A Chebyshev pseudo spectral method for solving fractional differential equations

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Received: July 2019 | Accepted: December 2019

Abstract:

The Chebyshev pseudo-spectral method is generalized for solving fractional differential equations with initial conditions. For this purpose, an appropriate representation of the solution is presented and the Chebyshev pseudo-spectral differentiation matrix of fractional order is derived. Then, by using Chebyshev pseudo-spectral scheme, the problem is reduced to the solution of a system of algebraic equations.

Keywords: Caputo derivative; Fractional differential equations; Chebyshev pseudo-spectral method.


Cite this article as (IEEE citation style):

1. Introduction

In the past century notable contributions have been made to the theory of the fractional calculus [2]. In recent decades, the fractional calculus provides an excellent instrument for the description of memory and hereditary properties of various materials and processes[3, 7]. Furthermore, the fractional order models of real systems are regularly more adequate than usually used integer order models. Consequently, the field of the fractional differential equations has attracted interest of researchers in several areas including physics, chemistry, engineering and even finance and social sciences[3, 7]. In this paper, we use Chebyshev pseudo-spectral method for numerical solution of fractional differential equations via collocation technique. Consider a fractional differential equations in the form:

\[
\begin{aligned}
D_0^\alpha C u + Lu &= f; 0 \leq t \leq T, \\
\ u^{(k)}(0) &= d_k, k = 0, \ldots, \lfloor \alpha \rfloor,
\end{aligned}
\]

(1.1)

where, \(L\) is the differential operator, \(f : [0, T] \rightarrow \mathbb{R}\) is a known function, \(u\) is the exact solution, and \(t\) is a variable. We use the floor function \(\lfloor \alpha \rfloor\) to denote the largest integer less than or equal to \(\alpha\) and the ceiling function \(\lceil \alpha \rceil\) to denote the smallest integer greater than or equal to \(\alpha\). Here, \(D_0^\alpha C u\) is the Caputo type fractional derivative of order \(\alpha > 0\), defined by

\[
D_0^\alpha C u = \frac{1}{\Gamma(s - \alpha)} \int_0^t (t - \alpha)^{s-\alpha-1} u^{(s)}(x)dx,
\]

where \(s = \lfloor \alpha \rfloor\). In order to applying our method in a simple manner, first we transform the problem (1.1) in to a problem with homogeneous initial conditions. For this, let us introduce an auxiliary function \(m(t)\), such that

\[
u(t) = m(t) + d_0 + d_1 t + \ldots + d_k t^k / k!
\]

(1.2)

By substituting above expression into the problem (1.1), we obtain the following problem for finding \(m(t)\):

\[
\begin{aligned}
D_0^\alpha C m(t) + Lm(t) &= h; 0 \leq t \leq T, \\
\ m^{(k)}(0) &= 0, k = 0, \ldots, \lfloor \alpha \rfloor,
\end{aligned}
\]

(1.3)

the \(h\) function is easily computable.
2. Preliminaries

In this section, we introduce the orthogonal polynomials and express some of their properties. The well-known Jacobi polynomials \( J_k^{(a,b)} \) with parameters \( a, b > -1 \) are probably the most widely used orthogonal polynomials for numerical solution of differential equations. The Jacobi polynomials may be given explicitly by

\[
J_k^{(a,b)} = \sum_{i=0}^{n} \frac{(-1)^{k-i}(1 + b)_k(1 + a + b)_i + k}{i!(k - i)!((1 + b)_i)(1 + b + a)_k} \left( \frac{1 + x}{2} \right)^i
\]

where \( \theta \in \mathbb{R} \), and

\[
(\theta)_0 = 1, (\theta)_j = (\theta)(\theta + 1), \ldots (\theta + j - 1), j = 1, 2, \ldots.
\]

