Sharp inequality of three point Gauss—Legendre quadrature rule

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Abstract:

An interesting identity for 3-point Gauss-Legendre quadrature rule using functions that are \( n \)-times differentiable. By applying the established identity, a sharp inequality which gives an error bound for 3-point Gauss-Legendre quadrature rule and some generalizations are derived. At the end, an application in numerical integration is given.

Keywords: Gauss quadrature formula; Hölder inequality.


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1. Introduction

In numerical analysis, inequalities play a main role in error estimations. A few years ago, by using modern theory of inequalities and Peano kernel approach a number of authors in [2, 3, 4, 6, 7] have considered an error analysis of some quadrature rules of Newton–Cotes type. In particular, the midpoint, trapezoid and Simpson’s rules have been investigated more recently with the view of obtaining bounds for the quadrature rules in terms of at most first derivative. The Newton–Cotes formulas use values of function at equally spaced points. The same practice when the formulas are combined to form the composite rules, but this restriction can significantly decrease the accuracy of the approximation. In fact, these methods are inappropriate when integrating a function on an interval that contains both regions with large functional variation and regions with small functional variation. If the approximation error is to be evenly distributed, a smaller step size is needed for the large–variation regions than for those with less variation. Among others Gaussian quadrature rules gives the highest possible degree of precision; so that it is recommended to be ‘almost’ the method of choice.

In order to investigate 2–point Gauss–Legendre quadrature rule, Ujević in [8], obtained bounds for absolutely continuous functions with derivatives which belong to $L_2(a,b)$, as follows:

**Theorem 1.1.** Let $f : [-1, 1] \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(-1,1)$. Then

\[
(1.1) \quad \left| f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) - \int_{-1}^{1} f(t) \, dt \right| \leq \sqrt{\frac{4 - 2\sqrt{3}}{3}} \cdot \sigma^{1/2}(f'),
\]

where

\[
\sigma(g) = (b - a) \cdot \mathcal{T}(g, g)
\]

and

\[
\mathcal{T}(g, g) = \frac{1}{(b-a)^2} \left\| g \right\|^2_2 - \frac{1}{(b-a)^4} \left( \int_a^b g(t) \, dt \right)^2.
\]

Inequality (1.1) is sharp in the sense that the constant $\sqrt{\frac{4 - 2\sqrt{3}}{3}}$ cannot be replaced by a smaller one.
Some Gaussian and Gaussian–like quadrature rules are considered in [9, 11, 12]. For recent Gauss–Legendre quadrature rule for Riemann–Stieltjes integral, see [5].

In this paper, we consider the approximation

\[
\int_{-1}^{1} f(t) \, dt \approx \frac{1}{9} \left[ 5f\left(-\frac{\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right) \right],
\]

which is the 3–point Gauss–Legendre quadrature rule. Error estimates for this approximation under various assumptions for the functions involved are proved. Interested readers see the references [1, 4, 10]. This approach allows us to investigate several quadrature rules that have restrictions on the behavior of the integrand and thus to deal with larger classes of functions.

Motivated by the above literatures, the main objective of this paper is to discover an interesting identity for 3–point Gauss–Legendre quadrature rule using functions that are \(n\)–times differentiable and applying it to obtain a sharp inequality of 3–point Gauss–Legendre quadrature rule for functions that are \(n\)–times differentiable on \([-1, 1]\), where \(n \geq 1\). Also, some generalizations of Theorems 2.3 and 2.5 will be given. At the end, to support our results and to mention which of them is better from the point of view of the best estimator and also to mention that for certain types of functions they may not be applicable an application in numerical integration is given. These results may motivate further research in different areas of pure and applied sciences.

2. Main results

In this section, first we establish in the following lemma a new identity for 3–point Gauss–Legendre quadrature rule using functions that are \(n\)–times differentiable on \([-1, 1]\), where \(n \geq 1\).
Lemma 2.1. Suppose $f : [-1, 1] \to \mathbb{R}$ be $n$–times differentiable function on $[-1, 1]$, where $n \geq 1$. Let $K_n : [-1, 1] \to \mathbb{R}$ be given by

$$K_n(t) = \begin{cases} 
\frac{(t + 1)^n}{n!}, & t \in \left[-1, -\frac{\sqrt{15}}{5}\right]; \\
\frac{(t + \frac{4}{5})^n}{n!}, & t \in \left(-\frac{\sqrt{15}}{5}, 0\right]; \\
\frac{(t - \frac{4}{5})^n}{n!}, & t \in \left(0, \frac{\sqrt{15}}{5}\right]; \\
\frac{(t - 1)^n}{n!}, & t \in \left(\frac{\sqrt{15}}{5}, 1\right].
\end{cases}$$

