Maps preserving the square zero of $\eta$-Lie product of operators

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Abstract:

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. In this paper, we identify the form of the unital surjective additive map $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ which preserves the square zero of $\eta$-Lie product of operators for some scalar $\eta$ with $\eta \neq 0, 1, -1$.

Keywords: Preserver problem; Square zero operator; $\eta$-Lie product.


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1. Introduction

Let $A$ be a Banach algebra, $A, B \in A$ and $\eta$ be a scalar. The Lie product, $\eta$-Lie product and triple Jordan products are defined as $[A, B] = AB - BA$, $[A, B]_\eta = AB + \eta BA$ and $A * B = ABA$, respectively. In last decade, many mathematician research on the preserving problems. Specially, maps preserving a certain property of products were often considered, see [1 - 4], [6], [8] and [10 - 12]. We point to some of them close to our purpose.

Authors in [10], considered the maps that strongly preserve the $\eta$-Lie product, that is $\phi(A)\phi(P) + \eta\phi(P)\phi(A) = AP + \eta PA$, for every $A$, some idempotent $P$ and some scalar $\eta$. Author in [12], identified the forms of bijective maps preserving Lie products from a factor von Neumann algebra into another factor von Neumann algebra.

Let $B(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. In [4], authors characterized the form of unital surjective maps on $B(\mathcal{X})$ preserving the nonzero idempotency of product of operators in both directions. Also in [11], authors characterized the form of linear surjective maps on $B(\mathcal{X})$ preserving the nonzero idempotency of either products of operators or triple Jordan products of operators.

We say an operator $A \in B(\mathcal{X})$ is a square zero operator, when $A^2 = 0$. Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. In this paper, we identify the form of surjective additive map $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ such that $\phi(I) = I$ and preserves the square zero of $\eta$-Lie product of operators for some scalar $\eta$ with $\eta \neq 0, 1, -1$. The complete form of our main result is as following:

**Main Theorem.** Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. Let $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ be an unital surjective additive map which satisfies

$$[A, B]^2_\eta = 0 \Leftrightarrow [\phi(A), \phi(B)]^2_\eta = 0,$$

for every $A, B \in B(\mathcal{H})$ and for some scalar $\eta$ with $\eta \neq 0, 1, -1$. Then there exists either a bounded invertible linear or a conjugate linear operator $T : \mathcal{H} \to \mathcal{H}$ such that

$$\phi(A) = TAT^{-1} \quad \text{or} \quad \phi(A) = TA^*T^{-1}$$

for every $A \in B(\mathcal{H})$. 
2. Proofs

First we recall some notations. We denote by \( \mathcal{I}(\mathcal{H}) \) the set of all idempotent operators in \( \mathcal{B}(\mathcal{H}) \). For every nonzero \( x, y \in \mathcal{H} \), the symbol \( x \otimes y \) stands for the rank one linear operator on \( \mathcal{H} \) defined by \( (x \otimes y)z = \langle z, y \rangle x \) for any \( z \in \mathcal{H} \). Note that every rank one operator in \( \mathcal{B}(\mathcal{H}) \) can be written in this way.

The rank one operator \( x \otimes y \) is idempotent if and only if \( \langle x, y \rangle = 1 \).

Let \( P, Q \in \mathcal{B}(\mathcal{H}) \) be idempotent operators. We say that \( P \) and \( Q \) are orthogonal if and only if \( PQ = QP = 0 \).

**Proposition 2.1.** Let \( A \in \mathcal{B}(\mathcal{H}) \), \( x, y \in \mathcal{H} \) such that \( \langle x, y \rangle = 1 \) and let \( \eta \) be a scalar such that \( \eta \neq 0, 1, -1 \). Then \( [A, x \otimes y]_\eta^2 = 0 \) if and only if only one of the following statements occurs:

(i) \( Ax < Ax, y > = -\eta x < A^2 x, y > \) and \( Ax = -\eta x < Ax, y > \).

(ii) \( A^* y = 0 \).

