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The t-pebbling number of Lamp graphs

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Abstract

Let G be a graph and some pebbles are distributed on its vertices. A pebbling move (step) consists of removing two pebbles from one vertex, throwing one pebble away, and moving the other pebble to an adjacent vertex. The t-pebbling number of a graph G is the least integer m such that from any distribution of m pebbles on the vertices of G, we can move t pebbles to any specified vertex by a sequence of pebbling moves. In this paper, we determine the t-pebbling number of Lamp graphs.

Keywords : *Pebbling number, t-Pebbling number, Lamp graphs.*

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1. Introduction

Pebbling in graphs was first considered by Chung [1]. Graph Pebbling is a network optimization model for the transportation of resources that are consumed in transit. The central problem in this model asks whether discrete pebbles from one set of vertices can be moved to another while pebbles are lost in the process. The graph pebbling model was born as a method for solving a combinatorial number theory conjecture of Paul Erdős [2, 3]. Lemke [4] has given different version of the conjecture. The conjecture has got applications to problems in combinatorial group theory and p-adic diophantine equations. Here, the term graph refers to a simple graph. A *configuration* C of pebbles on a graph $G = (V, E)$ can be thought of as a function $C : V(G) \to N \cup \{0\}$. The value $C(v)$ equals the number of pebbles placed at vertex v, and the *size* of the configuration is the number $|C| = \sum_{v \in V(G)} C(v)$ of pebbles placed in total on G. Suppose C is a configuration of pebbles on a graph G. A *pebbling move (step)* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex v, the *target vertex*, if we can apply pebbling moves repeatedly (if necessary) so that in the resulting configuration the vertex v has at least one pebble.

Definition 1.1. *[5] The pebbling number of a vertex v in a graph G, $f(v, G)$, is the smallest positive integer m such that however m pebbles are placed on the vertices of the graph, a pebble can be moved to v in finite number of pebbling moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. The pebbling number of G, $f(G)$, is defined to be the maximum of the pebbling numbers of its vertices.*

Thus the *pebbling number of a graph* G, $f(G)$, is the least m such that, for any configuration of m pebbles to the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex.

Definition 1.2. *[5] The t-pebbling number of a vertex v in a graph G, $f_t(v, G)$, is the smallest positive integer m such that however m pebbles are placed on the vertices of the graph, t pebbles can be moved to v in finite number of pebbling moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. The t-pebbling number of*

G, $f_t(G)$, is defined to be the maximum of the pebbling numbers of its vertices.

Thus the *t-pebbling number of a graph* G, $f_t(G)$, is the least m such that, for any configuration of m pebbles to the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex.

FIGURE 1. An illustration of moving one pebble to the end vertex of the path P_5 from a configuration of size 9.

Fact 1.3. *[12] For any vertex v of a graph G, $f(v,G) \geq n$ where $n = |V(G)|$.*

Fact 1.4. *[12] The pebbling number of a graph G satisfies*

$$f(G) \geq max\{2^{diam(G)}, |V(G)|\}.$$

Now, we state the known pebbling results of the Jahangir graph $J_{2,m}$ which will be used to prove the results of Section 2.

Definition 1.5. *[11] Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm+1$ vertices, consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.*

The pebbling number of Jahangir graph $J_{2,m}$ ($m \geq 3$) is as follows:

Theorem 1.6. *For the Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$.*

Theorem 1.7. *For the Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.*

Theorem 1.8. *For the Jahangir graph $J_{2,m}$ where $m \geq 8$, $f(J_{2,m}) = 2m+10$.*

Lourdusamy et al. determined the t-pebbling number of Jahangir graph $J_{3,m}$ (for $m \geq 3$) in [5]. And also they determined the t-pebbling number for squares of cycles ($t \geq 2$) in [6] and for some wheel related graphs in [8].