If $K_n f^{(n)}$ is Lebesgue integrable function on $[-1, 1]$, then we have

$$\int_{-1}^{1} f(t) dt = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \left[ a_k \left( f^{(k-1)} \left( -\frac{\sqrt{15}}{5} \right) \right) + (-1)^{k-1} f^{(k-1)} \left( \frac{\sqrt{15}}{5} \right) \right] + b_k f^{(k-1)}(0),$$

(2.1)

where

$$a_k = \left( \frac{5 - \sqrt{15}}{4} \right)^k - \left( \frac{20 - 9\sqrt{15}}{45} \right)^k, \quad b_k = \left( \frac{4}{5} \right)^k \left[ 1 - (-1)^k \right].$$

(2.2)

Proof. The proof will be done by induction. For $n = 1$, we get

$$(-1) \int_{-1}^{1} K_1(t) f'(t) dt = \int_{-1}^{1} f(t) dt - \frac{1}{9} \left[ 5f \left( -\frac{\sqrt{15}}{5} \right) + 8f(0) + 5f \left( \frac{\sqrt{15}}{5} \right) \right].$$

(2.3)

The (2.1) is valid for $n = 1$. Suppose by induction that this is true for $n = r$. Then, we have

$$(-1)^r \int_{-1}^{1} K_r(t) f^{(r)}(t) dt = \int_{-1}^{1} f(t) dt$$

$$+ \sum_{k=1}^{r} \frac{(-1)^{2r-k}}{k!} \left[ a_k \left( f^{(k-1)} \left( -\frac{\sqrt{15}}{5} \right) \right) + (-1)^{k-1} f^{(k-1)} \left( \frac{\sqrt{15}}{5} \right) \right] + b_k f^{(k-1)}(0),$$

(2.3)
where

\[(2.4) \quad a_k = \left(\frac{5-\sqrt{15}}{5}\right)^k - \left(\frac{20-9\sqrt{15}}{45}\right)^k, \quad b_k = \left(\frac{4}{9}\right)^k \left[1 - (-1)^k\right].\]

Now, we prove for \(n = r + 1\). From definition of \(K_n\), integrating by parts and using (2.3) and (2.4), we obtain

\[(-1)^{r+1} \int_{-1}^{1} K_{r+1}(t) f^{(r+1)}(t) dt = (-1)^{r+1} \left[ K_{r+1}(t) f^{(r)}(t) \right]_{-1}^{1} - \int_{-1}^{1} K_{r}(t) f^{(r)}(t) dt \]

\[= (-1)^{r+1} K_{r+1}(t) f^{(r)}(t) \bigg|_{-1}^{1} - (-1)^{r} \int_{-1}^{1} K_{r}(t) f^{(r)}(t) dt \]

\[= (-1)^{r+1} \left\{ \frac{(t+1)^{r+1}}{(r+1)!} f^{(r)}(t) \bigg|_{-1}^{1} + \frac{(t-1)^{r+1}}{(r+1)!} f^{(r)}(t) \bigg|_{-1}^{1} \right\} + \int_{-1}^{1} f(t) dt \]

\[+ \sum_{k=1}^{r+1} \frac{(-1)^2r-k}{k!} a_k \left[ f^{(k-1)} \left( -\frac{\sqrt{15}}{5} \right) + (-1)^{k-1} f^{(k-1)} \left( \frac{\sqrt{15}}{5} \right) \right] + b_k f^{(k-1)}(0) \]

\[= \int_{-1}^{1} f(t) dt \]

\[+ \sum_{k=1}^{r+1} \frac{(-1)^{2r-k} - 2r-k}{k!} a_k \left[ f^{(k-1)} \left( -\frac{\sqrt{15}}{5} \right) + (-1)^{k-1} f^{(k-1)} \left( \frac{\sqrt{15}}{5} \right) \right] + b_k f^{(k-1)}(0) \].

The proof of Lemma 2.1 is completed.

