in \mathbb{Z}_p\), \(T(n) + \lambda S(n) \in C\).

In the rest of this paper, for \(A \in B(X)\) be a spectral operator such that \(\sup_{n \in \mathbb{N}} \|A^n\|\) is finite, we assume that \(U_A = B(0, 1)\) where \(B(0, 1) = \{\lambda \in K : |\lambda| < 1\}\), and for all \(\lambda \in U_A\), \(R(\lambda, A) = (I - \lambda A)^{-1}\).

**Proposition 2.9.** Let \(X\) be a non-archimedean Banach space over \(K\), let \(A\) be a spectral operator and there is \(M \geq 1\) such that \(\sup_{n \in \mathbb{N}} \|A^n\| \leq M\). Then, for all \(\lambda \in C_A\), \(\|R(\lambda, A)\| \leq M\).
Proof. By Proposition 1.6, for all $\lambda \in C_A$, $\lim_{n \to \infty} |\lambda|^n \| A^n \| = 0$, hence

\[
\| R(\lambda, T) \| = \left\| \sum_{n=0}^{\infty} \lambda^n A^n \right\|
\leq M \max_{n \in \mathbb{N}} |\lambda|^n
= M.
\]

Proposition 2.10. Let $A \in B(X)$ be a spectral operator, let $(A^n)_{n \in \mathbb{N}}$ be a discrete semigroup of bounded linear operators on $X$ such that $\sup_{n \in \mathbb{N}} \| A^n \|$ is finite and $U_A = B(0, 1)$. Then, for all $\lambda, \mu \in C_A$,

\[
\lambda R(\lambda, A) - \mu R(\mu, A) = (\lambda - \mu) R(\lambda, A) R(\mu, A).
\]

Proof. Let $\lambda, \mu \in C_A$, we have

(2.1) $\lambda R(\lambda, A)(I - \mu A) R(\mu, A) - \mu R(\lambda, A)(I - \lambda A) R(\mu, A)$

\[
(2.1) = \lambda R(\lambda, A) R(\mu, A) - \lambda \mu R(\lambda, A) R(\mu, A) - \mu R(\lambda, A) R(\mu, A)
+ \lambda \mu R(\lambda, A) R(\mu, A);
\]

\[
= \lambda R(\lambda, A) R(\mu, A) - \mu R(\lambda, A) R(\mu, A);
= (\lambda - \mu) R(\lambda, A) R(\mu, A).
\]

Proposition 2.11. Let $A \in B(X)$ be a spectral operator such that $U_A = B(0, 1)$, let $(A^n)_{n \in \mathbb{N}}$ be a discrete semigroup of contractions on $X$, then for all $\lambda \in C_A$, $\| R(\lambda, A) - I \| \leq |\lambda|$.

Proof. Let $A \in B(X)$ be a spectral operator, then for all $\lambda \in C_A$, $R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^n A^n$. Hence, for all $\lambda \in C_A$,

(2.2) $\| R(\lambda, A) - I \| = \| \sum_{n=1}^{\infty} \lambda^n A^n \|$

(2.3) $\leq \sup_{n \geq 1} \| \lambda^n A^n \|$

(2.4) $\leq |\lambda|$. 

Proposition 2.12. Let $A \in B(X)$ be a spectral operator such that for all $n \in \mathbb{N}$, $\|A^n\| \leq 1$, then for all $n \in \mathbb{N}$, $\alpha \in C_A^*$, $\lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, A) = \frac{n!(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n}.$$  

Proof. Using Proposition 2.10, for all $\lambda, \mu \in \Omega_{|\alpha|}$ with $\alpha \in C_A^*$,

$$\left(\lambda I + (\mu - \lambda)I + (\lambda - \mu)R(\lambda, A)\right)R(\mu, A) = \lambda R(\lambda, A).$$

Then

$$\left(I - \frac{1}{\lambda}(\mu - \lambda)(R(\lambda, A) - I)\right)R(\mu, A) = R(\lambda, A).$$

The quantity in square brackets on the left of this equation is invertible for $|\lambda|^{-1}|\mu - \lambda||R(\lambda, A) - I| < 1$. Thus

$$R(\mu, A) = \sum_{n=0}^{\infty} \frac{(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n} (\mu - \lambda)^n.$$  

But it follows from Proposition 1.6 that $R(\mu, A)$, is analytic on $B(\lambda, |\alpha|)$. From $A \in B(X)$ is spectral operator, then for all $s, t \in \Omega_{|\alpha|}$, $R(\mu, A)$ can be written as follows:

$$R(\mu, A) = \sum_{n=0}^{\infty} \frac{R^{(n)}(\lambda, A)}{n!} (\mu - \lambda)^n.$$  

Hence, for all $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, A) = \frac{n!(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n}.$$  

We have the following theorem.