Remark 1.9. *Consider a graph G with n vertices and $f(G)$ pebbles are placed on its vertices. Suppose we choose a target vertex v from G to put a pebble on it. If $C(v) \geq 1$ or $C(u) \geq 2$ where $uv \in E(G)$, then we can move one pebble to v easily. So, we always assume that $C(v) = 0$ and $C(u) \leq 1$ for all $uv \in E(G)$ when v is the target vertex.*

Lemma 1. *If H is a spanning connected subgraph of G, then $f_t(G) \leq f_t(H)$.*

Proof. Let x be the t-pebbling number of the graph H. Obviously, G may or may not contain some more edges than H. Clearly, adding the remaining edges of G to H will not increase the t-pebbling number of G. Thus $f_t(G) \leq x = f_t(H)$. □

In the next section, we define the Lamp graph and then determine its t-pebbling number.

2. The t-pebbling number of Lamp graphs

Definition 2.1. *The join $G + H$ of two graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.*

Definition 2.2. *The wheel W_n is defined as the join $C_n + K_1$. The vertex K_1 is the apex vertex and the vertices on the underlying cycle are called rim vertices. The edges of the underlying cycle are called the rim edges and the edges joining the apex and the rim vertices are called spoke edges.*

Definition 2.3. *The Lamp graph is obtained from the wheel graph W_n ($n \geq 3$) by adding a new vertex corresponding to each rim edge of the wheel and make the new vertex adjacent to end vertices of corresponding rim edge. We denote the Lamp graph by L_n.*

Labeling for L_n: Let $n \geq 3$. Let v_{2n+1} be the label of the center vertex and then label the other vertices of L_n by $v_1, v_2, \cdots, v_{2n}$ such that $deg(v_1) = 5$, $deg(v_2) = 2$, $\cdots$ $deg(v_{2n-1}) = 5$, $deg(v_{2n}) = 2$. An illustration of the Lamp graph L_8 is shown in Figure 2.

We define the sets $S_1 = \{v_1, v_3, \cdots, v_{2n-1}\}$ and $S_2 = \{v_2, v_4, \cdots, v_{2n}\}$ from the labeling of L_n.

Theorem 2.4. *For L_3, the t-pebbling number is $f_t(L_3) = 4(t-1) + 8$.*

Proof. Let $C(v_4) = 4(t-1) + 3$, $C(v_6) = 3$, $C(v_7) = 1$ and $C(v_i) = 0$ for all $i \neq 4, 6, 7$. Then we cannot move t pebbles to v_2. Thus $f_t(L_3) \geq 4(t-1) + 8$.

Since $J_{2,3}$ is a spanning subgraph of L_3 and by Lemma 1, we have $f(L_3) \leq f(J_{2,3}) = 8$ (Theorem **??**). So the result is true for $t = 1$. Assume the result is true for $t' \geq 2$. Now consider the distribution of $4(t-1) + 8$ pebbles on the vertices of L_3. Clearly, we can move one pebble to any target vertex v_i at a cost of at most four pebbles, since $C(L_3) \geq 12$ and $f(L_3) = 8$. After moving one pebble to v_i, the remaining number of pebbles on the vertices of L_3 is at least $4(t-2)+8$. Hence we can move the additional $t - 1$ pebbles to v_i, by induction. Thus $f_t(L_3) \leq 4(t-1) + 8$. $\square$

Theorem 2.5. *For L_4, the t-pebbling number is $f_t(L_4) = 8(t-1) + 11$.*

Proof. Let $C(v_6) = 8(t-1) + 7$, $C(v_4) = C(v_8) = C(v_9) = 1$ and $C(v_i) = 0$ for all $i \neq 4, 6, 8, 9$. Then we cannot move t pebbles to v_2. Thus $f_t(L_4) \geq 8(t-1) + 11$.

We have three cases to prove $f(L_4) \leq 11$. Consider the distribution of 11 pebbles on the vertices of L_4.