By using Lemma 2.1, one can obtain the following upper bound of 3–point Gauss–Legendre quadrature rule for functions that are \(n\)–times differentiable on \([-1, 1]\), where \(n \geq 1\) using the well-known H"older’s inequality and the concept of norm in \(L_q[-1, 1]\).

**Theorem 2.2.** Let \(n \geq 1\), \(p \geq 1\) be two positive integers such that \(\frac{1}{p} + \frac{1}{q} = 1\).

Suppose \(f : [-1, 1] \rightarrow \mathbb{R}\) be \(n\)–times differentiable function on \([-1, 1]\) such that \(f^{(n)} \in L_q[-1, 1]\). Then

\[(2.5) \quad \left| (-1)^n \int_{-1}^{1} K_{n}(t) f^{(n)}(t) dt \right| \leq \sqrt{\frac{a_n(p)(1+(1-p)n)+b_n(p)}{(pn+1)(n)!^p}} \| f^{(n)} \|_q,\]
where

\[(2.6) \quad a_n(p) = \left(\frac{5-\sqrt{15}}{5}\right)^{pn+1} - \left(\frac{20-9\sqrt{15}}{45}\right)^{pn+1}\]

and

\[(2.7) \quad b_n(p) = \left(\frac{4}{9}\right)^{pn+1} \left[1 - (-1)^{pn+1}\right].\]

**Proof.** From Lemma 2.1, using definitions of $K_n$ and norm of $L_q[-1,1]$, Hölder’s inequality and properties of the modulus, we get

\[
\left|\frac{(-1)^n}{n} \int_{-1}^{1} K_n(t)f^{(n)}(t)dt\right| \leq \frac{1}{n} \int_{-1}^{1} |K_n(t)||f^{(n)}(t)|dt \leq \|K_n\|_p \|f^{(n)}\|_q
\]

\[
= \|f^{(n)}\|_q \times \left[ \int_{-1}^{\frac{\sqrt{15}}{5}} \left(\frac{(t+1)^n}{n!}\right)^p dt + \int_{\frac{\sqrt{15}}{5}}^{0} \left(\frac{(t+1)^n}{n!}\right)^p dt + \int_{0}^{\frac{\sqrt{15}}{5}} \left(\frac{(t-1)^n}{n!}\right)^p dt + \int_{\frac{\sqrt{15}}{5}}^{1} \left(\frac{(t-1)^n}{n!}\right)^p dt \right]^{\frac{1}{p}}
\]

\[
= \sqrt{a_n(p)(1+(-1)^{pn}) + b_n(p)} \|f^{(n)}\|_q.
\]

The proof of Theorem 2.2 is completed.

Let see now, a sharp inequality for absolutely continuous function whose derivative is in $L_2(-1,1)$.

**Theorem 2.3.** Let $f : [-1,1] \rightarrow \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(-1,1)$. Then

\[(2.8) \quad \left|\int_{-1}^{1} f(t)dt - \frac{1}{9} \left[5f\left(-\frac{\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right)\right]\right| \leq \frac{1}{9} \sqrt{108 - 26\sqrt{15} \cdot \sigma^{1/2}(f')},\]

where $\sigma$ is defined as in Theorem 1.1. Inequality (2.8) is sharp in the sense that the constant $\frac{1}{9} \sqrt{108 - 26\sqrt{15}}$ cannot be replaced by a smaller one.
Proof. Using (2.1), we have
\[ \int_{-1}^{1} K(t) f'(t) \, dt = \int_{-1}^{1} f(t) \, dt - \frac{1}{9} \left[ 5f \left( -\frac{\sqrt{15}}{3} \right) + 8f(0) + 5f \left( \frac{\sqrt{15}}{3} \right) \right] \]

Since \( \int_{-1}^{1} K(t) \, dt = 0 \), we get
\begin{align*}
\int_{-1}^{1} K(t) f'(t) \, dt &= \int_{-1}^{1} K(t) \, dt - \frac{1}{9} \int_{-1}^{1} f'(s) \, ds \\
&= 2T (K, f') \\
&= \mathcal{S}(K, f').
\end{align*}

Hence
\begin{align*}
\mathcal{S}^2(K, f') &= \left\{ \int_{-1}^{1} \left[ K(t) - \frac{1}{2} \int_{-1}^{1} K(s) \, ds \right] f'(t) \, dt - \frac{1}{2} \int_{-1}^{1} f'(s) \, ds \right\}^2 \\
&\leq \int_{-1}^{1} \left[ K(t) - \frac{1}{2} \int_{-1}^{1} K(s) \, ds \right]^2 f'(t) \, dt - \frac{1}{2} \int_{-1}^{1} f'(s) \, ds \right]^2 dt \\
&= \frac{108 - 26\sqrt{15}}{81} \mathcal{S}(f', f').
\end{align*}

(2.10)

From (2.9) and (2.10), we deduce the required result. The details to prove the sharpness are left to the interested reader. The proof of Theorem 2.3 is completed.

