Erdélyi-Kober fractional integrals on Hardy space and BMO

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Abstract:

The mapping properties of the multi. Erdélyi-Kober fractional integral operators on Hardy space and BMO. In particular, our main result gives the boundedness of the Erdélyi-Kober fractional integrals, the hypergeometric fractional integrals and the two-dimensional Weyl integrals on Hardy space and BMO.

Keywords: Fractional integral; Hardy spaces; Bounded mean oscillation; Erdélyi-Kober fractional integrals; Hypergeometric fractional integrals.


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1. Introduction

This paper establishes the boundedness of the multi. Erdélyi-Kober fractional integral operators on Hardy space.

The multi. Erdélyi-Kober fractional integral operators are extensions of the Erdélyi-Kober operators introduced in [1,11]. For some important applications of the Erdélyi-Kober operators on differential equations, such as nonlinear time-fractional diffusion equation, the reader is referred to [17, 18, 19, 20].

The multi. Erdélyi-Kober fractional integral operators include a number of important integral operators and integral transforms such as the Riemann-Liouville fractional integrals, the fractional Weyl integrals, the hypergeometric fractional integral and the two-dimensional Weyl integrals. These operators and transforms have several applications on applied mathematics, physics and statistics, see [10, 12, 15, 22].

This paper aims to study the boundedness of the multi. Erdélyi-Kober fractional integral operators on Hardy space $H^1$. The atomic decomposition is the most powerful method to establish the mapping properties of linear operators on Hardy spaces [2, 23]. We can also use extrapolation to establish the mapping properties of some sublinear operators on Hardy type spaces [7].

On the other hand, the results obtained in this paper do not rely on atomic decomposition and extrapolation. We obtain our result by using the basic definition of the Hardy space $H^1$.

Moreover, we also consider the mapping properties the adjoint multi. Erdélyi-Kober fractional integral operators. With these results, we establish the boundedness of the multi. Erdélyi-Kober fractional integral operators on the function space of bounded mean oscillation $BMO$.

This paper is organized as follows. Section 2 gives the definitions of the multi. Erdélyi-Kober fractional integral operators, the Hardy space $H^1$ and $BMO$. The main result of this paper is presented in Section 3.

2. Definitions and Preliminaries

Let $\sigma \in \mathbb{C}$, $p,q \in \mathbb{N}$, $m,n \in \mathbb{N} \cup \{0\}$, $0 \leq m \leq q$ and $0 \leq n \leq p$. The parameters $(a_k)^p_1$ and $(b_k)^q_1$ are chosen so that the set of the poles of $\Gamma(b_k - s)$, $\Gamma_1$ and the set of the poles of $\Gamma(1 - a_j + s)$, $\Gamma_2$ are disjoint. The Meijer $G$-function is defined by mean of the contour Mellin-Barnes type integrals
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\[
G_{m,n}^{p,q} \left[ \frac{(a_k)^p}{(b_k)^q} \right] = \frac{1}{2\pi i} \int_{\mathcal{C}} \prod_{k=m+1}^{n} \frac{\Gamma(b_k - s)}{\Gamma(1 - b_k + s)} \prod_{j=n+1}^{n} \frac{\Gamma(1 - a_j + s)}{\Gamma(a_j - s)} \sigma^s ds
\]

where the infinite contour \( \mathcal{C} \) separates the sets \( \Gamma_1 \) and \( \Gamma_2 \).

According to [10, (1.1.13)], we have the following asymptotic behaviours for \( G_{m,m}^{0,0} \). When \( \sigma \to 0 \), we have

\[
G_{m,m}^{0,0} \left[ \frac{(a_k)^p}{(b_k)^q} \right] = O(|\sigma|^\mu)
\]

where \( \mu = \min_{1 \leq k \leq m} (b_k) \).