Case 1: Let v_9 be the target vertex.
Clearly, $C(v_9) = 0$ and $C(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.9. Since, $C(S_2) \geq 7$, there exists a vertex, say v_2, such that $C(v_2) \geq 2$. If $C(v_1) = 1$ or $C(v_3) = 1$ or $C(v_4) \geq 2$ or $C(v_8) \geq 2$ then we can move one pebble to v_9 easily. Assume $C(v_1) = 0$, $C(v_3) = 0$, $C(v_4) \leq 1$, and $C(v_8) \leq 1$. Now, we can move one pebble to v_9 easily, since $C(v_2) \geq 4$ or $C(v_6) \geq 4$.

Case 2: Let v_1 be the target vertex.
Clearly, $C(v_1) = 0$, $C(v_2) \leq 1$, $C(v_3) \leq 1$, $C(v_7) \leq 1$, $C(v_8) \leq 1$ and $C(v_9) \leq 1$, by Remark 1.9. Let $C(v_5) \geq 2$. If $C(v_5) \geq 4$, then clearly we are done. If $C(v_5) = 2$ or 3 then either $C(v_4) \geq 2$ or $C(v_6) \geq 2$. So, we can move two pebbles to v_3 or v_7 and hence one pebble can be moved to v_1. Assume $C(v_5) \leq 1$. Without loss of generality, we assume $C(v_4) \geq 3$. If $C(v_4) \geq 4$ then clearly we are done. Assume $C(v_4) = 3$ and hence $C(v_6) \geq 2$. Clearly we are done if $C(v_5) = 1$ or $C(v_7) = 1$. Assume $C(v_5) = C(v_7) = 0$ and hence we get $C(v_6) \geq 4$. Thus we can move one pebble to v_1 easily.

Case 3: Let v_2 be the target vertex.
Clearly, $C(v_2) = 0$, $C(v_1) \leq 1$ and $C(v_3) \leq 1$, by Remark 1.9. Let $C(v_4) \geq 2$. If $C(v_4) \geq 4$ then clearly we are done. Assume $C(v_4) = 2$ or 3 then clearly $C(v_3) = 0$, $C(v_5) \leq 1$ and $C(v_9) \leq 1$ (otherwise, we can move one pebble to v_2). If $C(v_7) \geq 4$ or $C(v_8) \geq 4$ or $C(v_7) \geq 2$ and $C(v_8) \geq 2$ then we can move two pebbles to v_1 and then we move one pebble to v_2 from v_1. So, we assume $C(v_1) + C(v_7) + C(v_8) \leq 4$ such that we cannot move one pebble to v_2. This implies that $C(v_6) \geq 2$. Clearly we are done if $C(v_5) = 1$. Let $C(v_7) \geq 2$. Thus we can move one pebble to v_1 from v_7. If $C(v_9) = 1$ then we can move another one pebble to v_1 using the pebbles

at v_4 and v_6. If $C(v_9) = 0$ then we get $C(v_6) \geq 4$ and hence we can move another one pebble to v_1 using the pebbles at v_6. Assume $C(v_7) \leq 1$. In a similar way, we assume $C(v_8) \leq 1$. This implies that $C(v_6) \geq 3$. Clearly we are done if $C(v_5) = 1$. Otherwise, we get $C(v_6) \geq 4$ and hence we can move two pebbles to v_3 from v_6 and v_4. Thus we can move one pebble to v_2.