Theorem 2.3 can be generalize as follows:

**Theorem 2.4.** Let \( f : [-1, 1] \rightarrow \mathbb{R} \) be \( n \)-times differentiable function on \([-1, 1]\) for \( n \geq 1 \) and \( f \) is absolutely continuous whose derivatives \( f^{(n)} \in L^2(-1, 1) \). Then
\begin{align*}
\left| \int_{-1}^{1} f(t) \, dt - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \left[ a_k \left( f^{(k-1)} \left( -\frac{\sqrt{15}}{3} \right) + (-1)^{k-1} f^{(k-1)} \left( \frac{\sqrt{15}}{3} \right) \right) + b_k f^{(k-1)}(0) \right] \right| \\
&\leq \frac{1}{9} \sqrt{108 - 26\sqrt{15}} \cdot \sigma^{1/2} \left( f^{(n)} \right).
\end{align*}

(2.11)

Inequality (2.11) is sharp in the sense that the constant \( \frac{1}{9} \sqrt{108 - 26\sqrt{15}} \) cannot be replaced by a smaller one.

Proof. Using (2.1) and the fact that \( \int_{-1}^{1} K(t) \, dt = 0 \), we have
Theorem 2.5 can be generalized as follows:

Using (2.1), we have

\[
2T(K, f^{(n)}) = \mathcal{S}(K, f^{(n)}).
\]

(2.12)

So

\[
\mathcal{S}^2(K, f^{(n)}) = \left\{ \int_{-1}^{1} \left[ K(t) - \frac{1}{2} \int_{-1}^{1} K(s) \, ds \right] \left[ f^{(n)}(t) - \frac{1}{2} \int_{-1}^{1} f^{(n)}(s) \, ds \right] \, dt \right\}^2
\leq \int_{-1}^{1} \left[ K(t) - \frac{1}{2} \int_{-1}^{1} K(s) \, ds \right]^2 \, dt \int_{-1}^{1} \left[ f^{(n)}(t) - \frac{1}{2} \int_{-1}^{1} f^{(n)}(s) \, ds \right]^2 \, dt
= \frac{108 - 26 \sqrt{15}}{81} \mathcal{S}(f^{(n)}, f^{(n)}).
\]

(2.13)

From (2.12) and (2.13), we deduce the required result. The details to prove the sharpness are left to the interested reader. The proof of Theorem 2.4 is completed.

At the end, let prove an upper bound for bounded differentiable function whose derivative is in \(L_1(-1, 1)\).

**Theorem 2.5.** Let \(f : [-1, 1] \to \mathbb{R}\) be differentiable function on \((-1, 1)\). If \(f' \in L_1(-1, 1)\) and \(\gamma \leq f'(x) \leq \Gamma\) for all \(x \in (-1, 1)\), where \(\gamma, \Gamma > 0\), then

\[
\left| \int_{-1}^{1} f(t) \, dt - \frac{1}{6} \left[ 5f \left( -\frac{\sqrt{15}}{5} \right) + 8f(0) + 5f \left( \frac{\sqrt{15}}{5} \right) \right] \right| \leq \frac{(\Gamma - \gamma)}{90} \left( \frac{1051}{26} - 26\sqrt{15} \right).
\]

(2.14)

**Proof.** Using (2.1), we have

\[
\int_{-1}^{1} K(t) \, dt = f_1 \left[ 5f \left( -\frac{\sqrt{15}}{5} \right) + 8f(0) + 5f \left( \frac{\sqrt{15}}{5} \right) \right].
\]

Let \(C = \frac{\Gamma + \gamma}{2}\). Hence

\[
\int_{-1}^{1} K(t) \, dt = f_1 \left[ 5f \left( -\frac{\sqrt{15}}{5} \right) + 8f(0) + 5f \left( \frac{\sqrt{15}}{5} \right) \right].
\]

On the other hands,

\[
\left| \int_{-1}^{1} K(t) \, dt \right| \leq \max_{t \in (-1, 1)} |f'(t) - C| \int_{-1}^{1} |K(t)| \, dt
= \max_{t \in (-1, 1)} |f'(t) - C| \leq \frac{(\Gamma - \gamma)}{2} \cdot \frac{1}{\sqrt{2}} \left( \frac{1051}{26} - 26\sqrt{15} \right),
\]

which gives the required result. The proof of Theorem 2.5 is completed.