Moreover, by [10, (1.1.14)], when \( \nu^*_m \neq 0, \pm 1, \pm 2, \ldots \) and \( \sigma \to 1^- \), we have

\[
G_{m,m}^{0,0} \left[ \frac{(a_k)^p}{(b_k)^q} \right] \sim \frac{(1 - \sigma)^{\nu^*_m}}{\Gamma(\nu^*_m + 1)}
\]

where \( \nu^*_m = \sum_{k=1}^{m} (a_k - b_k) - 1 \).

**Definition 2.1.** Let \( m \in \mathbb{N} \), \( \beta > 0 \), \( \gamma_i \in \mathbb{R} \) and \( \delta_i > 0 \), \( i = 1, \ldots, m \).

The multi. Erdélyi-Kober fractional integral operator is defined as

\[
I_{m,m}^{(\gamma_k), (\delta_k)} f(x) = \int_0^1 G_{m,m}^{0,0} \left[ \frac{(a_k)^p}{(b_k)^q} \right] f(x \sigma^\beta) d\sigma.
\]

The multi. Erdélyi-Kober fractional integral operator of Weyl type is defined as

\[
W_{m,m}^{(\gamma_k), (\delta_k)} f(x) = \int_1^\infty G_{m,m}^{0,0} \left[ \frac{1}{\sigma} \frac{(a_k + \delta_k + 1)^m}{(a_k + 1)^m} \right] f(x \sigma^\beta) d\sigma.
\]

We now consider some important members of the multi. Erdélyi-Kober fractional integral operators.

Let \( \gamma \in \mathbb{R} \) and \( \delta > 0 \). When \( m = 1 \), we have

\[
G_{1,1}^{1,0} \left[ \frac{\gamma + \delta}{\gamma} \right] = \begin{cases} 
c \frac{(1-\sigma)^{\delta-1} \sigma^\gamma}{\Gamma(\delta)}, & 0 < \sigma < 1 \\
0, & \sigma > 1 
\end{cases}
\]
Let \( \beta > 0 \). According to (2.3) and (2.4) \( I_{\beta}^{\gamma,\delta} \) and \( W_{\beta}^{\gamma,\delta} \) become the multi. Erdélyi-Kober fractional integral operators

\[
I_{\beta}^{\gamma,\delta} f(t) = \frac{t^{-\beta(\gamma+\gamma)} \Gamma(\gamma)}{\Gamma(\gamma+\delta)} \int_0^t s^{-\beta(\gamma+1)-1}(t^\beta - s^\beta)^{\delta-1} f(s) ds,
\]

\[
K_{\beta}^{\gamma,\delta} f(t) = \frac{n^{\beta} t^{\delta} \Gamma(\gamma+\delta)}{\Gamma(\gamma+\delta+\delta)} \int_0^t (s^\beta - t^\beta)^{\delta-1} s^{-\beta(\gamma+\delta)+\beta-1} f(s) ds,
\]

respectively. Particularly, the multi. Erdélyi-Kober fractional integral operators include the Riemann-Liouville fractional integrals and the fractional Weyl integrals. Moreover, the Uspensky integral transform [24]

\[
\frac{1}{2} I_{2,1}^{\alpha+\frac{1}{2}} f(x) = \frac{1}{\Gamma(\alpha + \frac{1}{2})} \int_0^1 (1 - \sigma^2)^{\alpha-\frac{1}{2}} f(x\sigma) d\sigma
\]

is a member of the multi. Erdélyi-Kober fractional integral operators, see [10].

When \( m = 2 \), we have

\[
G_{2,2}^{1,0} \left( \begin{array}{c} \gamma_1 + \delta_1 + \delta_2 \\ \gamma_1 + \gamma_2 \\ \gamma_1 + \gamma_2 \\ \gamma_1 + \gamma_2 \end{array} \right) = \left\{ \begin{array}{ll} 2F_1(\gamma_2 + \delta_2 - \gamma_1, \delta_1 + \delta_2; 1 - \sigma), & \sigma < 1 \\ 0, & \sigma \geq 1 \end{array} \right.,
\]

where \( 2F_1 \) is the Gauss hypergeometric function [18, p. 64].