This expression shows that \( J_k^{(a,b)} \) are analytic functions of the parameters \( a \) and \( b \). The Jacobi polynomials are mutually orthogonal over the interval \((-1, 1)\) with respect to the weight function \((1 - x)^a(1 + x)^b\). The choice \( a = b = 0 \) yields the Legendre polynomials, while choosing \( a = b = -\frac{1}{2} \) gives Chebyshev polynomials. In practice, we can compute the Jacobi polynomials using the following three-term recurrence formula (\([6],[1],[11]\)).

\[
\begin{align*}
J_0^{(a,b)}(x) &= 1, \\
J_1^{(a,b)}(x) &= \frac{1}{2}[(a + b) + (a + b + 2)x], \\
a_{1,n}J_{n+1}^{(a,b)}(x) &= a_{2,n}(x)J_n^{(a,b)}(x) + a_{3,n}J_{n-1}^{(a,b)}(x),
\end{align*}
\]

where

\[
\begin{align*}
a_{1,n} &= 2(n + 1)(n + a + b + 1)(2n + a + b), \\
a_{2,n}(x) &= (2n + a + b + 1)(a^2 - b^2) + \frac{x\Gamma(2n + a + b + 3)}{\Gamma(2n + a + b)}, \\
a_{3,n} &= 2(n + a)(n + b)(2n + a + b + 2).
\end{align*}
\]

Let \(-1 < a - q < \alpha, \alpha \neq 0 \) and \( 0 \leq t \leq T \). Then
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This three-term recurrence relation is generally quite stable and can thus be conveniently employed in the numerical computation of orthogonal polynomials [5].

3. Pseudo-spectral method

In pseudo-spectral method, for problems on bounded domain, the unknown solution to the differential equation is expanded as global polynomial interpolants based on some suitable points. Also, the derivatives are approximated by discrete derivative operator (the differentiation matrix). So, the concept of interpolation and differentiation matrix are useful for understanding the pseudo-spectral method.

3.1. Approximation by interpolation

Let \( \rho_0, \rho_1, \ldots, \rho_n \) be \( n+1 \) distinct nodes in \([0, T]\), and \( \varphi_i(t), i = 0, \ldots, n \) be the Lagrange interpolation polynomials based on this nodes, that are expressed as:

\[
\varphi_i(t) = \prod_{j=0, j \neq i}^{n} \frac{t - \rho_j}{\rho_i - \rho_j},
\]

\[
f(t) \approx \sum_{i=0}^{n} f(\rho_i) \varphi_i(t).
\]

The above approximation can be reform to the following matrix form

\[
f(t) \approx \Phi_n^T(t) f,
\]

where \( f=[f(\rho_0), \ldots, f(\rho_n)]^{(T)} \) and \( \Phi_n(t) \) is a \((n+1)\)-dimensional vector function

\[
\Phi_n(t) = [\varphi_0(t), \ldots, \varphi_n(t)]^T.
\]

From the Kronecker property, we conclude

\[
\Phi_n(\rho_k) = e_k, k = 0, \ldots, n,
\]
where $e_k$ is a $(n+1)$-vector such that the $k$th entries of $e_k$ is equal to 1 and other entries are zero. In the other words, $e_k$ is the $k$th column of the identity matrix of dimension $n + 1$. It is a well-established fact in the numerical analysis that proper distribution of nodes is crucial for both the accuracy of the approximating solution and for the computational effort. A good choice for these nodes is the well-known Gauss and Gauss-Lobatto points [4], where lie inside $[0, T]$ and are cluster near the endpoints. In this sequel we use Chebyshev-Gauss nodes associated with interval $[0, T]$, i.e

$$\rho_i = \frac{T}{2} - \frac{T}{2} \cos\left(\frac{2i + 1}{n + 1} \frac{\pi}{2}\right), i = 0, \cdots, n. \quad (3.5)$$