Theorem 2.5 can be generalize as follows:
Theorem 2.6. Let \( f : [-1,1] \to \mathbb{R} \) \( n \)–times differentiable function on \((-1,1)\). If \( f^{(n)} \in L_1(-1,1) \) and \( \gamma \leq f^{(n)}(x) \leq \Gamma \) for all \( x \in (-1,1) \), where \( \gamma, \Gamma > 0 \), then
\[
\left| \int_{-1}^{1} f(t) dt - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \left[ a_k \left( f^{(k-1)} \left( -\frac{\sqrt{15}}{5} \right) + (-1)^{k-1} f^{(k-1)} \left( \frac{\sqrt{15}}{5} \right) \right) \right] \right| 
\leq \frac{\Gamma - \gamma}{90} \left( \frac{1051}{9} - 26\sqrt{15} \right).
\]

Proof. Using (2.1) and denoted \( C \) as in Theorem 2.5, we obtain
\[
\int_{-1}^{1} K(t) \left[ f^{(n)}(t) - C \right] dt = \int_{-1}^{1} f(t) dt 
- \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \left[ a_k \left( f^{(k-1)} \left( -\frac{\sqrt{15}}{5} \right) + (-1)^{k-1} f^{(k-1)} \left( \frac{\sqrt{15}}{5} \right) \right) \right] + b_k f^{(k-1)}(0).
\]
From the other side, we have
\[
\left| \int_{-1}^{1} K(t) \left[ f^{(n)}(t) - C \right] dt \right| \leq \max_{t \in (-1,1)} \left| f^{(n)}(t) - C \right| \int_{-1}^{1} |K(t)| dt 
= \max_{t \in (-1,1)} \left| f^{(n)}(t) - C \right| \leq \frac{\Gamma - \gamma}{2} \cdot \frac{1}{45} \left( \frac{1051}{9} - 26\sqrt{15} \right),
\]
which gives the required result. The proof of Theorem 2.6 is completed.

3. Application

In this section, to support our results we give the following numerical example.

Example 3.1. Let us consider the integral \( \int_{0}^{1} \sqrt[3]{\sin(t^4)} dt \). We have
\[
f(t) = \sqrt[3]{\sin(t^4)} \quad \text{and} \quad f'(t) = \frac{4t^3 \cos(t^4)}{3 \sqrt[3]{\sin(t^4)}},
\]
This mean that \( f'(t) \to \infty, \quad t \to 0 \) and we cannot apply the estimate (2.14). On the other hand, we get
\[
\int_{0}^{1} \left[ f'(t) \right]^2 dt \leq \frac{16}{9} \cdot \max_{t \in [0,1]} \frac{t^6 \cos(t^4)}{\sqrt[3]{\sin(t^4)}} \int_{0}^{1} \frac{dt}{\sqrt[3]{\sin(t^4)}} \leq \frac{64}{9},
\]
i.e. \( ||f'||_2 \leq \frac{8}{3} \) and we can apply the estimate (2.8).

Remark 3.2. Hence, although the estimate (2.14) is better than the estimate (2.8) the last mentioned estimate have their field of application. Furthermore, this is sharp and it is proved for a larger class of functions. An example that shows that the estimator (2.14) is better than (2.8) for a
type of function to which both results apply is the function \( f(t) = \exp(-t^2) \) for all \( t \in [-1, 1] \). We omit the proof and the details are left to the interested reader.

Conclusion

The problem of introducing quadrature rules was studied by many authors via theory of inequalities, where two famous real inequalities are the well-known Hermite–Hadamard and Ostrowski type inequalities and their modifications. The author hope that these results about 3-point Gauss–Legendre quadrature rule has large applications and may motivate further research in different areas of pure and applied sciences.

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