Consequently, the corresponding multi. Erdélyi-Kober fractional integral operator

\[
I_{\beta,2}^{(\gamma_1,\gamma_2), (\delta_1,\delta_2)} f(x) = \int_0^1 \frac{\sigma^{\gamma_2(1-\sigma)^{\delta_1+\delta_2-1}}}{(1(\delta_1+\gamma_2))} 2F_1(\gamma_2 + \delta_2 - \gamma_1, \delta_1 + \delta_2; 1 - \sigma) f(x\sigma^\beta) d\sigma
\]

is the hypergeometric fractional integral in [12].

An example of the multi. Erdélyi-Kober fractional integral operator of Weyl type is the two-dimensional Weyl integral given by

\[
W^{\theta,\tau} f(x) = \int_1^\infty \int_1^\infty \frac{(\sigma_1-1)^{\theta-1}(\sigma_2-1)^{-1-\tau}}{\Gamma(\theta+\tau)} f(x\sigma_1\sigma_2) d\sigma_1 d\sigma_2 = W_{1,2}^{(-\theta,\tau), (\theta,\tau)} f(x).
\]

Next, we turn to the definition of the Hardy space \( H^1 \).

Let \( M \) and \( L_{loc} \) denote the set of Lebesgue measurable functions and the set of locally integrable function on \( \mathbb{R} \), respectively. Let \( S \) and \( S' \) denote the class of Schwartz functions and the class of tempered distributions on \( \mathbb{R} \), respectively. We say that \( f \) is a bounded distribution if \( f \ast \Phi \in L^\infty \) for any \( \Phi \in S \). For any \( \lambda > 0 \) and \( f \in L_{loc}(\mathbb{R}) \), define \( D_\lambda f(x) = f(x/\lambda) \), \( x \in \mathbb{R} \).
Let $P(x) = \frac{1}{\pi} \frac{1}{(1+|x|^2)^{\frac{3}{2}}}$ be the Poisson kernel on $\mathbb{R}$. For any bounded distribution $f$, define $M_P f$ by

$$M_P f(x) = \sup_{t>0} |(f * P_t)(x)|$$

where $P_t(x) = t^{-1} P(x/t)$, $t > 0$.

Notice that $f * P_t$ is well defined whenever $f$ is a bounded distribution, see [26, p. 89-90].

We recall the definition of the Hardy space $H^1$ from [23].

**Definition 2.2.** The Hardy space $H^1$ consists of all bounded $f \in \mathcal{S}'$ satisfying

$$\|f\|_{H^1} = \|M_P f\|_{L^1} < \infty.$$  

There are a number of characterizations of Hardy spaces such as Littlewood-Paley characterizations, grand maximal function characterizations. For the studies of the Hardy spaces, the reader is referred to [26, Chapters III and IV]. One of the remarkable features is that

$$H^1 \subset L_1,$$

see [26, Chapter III, Section 2.3.3]. The above inclusion property is crucial for our study on fractional integrals as it guarantees that the multi-Erdélyi-Kober fractional integral operators are well-defined on $H^1$.

There are several generalizations of the classical Hardy spaces such as Hardy-Herz spaces, Hardy-Morrey spaces, Hardy spaces with variable exponent and Hardy spaces built on ball quasi-Banach function spaces [3, 4, 5, 6, 7, 9, 13, 14, 16, 21, 25].

Another celebrated result for Hardy space $H^1$ is the duality between it and the function space of bounded mean oscillation [26, Chapter IV, Theorem 1]. Recall that a locally integrable function $f$ is a function of bounded mean oscillation if

$$\|f\|_{BMO} = \sup_{B \in \mathcal{B}} \frac{\|\chi_B(f - f_B)\|_{L^1}}{|B|} < \infty$$

where $\mathcal{B}$ denote the set of connected intervals in $\mathbb{R}$, $|B|$ is the Lebesgue measure of $B$ and $f_B = \frac{1}{|B|} \int_B f(y)dy$. We have

$$\langle H^1 \rangle^* = BMO$$
where $(H^1)^*$ denote the dual space of $H^1$. For any $g \in BMO$, the pairing of the above duality is give by

$$l_g(f) = \int_{\mathbb{R}} f(x)g(x)dx, \quad f \in H^1,$$

see [26, Chapter IV, (3)].