Theorem 2.13. Let $X$ be a non-archimedean Banach space over $\mathbb{C}_p$, and $A \in B(X)$ be a spectral operator, then for all $n \in \mathbb{N}$, $\|A^n\| \leq 1$ if and only if

$$\|\left(R(\lambda, A) - I\right)^n R(\lambda, A)\| \leq |\lambda|^p,$$

for all $\lambda \in \Omega_{|\alpha|}$ where $\alpha \in C_A^*$ and $R(\lambda, A) = (I - \lambda A)^{-1}$.  

□
Proof. Assume that for all \( n \in \mathbb{N} \), \( \|A^n\| \leq 1 \), let \( \alpha \in C_A^* \), by Proposition 1.6, \( R(\lambda, A) = (I - \lambda A)^{-1} = \sum_{k=0}^{\infty} \lambda^k A^k \) is analytic on \( \Omega_{|\alpha|} \). Using Proposition 2.12, for all \( n \in \mathbb{N} \), \( \lambda \in \Omega_{|\alpha|} \),

\[
R^{(n)}(\lambda, A) = \frac{n!(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n},
\]

and

\[
R^{(n)}(\lambda, A) = \sum_{k=n}^{\infty} k(k-1) \cdots (k-n+1) \lambda^{k-n} A^k = \sum_{k=n}^{\infty} n! \binom{k}{n} \lambda^{k-n} A^k,
\]

then for all \( n \in \mathbb{N} \) and \( \lambda \in \Omega_{|\alpha|} \),

\[
\left\| R^{(n)}(\lambda, A) \right\| = \left\| \sum_{k=n}^{\infty} \binom{k}{n} \lambda^{k-n} A^k \right\|,
\]

\[
\leq \sup_{k \geq n} \binom{k}{n} p|\lambda|^{k-n} \|A^k\|,
\]

\[
\leq \sup_{k \geq n} |\lambda|^{k-n} \|A^k\|,
\]

\[
\leq 1.
\]

Thus, for all \( n \in \mathbb{N} \) and \( t \in \Omega_{|\alpha|} \),

\[
\left\| \frac{R^{(n)}(\lambda, A)}{n!} \right\| \leq 1.
\]

From 2.9 and 2.14, we have for all \( n \in \mathbb{N} \), \( \lambda \in \Omega_{|\alpha|} \),

\[
\| (R(\lambda, A) - I)^n R(\lambda, A) \| \leq |\lambda|_p^n.
\]

Conversely, let \( A \in B(X) \) be a spectral operator, we assume that 2.8, for all \( \lambda \in \Omega_{|\alpha|} \), \( R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^n A^n \). Set for all \( \lambda \in \Omega_{|\alpha|} \), \( k \in \mathbb{N} \), \( S_k(\lambda) = \lambda^{-k}(R(\lambda, A) - I)^k R(\lambda, A) \), then for all \( \lambda \in \Omega_{|\alpha|} \), \( k \in \mathbb{N} \), \( \|S_k(\lambda)\| \leq 1 \). Since \( A \) and \( R(\lambda, A) \) commute, we have:

\[
S_k(\lambda) = \lambda^{-k} \left( (I - (I - \lambda A)) R(\lambda, A) \right)^k R(\lambda, A),
\]

\[
= \lambda^{-k} (\lambda A R(\lambda, A))^k R(\lambda, A),
\]

\[
= A^k R(\lambda, A)^{k+1}.
\]
Then for all $\lambda \in \Omega_{[a]}$, $k \in \mathbb{N}$,

\begin{align}
(2.19) \quad \|A^k\| &= \|(I - \lambda A)^{k+1}S_k(\lambda)\|, \\
(2.20) \quad &\leq \|(I - \lambda A)^{k+1}\|\|S_k(\lambda)\|, \\
(2.21) \quad &\leq \|\sum_{j=0}^{k+1} \binom{k+1}{j}(-\lambda A)^j\|, \\
(2.22) \quad &\leq \max\{1, \|\lambda A\|, \|\lambda^2 A^2\|, \ldots, \|\lambda^{k+1} A^{k+1}\|\},
\end{align}

for $\lambda \to 0$, we have for all $k \in \mathbb{N}$, $\|A^k\| \leq 1$. \hfill \Box