Assume $C(v_4) \leq 1$. Similarly, we assume $C(v_8) \leq 1$. Assume $C(v_9) \geq 2$. Clearly, we are done if $C(v_9) \geq 4$ or $C(v_3) = 1$ or $C(v_1) = 1$. Assume $C(v_3) = 0$ or $C(v_1) = 0$. Let $C(v_9) = 2$ or 3. Since either $C(v_6)+C(v_5) \geq 4$ or $C(v_6) + C(v_7) \geq 4$, we can move one pebble to either v_3 or v_1. Since $C(v_9) \geq 2$, we can move another one pebble to v_1 or v_3 and hence we can move one pebble to v_2. So, we assume $C(v_9) \leq 1$. Also, we must have $C(v_5) \leq 1$ and $C(v_7) \leq 1$ (otherwise one pebble could be moved to v_2). Thus we have $C(v_6) \geq 4$. Clearly, we are done if $C(v_1) = 1$ or $C(v_3) = 1$. Assume $C(v_1) = 0$ and $C(v_3) = 0$. Thus we get $C(v_6) \geq 6$. Clearly, we are done if $C(v_5) = 1$ or $C(v_7) = 1$. Assume $C(v_5) = 0$ and $C(v_7) = 0$. Now, we have $C(v_6) \geq 8$ and hence we can move one pebble to v_2 easily.

So, the result is true for $t = 1$. Assume the result is true for $t' \geq 2$. Now consider the distribution of $8(t-1) + 11$ pebbles on the vertices of L_4. Clearly, we can move one pebble to any target vertex v_i at a cost of at most eight pebbles, since $C(L_4) \geq 19$ and $f(L_4) = 11$. After moving one pebble to v_i, the remaining number of pebbles on the vertices of L_4 is at least $8(t-2) + 11$. Hence we can move the additional $t-1$ pebbles to v_i, by induction. Thus $f_t(L_4) \leq 8(t-1) + 11$. □

Theorem 2.6. *For L_5, the t-pebbling number is $f_t(L_5) = 8(t-1) + 13$.*

Proof. Let $C(v_8) = 8(t-1) + 7$, $C(v_4) = C(v_5) = C(v_6) = C(v_{10}) = C(v_{11}) = 1$ and $C(v_i) = 0$ for all $i \neq 4, 5, 6, 8, 10, 11$. Then we cannot move t pebbles to v_2. Thus $f_t(L_5) \geq 8(t-1) + 13$.

We have three cases to prove $f(L_5) \leq 13$. Consider the distribution of 13 pebbles on the vertices of L_5.

Case 1: Let v_{11} be the target vertex.

Clearly, $C(v_{11}) = 0$ and $C(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.9. Since, $C(S_2) \geq 8$, there exists a vertex, say v_2, such that $C(v_2) \geq 2$. Clearly, we are done if $C(v_2) \geq 4$. If $C(v_1) = 1$ or $C(v_3) = 1$ or $C(v_4) \geq 2$ or $C(v_{10}) \geq 2$ then we can move one pebble to v_{11} easily. Assume $C(v_1) = 0$,

$C(v_3) = 0$, $C(v_4) \leq 1$, and $C(v_{10}) \leq 1$. Now, we can move one pebble to v_{11} easily, since $C(v_6) \geq 4$ or $C(v_8) \geq 4$.