In order to obtain the mapping properties of the multi. Erdélyi-Kober fractional integral operator on $BMO$, we need to study the adjoint operators of $I^{(\gamma_k,),(\delta_k)}_{\beta,m}$ and $W^{(\gamma_k,),(\delta_k)}_{\beta,m}$.

**Definition 2.3.** Let $m \in \mathbb{N}$, $\beta > 0$, $\gamma_i \in \mathbb{R}$ and $\delta_i > 0$, $i = 1, \cdots, m$.

The adjoint multi. Erdélyi-Kober fractional integral operator is defined as

$$J^{(\gamma_k,),(\delta_k)}_{\beta,m} f(x) = \int_0^1 G_{\beta,m,m}^{(\gamma_k,),(\delta_k)} \left[ \frac{(\gamma_k + \delta_k)}{(\gamma_k)^{\beta/m}} \right] \sigma^{-\frac{1}{\beta}} f(x \sigma^{-\frac{1}{\beta}}) d\sigma.$$

The adjoint multi. Erdélyi-Kober fractional integral operator of Weyl type is defined as

$$V^{(\gamma_k,),(\delta_k)}_{\beta,m} f(x) = \int_1^\infty G_{\beta,m,m}^{(\gamma_k,),(\delta_k)} \left[ \frac{1}{\sigma} \frac{(\gamma_k + \delta_k + 1)}{(\gamma_k + 1)^{\beta/m}} \right] \sigma^{-\frac{1}{\beta}} f(x \sigma^{-\frac{1}{\beta}}) d\sigma.$$

In view of (2.6) and (2.7), we see that whenever we obtain the mapping properties for $J^{(\gamma_k,),(\delta_k)}_{\beta,m}$ and $V^{(\gamma_k,),(\delta_k)}_{\beta,m}$ on $H^1$, we can define the action of the operators $I^{(\gamma_k,),(\delta_k)}_{\beta,m}$ and $W^{(\gamma_k,),(\delta_k)}_{\beta,m}$ for $f \in BMO$ by the followings

$$\int_{\mathbb{R}} (I^{(\gamma_k,),(\delta_k)}_{\beta,m} f)(x)g(x)dx = \int_{\mathbb{R}} f(x)(J^{(\gamma_k,),(\delta_k)}_{\beta,m} g)(x)dx, \quad \forall g \in H^1,$$

$$\int_{\mathbb{R}} (W^{(\gamma_k,),(\delta_k)}_{\beta,m} f)(x)g(x)dx = \int_{\mathbb{R}} f(x)(V^{(\gamma_k,),(\delta_k)}_{\beta,m} g)(x)dx, \quad \forall g \in H^1.$$
3. Main result

This section gives the main result of this paper. We establish the mapping properties of the multi-Erdélyi-Kober fractional integral operators on the Hardy space $H^1$.

**Theorem 3.1.** Let $m \in \mathbb{N}$, $\beta > 0$, $\gamma_i \in \mathbb{R}$ and $\delta_i > 0$, $i = 1, \cdots, m$.

1. If $\beta$, $(\delta_k)_{k=1}^m$ and $(\gamma_k)_{k=1}^m$ satisfy

$$\min_{1 \leq k \leq m} \gamma_k > \frac{1}{\beta} - 1,$$

then for any $f \in H^1$

$$\|I_{\beta,m}^{(\gamma_k), (\delta_k)} f\|_{H^1} \leq \left( \int_0^1 \sigma^{-\frac{1}{\beta}} d\sigma \right) \|f\|_{H^1}.$$  

2. If $\beta$ and $(\gamma_k)_{k=1}^m$ satisfy

$$\min_{1 \leq k \leq m} \gamma_k + \frac{1}{\beta} > 0,$$

then for any $f \in H^1$

$$\|W_{\beta,m}^{(\gamma_k), (\delta_k)} f\|_{H^1} \leq \left( \int_1^{\infty} \sigma^{-\frac{1}{\beta}} d\sigma \right) \|f\|_{H^1}.$$  