**Proof.** Assume first that \( Ax < Ax, y > = -\eta x < A^2 x, y > \) and \( Ax = -\eta x < Ax, y > \). Hence

\[
[A, x \otimes y]_\eta^2 = (Ax \otimes y + \eta x \otimes A^* y)^2 \\
= \langle Ax, y \rangle Ax \otimes y + \eta Ax \otimes A^* y \\
+ \eta^2 < Ax, y > x \otimes A^* y + \eta < A^2 x, y > x \otimes y \\
= -\eta x < A^2 x, y > \otimes y - \eta^2 x < Ax, y > \otimes y A \\
+ \eta^2 < Ax, y > x \otimes A^* y + \eta < A^2 x, y > x \otimes y = 0.
\]

Now if \( A^* y = 0 \), then

\[
[A, x \otimes y]_\eta^2 = (Ax \otimes y + \eta x \otimes A^* y)^2 \\
= (Ax \otimes y)^2 = \langle Ax, y \rangle Ax \otimes y \\
= \langle x, A^* y \rangle Ax \otimes y = 0.
\]

Conversely, Assume that \( [A, x \otimes y]_\eta^2 = 0 \). It is clear that

\( B^2 = 0 \) if and only if \( B(Bx) = 0 \), \( \forall x \in X \) if and only if \( \text{Im} \ B \subseteq \ker \ B \).

This together with assumption implies

\( [A, x \otimes y]_\eta^2 = 0 \) if and only if \( \text{Im}[A, x \otimes y]_\eta \subseteq \ker[A, x \otimes y]_\eta \).
Let $A^* y \neq 0$. If $A^* y$ and $y$ are linearly independent, then In the following lemmas, assume that $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is an unital surjective additive map which satisfies $[A, B]_\eta^2 = 0 \iff [\phi(A), \phi(B)]_\eta^2 = 0$, for every $A, B \in \mathcal{B}(\mathcal{H})$ and for some scalar $\eta$ with $\eta \neq 0, 1, -1$.

**Lemma 2.2.** $\phi$ preserves the square zero operators in both directions.

**Proof.** Let $A \in \mathcal{B}(\mathcal{H})$. By assumptions we have

\[
A^2 = 0 \iff (1 + \eta)^2 A^2 = [A, I]_\eta^2 = 0 \\
\iff [\phi(A), I]_\eta^2 = 0 \\
\iff (1 + \eta)^2 \phi(A)^2 = 0 \\
\iff \phi(A)^2 = 0.
\]

The following theorem is a straightforward consequence of Theorem 2.1 in [7].

**Theorem 2.3.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a surjective additive map satisfying $\phi(I) = I$. Assume that $\phi$ preserves the square zero operators in both directions. Then $\phi$ is injective and preserves the idempotent operators in both directions.

**Lemma 2.4.** $\phi$ is injective and preserves the idempotent operators in both directions.

**Proof.** It is clear by assumptions and Theorem 2.3.

**Lemma 2.5.** There exists either a bounded invertible linear or a conjugate linear operator $T : \mathcal{H} \to \mathcal{H}$ such that

\[
\phi(P) = TPT^{-1}
\]

or

\[
\phi(P) = TP^*T^{-1}
\]

for every $P \in \mathcal{I}(\mathcal{H})$. 
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Proof. Since $\phi$ is additive and by Lemma 2.4 preserves idempotent operators in both directions, then $\phi$ preserves the orthogonality of idempotent operators in both directions. Thus we can obtain the form of $\phi$ on idempotents by Lemma 3.1 in [5].

Remark 2.6. Let $T$ be the same operator defined in Lemma 2.5. It is clear that $\Psi = T^{-1} \phi T : B(H) \to B(H)$ satisfies the assumptions on $\phi$. Therefore, without loss of generality we can assume that $\phi(P) = P$ or $\phi(P) = P^*$ for every $P \in \mathcal{I}(H)$.

Now we are in a position to prove our main result.