### 3.2. Differentiation matrix

In pseudo-spectral methods, it is crucial to express the derivatives $f^{(p)}(t)$ in term of $f(t)$ at the collocation points $\rho_k$. This can be done by using the so-called differentiation matrix. Let $f$ be a function with sufficient degree of smoothness and $p$ be a positive integer number. The $p$th derivative of $f(t)$ can be approximated by

$$f^{(p)}(t) \approx D^{(p)}_n f, \quad (3.6)$$

where $D^{(p)}_n$ is the so-called differentiation matrix of order $p$. Note that, by using differentiation matrix, the numerical differentiation process can be performed by matrix-vector product. According to [12], the entries of differentiation matrix $D^{(p)}_n$ are given by

$$d^{(p)}_{ij} = \varphi^{(p)}_j(\rho_i), \quad i, j = 0, \cdots, n. \quad (3.7)$$

So, the entries of differentiation matrix are computed by taking the analytical derivative of $\varphi_j(t)$ of order $p$ and evaluating it at points $\rho_i$ for $i, j = 0, \cdots, n$. More alternative computational approaches for deriving these entries, in accurate and stable manner, can be found in [10, 12].

### 4. Description of the method

In this section, we concentrate on generalizing the pseudo-spectral method in order to solve the initial value problem of fractional differential equations expressed in (1.3) with zero initial conditions. Note that this problem is obtained from the main problem (1.1) by applying the transformation (1.2). In pseudo-spectral method, at first, a set of suitable nodes must be chosen.
We use the Chebyshev-Gauss nodes (3.5), that correspond to the interval $[0, T]$. Also zero does not belong to this set of nodes. In the second step, we must approximate the unknown solution based on the chosen nodes. In the traditional pseudo-spectral method, the unknown solution is approximated by an interpolating polynomial as (1.2). But in the presented method, we approximated the unknown solution $m(t)$ by $m_n(t)$ as follows

\begin{equation}
  m_n(t) = t^\alpha \sum_{k=0}^{n} v_k \varphi_k(t) = t^\alpha \Phi_n^T(t)V,
\end{equation}

where $V = [v_0, \ldots, v_n]^T$ is the unknown vector. The above approximation satisfies the zero initial condition(s) (1.3). Furthermore,

\begin{equation*}
  m_n(t) \in t^\alpha P_n = \{t^\alpha p(t) : p \in P_n\}.
\end{equation*}

The Caputo fractional derivative of order $q > 0$ of $m_n(t)$ can be expressed as follows

\begin{equation}
  D^q_Cm_n(t) = t^{\alpha-q} \sum_{k=0}^{n} w_k \varphi_k(t) = t^{\alpha-q} \Phi_n^T(t)W,
\end{equation}

where $W = [w_0, \ldots, w_n]^T$. There is a linear connection between vectors $V$ and $W$, which is established by fractional differentiation matrix.

**4.1. Fractional differentiation matrix**

In the following, we extend the concept of differentiation matrix of integer order to fractional order. In our approach, the fractional differentiation matrix connects the coefficient vectors $V$ and $W$ in (4.1) and (4.2) as

\begin{equation}
  W = D^{(q, \alpha)}_n V.
\end{equation}

From Eqs. 4.1 – 4.3, it can be shown that, the $(i, j)$th component of the matrix $D^{(q, \alpha)}_n$ is obtained by

\begin{equation*}
  \frac{D^q_C[t^{\alpha} \varphi_j(t)]}{t^{\alpha-q}}|_{t=\rho_i}.
\end{equation*}

However, the direct calculation of the above expression is almost impossible (in terms of computation, it’s almost impossible). So, in the following, we
suggest an efficient alternative approach. Let $\psi_j(t)$ be a linearly independent subset of $t^\alpha P_n$, then there are scalars $s_{ij}$ such that

$$\psi_j(t) = t^\alpha \sum_{i=0}^{n} s_{ij} \varphi_i(t),$$

where

$$s_{ij} = \rho_i^{-\alpha} \psi_j(\rho_i).$$

Now we assume that $D^q_{C}\psi_j(t)$ can be computed as following

$$D^q_{C}\psi_j(t) = t^{\alpha-q} \sum_{i=0}^{n} u_{ij} \varphi_i(t),$$

where

$$u_{ij} = \rho_i^{\alpha-q} \psi_j(\rho_i).$$

If the $s_{ij}$ and $u_{ij}$ are replaced as entries in matrices $S = (s_{ij})$ and $U = (u_{ij})$, then the fractional differentiation matrix can be computed as follows matrix equation

$$U = D^{(q,\alpha)} S.$$  (4.4)