**Proof:** Let $f \in H^1$. For any fixed $t > 0$, we consider

$$\int_\mathbb{R} \int_0^1 P_t(y-x) G_{m,m,0}^{m,0} \left[ \sigma \left( \frac{(\gamma_k + \delta_k)_{1}^{m}}{(\gamma_k)_{1}^{m}} \right) \right] f(x \sigma^{\frac{1}{\beta}}) d\sigma dx.$$  

As $|P_t(x-y)| \leq Ct^{-n}$, $\forall x, y \in \mathbb{R}$, by using the substitution $z = x \sigma^{\frac{1}{\beta}}$, we find that
\[
\int_{\mathbb{R}} \int_{0}^{1} P_t(y - x) G_{m,m} \begin{bmatrix} \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_1^m \end{bmatrix} f(x \sigma^\frac{1}{\beta}) \, d\sigma dx \\
\leq Ct^{-n} \int_{\mathbb{R}} \int_{0}^{1} G_{m,m} \begin{bmatrix} \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_1^m \end{bmatrix} f(x \sigma^\frac{1}{\beta}) \, d\sigma dx \\
\leq Ct^{-n} \int_{\mathbb{R}} \int_{0}^{1} G_{m,m} \begin{bmatrix} \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_1^m \end{bmatrix} \sigma^{-\frac{1}{\beta}} f(z) \, d\sigma dz.
\]

Since \( f \in H^1 \subset L_1 \), we have
\[
\int_{\mathbb{R}} \int_{0}^{1} P_t(y - x) G_{m,m} \begin{bmatrix} \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_1^m \end{bmatrix} f(x \sigma^\frac{1}{\beta}) \, d\sigma dx \\
\leq Ct^{-n} \| f \|_{L_1} (I + II)
\]
where
\[
I = \int_{0}^{\frac{1}{\beta}} G_{m,m} \begin{bmatrix} \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_1^m \end{bmatrix} \sigma^{-\frac{1}{\beta}} \, d\sigma, \\
II = \int_{\frac{1}{\beta}}^{1} G_{m,m} \begin{bmatrix} \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_1^m \end{bmatrix} \sigma^{-\frac{1}{\beta}} \, d\sigma.
\]

For \( I \), (2.1) and (3.1) guarantee that
\[
I \leq C \int_{0}^{\frac{1}{\beta}} \sigma^{\mu_0} \sigma^{-\frac{1}{\beta}} d\sigma < \infty
\]
where \( \mu_0 = \min_{1 \leq k \leq m} (\gamma_k) \). For \( II \), (2.2) assures that
\[
II \leq C \int_{\frac{1}{\beta}}^{1} (1 - \sigma)^{\nu_0} \sigma^{-\frac{1}{\beta}} d\sigma < \infty
\]
where \( \nu_0 = \sum_{k=1}^{m} \delta_k - 1 \). The above inequalities conclude that
\[
P_t(y - x) G_{m,m} \begin{bmatrix} \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_1^m \end{bmatrix} f(x \sigma^\frac{1}{\beta})
\]
is absolutely integrable with respect to \( d\sigma dx \), therefore, we are allowed to use Fubini’s theorem to obtain
\[
P_t \ast i_{\beta,m}^{(\gamma_k), (\delta_k)}(f) = \int_{0}^{1} G_{m,m} \begin{bmatrix} \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_1^m \end{bmatrix} P_t \ast (D_{\sigma^\frac{1}{\beta}} f) d\sigma.
\]
Consequently,
Therefore, \( f \) is absolutely integrable with respect to the product measure \( M \).

For any \( \lambda \in (0, \infty) \) and \( f \in H^1 \), by using the substitution \( y = \lambda z \), we get

\[
D_\lambda f \ast P_t(x) = \int_\mathbb{R} f((y-x)/\lambda) P_t(y) dy = \int_\mathbb{R} f(z - (x/\lambda)) P_t(\lambda z) \lambda dz
\]

= \int_\mathbb{R} f(z - (x/\lambda)) P_t(z) dz = (f \ast P_t/\lambda)(x/\lambda).