Proof of Main Theorem. Let $A \in B(H)$ such that $\ker A \neq 0$. Let $x \in \ker A$ be nonzero. Hence there exists a nonzero vector $y \in H$ such that $< x, y > = 1$. Let the first case of Lemma 2.5 occurs. So by Remark 2.6, $\phi(x \otimes y) = x \otimes y$. By $Ax = 0$ and Proposition 2.1 we infer that $[\phi(A), x \otimes y]_\eta = 0$ and by assumption

$$[\phi(A), \phi(x \otimes y)]_\eta^2 = [\phi(A), x \otimes y]_\eta^2 = 0.$$ Using again Proposition 2.1 implies

(1) $\phi(A)x < \phi(A)x, y > = -\eta x < \phi(A)^2 x, y >$

and

(2) $\phi(A)x = -\eta x < \phi(A)x, y >$

or $\phi(A)^*y = 0$. We assert that $\phi(A)x = 0$. We assume on the contrary that $\phi(A)x \neq 0$. Let us first assume that (1) and (2) occur.

Thus

$$-\eta x < \phi(A)x, y >^2 = -\eta x < \phi(A)^2 x, y >$$

and since $\eta \neq 0$, $< \phi(A)x, y >^2 = < \phi(A)^2 x, y >$. It easily follows that $x$, $\phi(A)x$ and $\phi(A)^2 x$ are linearly dependent, because otherwise, there exists a vector $y$ such that $< x, y > = 1$ and $< \phi(A)x, y >^2 \neq < \phi(A)^2 x, y >$.

If $x$ and $\phi(A)x$ are linearly dependent, then $\phi(A)x = \alpha x$ for some nonzero scalar $\alpha$. From (2) we obtain $\alpha x < x, y > = -\eta \alpha x < x, y >$ which implies that $\eta = -1$, that is a contradiction. If $x$ and $\phi(A)x$ are linearly independent, then we conclude that $\phi(A)^2 x \in \text{span}\{\phi(A)x, x\}$ and so $\phi(A)^2 x = \alpha \phi(A)x + \beta x$ for some scalars $\alpha, \beta$. It implies that $< \phi(A)^2 x, y > = \alpha < \phi(A)x, y > + \beta$. According to (1)

$$\phi(A)x < \phi(A)x, y > = -\eta x (\alpha < \phi(A)x, y > + \beta).$$
Since $x$ and $\phi(A)x$ are linearly independent, $<\phi(A)x, y> = 0$ which by (2) implies, $<x, y> = 0$, a contradiction.

Now let $\phi(A)^*y = 0$. Since we assume that $\phi(A)x \neq 0$, there exists a vector $y$ such that $<x, y> = 1$ and $<\phi(A)x, y> \neq 0$. This implies $<x, \phi(A)^*y> \neq 0$. It is a contradiction, because $\phi(A)^*y = 0$. The proof of assertion is completed and so $\ker A \subseteq \ker \phi(A)$, when $\ker A \neq 0$. This implies that if $\ker A \neq 0$, then $\ker \phi(A) \neq 0$ and this with a similar discussion as above yields that $\ker \phi(A) \subseteq \ker A$. Therefore, $\ker \phi(A) = \ker A$ for every operator $A$ such that $\ker A \neq 0$. Moreover, this implies that $\ker A \neq 0$ if and only if $\ker \phi(A) \neq 0$ which yields that $\ker A = 0$ if and only if $\ker \phi(A) = 0$. Hence $\ker \phi(A) = \ker A$ for every operator $A$ and so $F(\phi(A)) = F(A)$ and since $\phi$ is additive, $F(\phi(A) + \phi(B)) = F(A + B)$ for every $A, B \in \mathcal{B}(H)$. The form of such $\phi$ has been given in [9], Theorem 3.5. By this theorem, $\phi(A) = U_A + R$ such that $U = I - 2\phi(0)$ and $R = \phi(0)$. Since $\phi$ is additive, $\phi(A) = A$. With a similar discussion, we obtain $\phi(A) = A^*$, when the second case in Lemma 2.5 occurs. These together with Remark 2.6 complete the proof.

References