For $A$ densely closed linear operator on $X$, the resolvent set $\rho(A)$ is the set of all $\lambda \in \mathbb{K}$ such that the range $Im(\lambda - A)$ is dense in $X$ and that $\lambda I - A$ has the continuous inverse $(\lambda I - A)^{-1}$ on $D((\lambda I - A)^{-1}) = Im(\lambda I - A)$, where $Im(\lambda I - A)$ denote the range of $(\lambda I - A)$. In the next statements, we assume that $\mathbb{K}$ is a non-archimedean non trivially complete valued field with valuation $|\cdot|$.

**Theorem 2.14.** Let $X$ be a non-archimedean Banach space of countable type over $\mathbb{K}$, let $(A^n)_{n \in \mathbb{N}}$ be a discrete semigroup of generator $A$ be densely defined closed linear operator on $X$ such that $Im(A) \subset D(A)$. Suppose that $\rho(A) \neq \emptyset$, then $A$ is bounded.

**Proof.** Let $(A^n)_{n \in \mathbb{N}}$ be a discrete semigroup, suppose that $\rho(A) \neq \emptyset$, let $\lambda \in \rho(A)$, hence $(\lambda I - A)^{-1} \in B(X)$, then there exists $M > 0$ such that

\begin{equation}
(2.23) \quad \text{for all } x \in D(\lambda I - A) = D(A), \quad \|(\lambda I - A)x\| \geq M\|x\|.
\end{equation}

Or $A$ be densely closed linear operator, then $Im(\lambda I - A)$ is closed. In fact, $x_n \in D(A)$ and $z \in X$, $(\lambda I - A)x_n \to z$. By inequality 2.23, $(x_n)$ is a Cauchy sequence in $X$. Or $X$ is complete, then $x_n \to x$, for some $x \in X$. Thus, $x_n \to x$ and $Ax_n \to \lambda x - z$. By the closedness of $A$, we have $x \in D(A)$ and $Ax = \lambda x - z$. Since $\lambda \in \rho(A)$, $Im(\lambda I - A)$ is dense in $X$, then $Im(\lambda I - A) = X$. Consequently, $X = Im(\lambda I - A) \subseteq D(A)$, hence $D(A) = X$, then $A$ is bounded. \hfill \Box

**Proposition 2.15.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$ such that $\|A\| < 1$. Let $q(\lambda)$ be an arbitrary polynomial and set $p(\lambda) = 1 - (1 - \lambda)q(\lambda)$. Then we have

\[ \|p(A)\| = \|(I - A)^{-1} - q(A)\|. \]
Proof. Let $A \in B(X)$ such that $\|A\| < 1$. Let $q(\lambda)$ be an arbitrary polynomial and set $p(\lambda) = 1 - (1 - \lambda)q(\lambda)$, then

$$\begin{align*}
(I - A)^{-1} - q(A) & = (I - A)^{-1} \left( I - (I - A)q(A) \right) \\
& = (I - A)^{-1} p(A).
\end{align*}$$

(2.24)

Thus,

$$\begin{align*}
\| (I - A)^{-1} - q(A) \| & = \| (I - A)^{-1} p(A) \|, \\
\| p(A) \| & \leq \| p(A) \|.
\end{align*}$$

(2.25)

On the other hand,

$$\begin{align*}
p(A) & = (I - A) \left( (I - A)^{-1} - q(A) \right).
\end{align*}$$

(2.26)

Hence,

$$\begin{align*}
\| p(A) \| & = \| (I - A) \left( (I - A)^{-1} - q(A) \right) \|, \\
& \leq \| (I - A) \| \| (I - A)^{-1} - q(A) \|, \\
& \leq \| (I - A)^{-1} - q(A) \|.
\end{align*}$$

(2.27)

Then,

$$\begin{align*}
\| p(A) \| & = \| (I - A)^{-1} - q(A) \|.
\end{align*}$$

(2.28)
References


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