Consequently,

\[
M_P D_\lambda f(x) = \sup_{t>0} |D_\lambda f \ast P_t(x)| = \sup_{t>0} |(f \ast P_t/\lambda)(x/\lambda)|
\]

= \( M_P f(x/\lambda) = D_\lambda M_P f(x) \).

Therefore,

\[
|P_t \ast I^{(\gamma_k),(\delta_k)}_{\beta,m}(f)| \leq \int_0^1 G_{m,m}^{0,0} \left[ \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_n \right] |D_{\sigma - \frac{1}{\beta}} M_P f| d\sigma.
\]

By taking supremum over \( t > 0 \) on both side of the above inequality, we have

\[
M_P I^{(\gamma_k),(\delta_k)}_{\beta,m}(f) = \sup_{t>0} |P_t \ast I^{(\gamma_k),(\delta_k)}_{\beta,m}(f)|
\]

\[
\leq \int_0^1 G_{m,m}^{0,0} \left[ \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_n \right] |D_{\sigma - \frac{1}{\beta}} M_P f| d\sigma.
\]

Next, we show that

\[
G_{m,m}^{0,0} \left[ \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_n \right] D_{\sigma - \frac{1}{\beta}} M_P f(y)
\]

is absolutely integrable with respect to the product measure \( d\sigma dx \).

By using the substitution \( z = \frac{1}{\beta} y \), we have

\[
\int_\mathbb{R} \int_0^1 G_{m,m}^{0,0} \left[ \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_n \right] |D_{\sigma - \frac{1}{\beta}} M_P f(y)| d\sigma dy
\]

\[
= \int_\mathbb{R} \int_0^1 G_{m,m}^{0,0} \left[ \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_n \right] \sigma^{-\frac{1}{\beta}} |M_P f(y)| d\sigma dy
\]

\[
= \left( \int_0^1 G_{m,m}^{0,0} \left[ \sigma \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)_n \right] \sigma^{-\frac{1}{\beta}} d\sigma \right) \|f\|_{H^1}
\]

where the last identity follows from the definition of \( H^1 \).
According to (3.3)-(3.6), we have

\[
\int_\mathbb{R} \int_0^1 G_{m,0}^{m,m} \left[ \sigma \left( \frac{\gamma_k + \delta_k)^m}{(\gamma_k + 1)^m} \right) \right] \left| D_{\sigma - \frac{1}{\sigma}} \frac{1}{M} f(y) \right| d\sigma dy < \infty.
\]

Thus, by applying Fubini’s theorem on (3.7), (3.8) gives

\[
\| I_{\beta,m}^{(\gamma_k), (\delta_k)}(f) \|_{H^1} = \int_\mathbb{R} | M_{\beta,m}^{(\gamma_k), (\delta_k)}(f)(y) | dy \leq \int_\mathbb{R} \int_0^1 G_{m,0}^{m,m} \left[ \sigma \left( \frac{\gamma_k + \delta_k)^m}{(\gamma_k + 1)^m} \right) \right] \left| D_{\sigma - \frac{1}{\sigma}} \frac{1}{M} f(y) \right| d\sigma dy = \int_\mathbb{R} \int_0^1 G_{m,0}^{m,m} \left[ \sigma \left( \frac{\gamma_k + \delta_k)^m}{(\gamma_k + 1)^m} \right) \right] \left| D_{\sigma - \frac{1}{\sigma}} \frac{1}{M} f(y) \right| d\sigma dy = III + IV.
\]

We find that

\[
III \leq C \int_1^\infty \left( \sigma - 1 \right)^{\nu_1} \sigma^{-\frac{1}{\sigma^1}} d\sigma = C \int_1^\infty \frac{1}{\sigma} \left( \sigma - 1 \right)^{\nu_1} \sigma^{-\frac{1}{\sigma^1}} d\sigma < \infty
\]

because \( \delta_k > 0, k = 1, \ldots, m \) where \( \nu_1 = \sum_{k=1}^m \delta_k \).