So, in this scheme, the fractional derivative of test functions $\{\psi_j\}$ must be easily computed. Naturally, the test functions $\psi_j(t)$ may be chosen as $\psi_j(t) = t^\alpha v^j$. Consequently,

$$D^q_{C}\psi_j(t) = \frac{\Gamma(\alpha + j + 1)}{\Gamma(\alpha + j + 1 - q)} t^{\alpha+j+q}.$$

Unfortunately, by using these test functions, for large $n$, the resulting matrix $S$ becomes ill-conditioned. So, in practice, the mentioned choice is not suitable and leads to inaccurate results.

In Summary, we assume $\psi_k(t) = t^\alpha J_k^{(\alpha-q,\alpha)}(\frac{\pi}{2} - 1)$ and by using test function 2.2, we are able to resolve the ill-conditioned problem and we can directly derive the matrices $S$ and $U$ and then by using the matrix Eq.(4.4), the fractional differentiation matrix $D^{(q,\alpha)}_n$ could be obtained.
5. Numerical Example

Example 5.1. Consider the following fractional differential equation

\[
(5.1) \quad u'(t) + D^{(1)}_{C^2} u(t) - 2u(t) = 0, \quad t > 0,
\]

which arises, for instance, in the study of the generalized Basset force occurring when a sphere sinks in a (relatively less dense) viscous fluid [8, 9]. The analytical solution, obtained with the help of Laplace transformation of Caputo fractional derivatives, under the initial condition \(u(0) = 1\), is given by

\[
u(t) = \frac{2}{3\sqrt{t}} E_{\frac{3}{2}, \frac{1}{2}}(\sqrt{t}) - \frac{1}{6\sqrt{t}} E_{\frac{3}{2}, \frac{1}{2}}(-2\sqrt{t}) - \frac{1}{2\sqrt{\pi}t},
\]

where \(E_{\lambda,\mu}\) is Mittag-Leffler function [10, 3] with parameters \(\lambda, \mu > 0\),

\[
E_{\lambda,\mu}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\lambda k + \mu)}.
\]

We present the numerical solution for 5.1 with \(T = 5\) in some values of \(t\) and \(n = 5, 10, 15, 20\) that shown in the Table 5.1.

From the comparison between analytical solution and our method, an almost good approximation is observable. In order to see the results clearly, we also present the Fig. 1

<table>
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<tr>
<th>(n)</th>
<th>(t = 1)</th>
<th>(t = 2)</th>
<th>(t = 3)</th>
<th>(t = 4)</th>
<th>(t = 5)</th>
</tr>
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<td>2.714336</td>
<td>8.922571</td>
<td>24.59981</td>
<td>65.78029</td>
<td>180.1481</td>
</tr>
<tr>
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<td>9.696794</td>
<td>26.65929</td>
<td>72.72038</td>
<td>197.8994</td>
</tr>
<tr>
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<td>3.424960</td>
<td>9.62754</td>
<td>26.64683</td>
<td>72.68787</td>
<td>197.8112</td>
</tr>
<tr>
<td>20</td>
<td>3.424807</td>
<td>9.691706</td>
<td>26.64381</td>
<td>72.67936</td>
<td>197.7879</td>
</tr>
<tr>
<td>Exact</td>
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<td>9.69088</td>
<td>26.6414</td>
<td>72.6729</td>
<td>197.770</td>
</tr>
</tbody>
</table>

Table 5.1: The resulting values of the presented method on example 5.1, with \(T = 5\) in some values of \(t\).
Conclusion

In this paper, the Chebyshev pseudo-spectral method is generalized for solving fractional differential equations with initial conditions. The numerical results are given to demonstrate the validity and applicability of the method. We note that similar techniques can be applied to tau and collocation methods using Legendre polynomials or other Jacobi polynomials.
References