Case 2: Let v_1 be the target vertex.
Clearly, $C(v_1) = 0$, $C(v_2) \leq 1$, $C(v_3) \leq 1$, $C(v_9) \leq 1$, $C(v_{10}) \leq 1$ and $C(v_{11}) \leq 1$, by Remark 1.9. Let $C(v_5) \geq 2$. If $C(v_5) \geq 4$, then clearly we are done. Assume $C(v_5) = 2$ or 3. If $C(v_3) = 1$ or $C(v_{11}) = 1$ or $C(v_4) \geq 2$ or $C(v_7) \geq 2$ then we can move one pebble to v_1 easily. So, we assume $C(v_3) = 0$ or $C(v_{11}) = 0$ or $C(v_4) \geq 1$ or $C(v_7) \geq 1$. Then we have $C(v_6)+C(v_8) \geq 5$. Clearly we can move one pebble to v_{11} using the pebbles at v_6 and v_8 and also we move one more pebble to v_{11} from v_5. Thus we can move one pebble to v_1 through v_{11}. Assume $C(v_5) \leq 1$. In a similar way, we assume $C(v_7) \leq 1$. Now, we have $C(v_4) + C(v_6) + C(v_8) \geq 6$. Assume $C(v_4) \geq 2$. Clearly, we are done if $C(v_4) \geq 4$. Let $C(v_4) = 2$ or 3. Clearly, we are done if $C(v_3) = 1$ or both $C(v_5) = C(v_{11}) = 1$; if not then we have either $C(v_6) \geq 4$ or $C(v_8) \geq 4$. In either cases, we can move one pebble to v_1. So, we assume $C(v_4) \leq 1$. In a similar way, we assume that $C(v_7) \leq 1$. Thus we have $C(v_6) \geq 4$. Clearly, we are done if $C(v_3) = 1$ or $C(v_9) = 1$ or $C(v_{11}) = 1$. Assume $C(v_3) = C(v_9) = C(v_{11}) = 0$. Now we have $C(v_6) \geq 7$. We are done if $C(v_5) = 1$ or $C(v_7) = 1$. If not, then we get $C(v_6) \geq 8$ and hence we can move one pebble to v_1.

Case 3: Let v_2 be the target vertex.
Clearly, $C(v_2) = 0$, $C(v_1) \leq 1$ and $C(v_3) \leq 1$, by Remark 1.9. Let $C(v_4) \geq 2$. If $C(v_4) \geq 4$ then clearly we are done. Assume $C(v_4) = 2$ or 3 then clearly $C(v_3) = 0$, $C(v_5) \leq 1$ and $C(v_{11}) \leq 1$ (otherwise, we can move one pebble to v_2). If $C(v_6) \geq 4$ or $C(v_7) \geq 4$ or both $C(v_6) \geq 2$ and $C(v_7) \geq 2$ then we can move two pebbles to v_3 and then we move one pebble to v_2 from v_3. So, we assume $C(v_6) + C(v_7) \leq 4$ such that we cannot move one pebble to v_3. Let $C(v_6) \geq 2$. Clearly we are done if $C(v_5) = 1$ or both $C(v_7) = 1$ and $C(v_{11}) = 1$. Otherwise, we get $C(v_8) + C(v_9) + C(v_{10}) \geq 5$. Clearly, we are done if $C(v_9) \geq 4$ or $C(v_{10}) \geq 4$ or both $C(v_9) \geq 2$ and $C(v_{10}) \geq 2$. Assume $C(v_{10}) \geq 2$. If $C(v_1) = 1$ then we can move one pebble to v_2 easily. Let $C(v_1) = 0$. Assume $C(v_7) = 1$ and $C(v_{11}) = 0$. We can move one pebble to v_9. Clearly we can move one pebble to v_1 through v_9, since either $C(v_9) = 1$ or $C(v_8) \geq 2$ and hence one pebble could be moved to v_2. Now, assume $C(v_7) = 0$ and $C(v_{11}) = 1$. Clearly we can move one pebble to v_{11} from the pebbles at the vertices v_6 and v_8 and hence one pebble could be moved to v_1. Then we can move one pebble to v_2 easily.