For IV, according to (2.1) and (3.2), we get

\[
IV \leq C \int_2^\infty \sigma^{-\nu_1 - \frac{1}{\sigma}} d\sigma < \infty
\]

where \( \nu_1 = \min_{1 \leq k \leq m} (\gamma_k + 1) \).

Consequently,

\[
\int_1^\infty \left| G_{m,0}^{m,m} \left[ \sigma \left( \frac{\gamma_k + \delta_k)^m}{(\gamma_k + 1)^m} \right) \right] \right| \sigma^{-\frac{1}{\sigma^1}} d\sigma < \infty.
\]

Similar to the proof of (i), this inequality assures that we are allowed to use Fubini’s theorem to obtain
\[ P_t \ast W^{(\gamma_k), (\delta_k)}_{\beta, m}(f) = \int_0^1 G_{m,m}^{m,0} \left[ \frac{1}{\sigma} \left( \frac{\gamma_k + \delta_k + 1}{\gamma_k + 1} \right)^m \right] P_t \ast \left( D_{\sigma} - \frac{1}{\sigma} f \right) d\sigma \]

and
\[ MPW^{(\gamma_k), (\delta_k)}_{\beta, m}(f) = \sup_{t > 0} \left| P_t \ast W^{(\gamma_k), (\delta_k)}_{\beta, m}(f) \right| \leq \int_0^1 G_{m,m}^{m,0} \left[ \frac{1}{\sigma} \left( \frac{\gamma_k + \delta_k + 1}{\gamma_k + 1} \right)^m \right] \left| D_{\sigma} - \frac{1}{\sigma} M_P f \right| d\sigma. \]

Moreover, (3.9) and Fubini’s theorem yield
\[ \left\| W^{(\gamma_k), (\delta_k)}_{\beta, m}(f) \right\|_{H^1} = \int_R \left| MPW^{(\gamma_k), (\delta_k)}_{\beta, m}(f)(y) \right| dy \]
\[ \leq \int_0^1 \int_0^1 G_{m,m}^{m,0} \left[ \frac{1}{\sigma} \left( \frac{\gamma_k + \delta_k + 1}{\gamma_k + 1} \right)^m \right] \left| D_{\sigma} - \frac{1}{\sigma} M_P f(y) \right| d\sigma dy \]
\[ = \left( \int_0^1 G_{m,m}^{m,0} \left[ \frac{1}{\sigma} \left( \frac{\gamma_k + \delta_k + 1}{\gamma_k + 1} \right)^m \right] \left| D_{\sigma} - \frac{1}{\sigma} M_P f(y) \right| d\sigma \right) \left\| f \right\|_{H^1}. \]

We also have the mapping properties for the corresponding adjoint operators.

**Theorem 3.2.** Let \( m \in \mathbb{N}, \beta > 0, \gamma_i \in \mathbb{R} \) and \( \delta_i > 0, i = 1, \cdots, m. \)

1. If \((\gamma_k)_{k=1}^m\) satisfy
\[ \min_{1 \leq k \leq m} \gamma_k > -1, \]
\[ \sum_{k=1}^m \delta_k > 0, \]
then for any \( f \in H^1 \)
\[ \left\| J_{\beta, m}^{(\gamma_k)} f \right\|_{H^1} \leq \left( \int_0^1 G_{m,m}^{m,0} \left[ \frac{1}{\sigma} \left( \frac{\gamma_k + \delta_k}{\gamma_k} \right)^m \right] d\sigma \right) \left\| f \right\|_{H^1}. \]

2. If \((\gamma_k)_{k=1}^m\) satisfy
\[ \min_{1 \leq k \leq m} \gamma_k > 0, \]
then for any \( f \in H^1 \)
\[ \left\| V_{\beta, m}^{(\gamma_k), (\delta_k)} f \right\|_{H^1} \leq \left( \int_1^\infty G_{m,m}^{m,0} \left[ \frac{1}{\sigma} \left( \frac{\gamma_k + \delta_k + 1}{\gamma_k + 1} \right)^m \right] d\sigma \right) \left\| f \right\|_{H^1}. \]
With some simple modifications, the proof of the previous result follows from the proof of Theorem 3.1, therefore, for simplicity, we leave the details to the readers.

By using (2.6), (2.8) and (2.9), Theorem 3.2 gives the boundedness of the multi-Erdélyi-Kober fractional integral operators on $BMO$.

**Theorem 3.3.** Let $m \in \mathbb{N}$, $\beta > 0$, $\gamma_i \in \mathbb{R}$ and $\delta_i > 0$, $i = 1, \ldots, m$.

1. If $(\gamma_k)_{k=1}^m$ satisfy

\begin{equation}
\min_{1 \leq k \leq m} \gamma_k > -1,
\end{equation}

\begin{equation}
\sum_{k=1}^m \delta_k > 0,
\end{equation}

then for any $f \in BMO$

$$
\|I_{\beta, m}^{(\gamma_k), (\delta_k)} f\|_{BMO} \leq \left( \int_0^1 \left| G_{m, m}^{\gamma_k, 0} \left[ \frac{1}{\sigma} \left( \frac{(\gamma_k + \delta_k)^m}{m!} \right) \right] \right| d\sigma \right) \|f\|_{BMO}.
$$

2. If $(\gamma_k)_{k=1}^m$ satisfy

\begin{equation}
\min_{1 \leq k \leq m} \gamma_k > 0,
\end{equation}

then for any $f \in BMO$

$$
\|W_{\beta, m}^{(\gamma_k), (\delta_k)} f\|_{BMO} \leq \left( \int_1^\infty \left| G_{m, m}^{\gamma_k, 0} \left[ \frac{1}{\sigma} \left( \frac{(\gamma_k + \delta_k + 1)^m}{m!} \right) \right] \right| d\sigma \right) \|f\|_{BMO}.
$$

The followings are applications of Theorem 3.1. We have the boundedness of the Erdélyi-Kober fractional integrals, the hypergeometric fractional integrals and the two-dimensional Weyl integrals on the Hardy space $H^1$.

**Corollary 3.4.** Let $\gamma \in \mathbb{R}$ and $\delta, \beta > 0$.

1. If $\gamma > \frac{1}{\beta} - 1$, then $I_{\beta}^{\gamma, \delta}$ is bounded on $H^1$.

2. If $\gamma + \frac{1}{\beta} > 0$, then $K_{\beta}^{\gamma, \delta}$ is bounded on $H^1$. 
Corollary 3.5. Let $\beta > 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\delta_1, \delta_2 > 0$. If $\min(\gamma_1, \gamma_2) > \frac{1}{3} - 1$, then the hypergeometric fractional integral $I_{\beta,2}^{(\gamma_1, \gamma_2), (\delta_1, \delta_2)}$ is bounded on $H^1$.

Corollary 3.6. Let $\theta, \tau > 0$. If

$$\min(-\theta, -\tau) + 1 > 0,$$

then the two-dimensional Weyl integral $W^{\theta, \tau}$ is bounded on $H^1$.

Similarly, in view of Theorem 3.3, we have the following results for the Erdélyi-Kober fractional integrals, the hypergeometric fractional integrals on $BMO$.

Corollary 3.7. Let $\gamma \in \mathbb{R}$ and $\delta, \beta > 0$. If $\gamma > 0$, then $I_{\beta}^{\gamma, \delta}$ and $K_{\beta}^{\gamma, \delta}$ are bounded on $BMO$.

Corollary 3.8. Let $\beta > 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\delta_1, \delta_2 > 0$. If $\min(\gamma_1, \gamma_2) > -1$, then the hypergeometric fractional integral $I_{\beta,2}^{(\gamma_1, \gamma_2), (\delta_1, \delta_2)}$ is bounded on $BMO$.

The reader is referred to [8] for the studies of some other integral operators such as Hadamard fractional integrals on $BMO$ and ball Banach function spaces.

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References