Assume $C(v_7) = C(v_{11}) = 0$. Now we have $C(v_8) + C(v_9) \geq 4$ and hence we can move one pebble to v_1. Thus we can move one pebble to v_2 easily. Assume $C(v_{10}) \leq 1$. In a similar way, we assume that $C(v_9) \leq 1$. Clearly, $C(v_8) \geq 3$. We can move one pebble to v_{11} or v_1. Thus we are done if $C(v_1) = 1$ or $C(v_{11}) = 1$. Otherwise, we get $C(v_8) \geq 4$. Thus we move one pebble to v_5 from v_8 and hence we are done. Thus we can move one pebble to v_2 when $C(v_6) \geq 2$. So we assume that $C(v_6) \leq 1$. In a similar way, we could assume that $C(v_7) \leq 1$. Again we get $C(v_8) + C(v_9) + C(v_{10}) \geq 5$. By our previous discussions, we could send one pebble to v_1 or v_{11} easily. Clearly, we are done if $C(v_1) = 1$ or $C(v_{11}) = 1$. Assume $C(v_1) = 0$ and $C(v_{11}) = 0$. Either we can move one pebble to v_3 or two pebbles to v_1 from the remaining distributions when $C(v_4) = 2$ or 3. So, we assume that $C(v_4) \leq 1$. In a similar way, we can assume that $C(v_{10}) \leq 1$, $C(v_5) \leq 1$ and $C(v_9) \leq 1$. Assume $C(v_{11}) \geq 2$. We can move one pebble to v_2 easily, if $C(v_{11}) = 2$ or 3 or $C(v_{11}) \geq 4$. Assume $C(v_{11}) \leq 1$. Now, we have $C(v_8) + C(v_9) \geq 5$. Clearly, we can move one pebble to v_2 if $C(v_1) = 1$ or both $C(v_{11}) = 1$ and $C(v_3) = 1$. If not, we have $C(v_8) + C(v_9) \geq 7$. We can move one pebble to v_2 easily except the distribution $C(v_8) = 7$ and $C(v_9) = 0$. We have either $C(v_7) = C(v_{11}) = 1$ or $C(v_7) = C(v_3) = 1$. Hence we can move one pebble to v_2 for this distribution too. Thus we can always move one pebble to v_2.

So, the result is true for $t = 1$. Assume the result is true for $t' \geq 2$. Now consider the distribution of $8(t-1) + 13$ pebbles on the vertices of L_5. Clearly, we can move one pebble to any target vertex v_i at a cost of at most eight pebbles, since $C(L_5) \geq 21$ and $f(L_5) = 13$. After moving one pebble to v_i, the remaining number of pebbles on the vertices of L_5 is at least $8(t-2) + 13$. Hence we can move the additional $t - 1$ pebbles to v_i, by induction. Thus $f_t(L_5) \leq 8(t-1) + 13$. □

Theorem 2.7. *For L_6, the t-pebbling number is $f_t(L_6) = 16(t-1) + 20$.*

Proof. Let $C(v_8) = 16(t-1)+15$, $C(v_4) = C(v_6) = C(v_{10}) = C(v_{12}) = 1$ and $C(v_i) = 0$ for all $i \neq 4, 6, 8, 10, 12$. Then we cannot move t pebbles to v_2. Thus $f_t(L_6) \geq 16(t-1) + 20$. We have three cases to prove $f(L_6) \leq 20$. Consider the distribution of 20 pebbles on the vertices of L_6.

Case 1: Let v_{13} be the target vertex.

Clearly, $C(v_{13}) = 0$ and $C(v_i) \leq 1$ for all $v_i \in S_1$ by Remark 1.9. Since, $C(S_2) \geq 14$, there exists a vertex, say v_2, such that $C(v_2) \geq 3$. Clearly, we are done if $C(v_2) \geq 4$. If $C(v_1) = 1$ or $C(v_3) = 1$ or $C(v_4) \geq 2$ or $C(v_{12}) \geq 2$ then we can move one pebble to v_{13} easily. Assume $C(v_1) = 0$, $C(v_3) = 0$, $C(v_4) \leq 1$, and $C(v_{12}) \leq 1$. Now, we can move one pebble to v_{13} easily, since $C(v_6) \geq 4$ or $C(v_8) \geq 4$ or $C(v_{10}) \geq 4$.

Case 2: Let v_1 be the target vertex.

Clearly, $C(v_1) = 0$ and $C(v_i) \leq 1$ for all $i \in \{2, 3, 11, 12, 13\}$ by Remark 1.9. Let $C(v_5) \geq 2$. If $C(v_3) = 1$ or $C(v_{13}) = 1$ or a vertex of $S_1 - \{v_1, v_3, v_{11}\}$ has more than one pebble then we can move one pebble to v_1 easily. Otherwise, there exists a vertex, say v_6, of $S_2 - \{v_2, v_{12}\}$, contains more than three pebbles and hence we are done. Assume $C(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3, v_{11}\}$. Clearly, $S_2 - \{v_2, v_{12}\} \geq 13$, and so we can move two pebbles to v_{13} and hence we are done.

Case 3: Let v_2 be the target vertex.
Clearly, $C(v_2) = 0$, $C(v_1) \leq 1$ and $C(v_3) \leq 1$ by Remark 1.9. If $C(v_5) \geq 4$ then clearly we are done. So, we assume that $C(v_5) \leq 3$. Similarly, we assume that $C(v_{11}) \leq 3$. Let $C(v_7) \geq 4$. If $C(v_1) = 1$ or $C(v_3) = 1$ or $C(v_4) \geq 2$ or $C(v_5) \geq 2$ or $C(v_{11}) \geq 2$ or $C(v_{12}) \geq 2$ or $C(v_{13}) \geq 2$ then we can move one pebble to v_2 easily. Assume that $C(v_1) = 0$, $C(v_3) = 0$, $C(v_4) \leq 1$, $C(v_5) \leq 1$, $C(v_{11}) \leq 1$, $C(v_{12}) \leq 1$ and $C(v_{13}) \leq 1$. Also we can move one pebble to v_2 if $C(v_9) \geq 4$. So, we assume $C(v_9) \leq 3$. Let $C(v_7) = 6$ or 7. Clearly we are done if $C(v_9) \geq 2$ or $C(v_6) \geq 2$ or $C(v_{13}) = 1$ or $C(v_5) = 1$. So, we assume that $C(v_9) \leq 1$, $C(v_6) \leq 1$, $C(v_{13}) = 0$ and $C(v_5) = 0$. This implies that, we have both $C(v_8) \geq 2$ and $C(v_{10}) \geq 2$ and hence we can move one pebble to v_2 by moving four pebbles to v_{13} using the pebbles at the vertices v_7, v_8 and v_{10}. Let $C(v_7) = 4$ or 5. Assume $C(v_9) = 2$ or 3. Clearly we are done if $C(v_{11}) = 1$ or $p(v_{13}) = 1$. Now, we have $C(v_6) + C(v_8) + C(v_{10}) \geq 10$. Clearly, we are done if $C(v_6) \geq 4$ or $C(v_{10}) \geq 4$. If not, then we have $C(v_8) \geq 4$. If $C(v_6) \geq 2$, then we move one pebble to v_5 and then we move three more pebbles to v_5 from the vertices v_7 and v_8 and hence we are done. So, we assume that $C(v_6) \leq 1$ and also $C(v_{10}) \leq 1$ in a similar way. Thus we have $C(v_8) \geq 8$ and hence we can one pebble to v_2 by moving four pebbles to v_{13} using the pebbles at the vertices v_7, v_8. Now we assume that $C(v_7) \leq 3$. In a similar way, we assume that $C(v_9) \leq 3$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more

pebbles each then clearly we can move four pebbles to v_{13} and hence one pebble can be moved to v_2 from v_{13}.

Subcase 3.1: Three vertices of $S_1-\{v_1,v_3\}$ have two or more pebbles each.

Clearly we are done if $C(v_1)=1$ or $C(v_3)=1$ or $C(v_{13})=1$ or $C(v_4)\geq 2$ or $C(v_5)\geq 2$ or $C(v_{11})\geq 2$ or $C(v_{12})\geq 2$. Assume $C(v_1)=C(v_3)=C(v_{13})=0$ and $C(v_4)\leq 1$, $C(v_5)\geq 2$, $C(v_{11})\geq 2$, and $C(v_{12})\leq 1$. Clearly, $C(v_6)+C(v_8)+C(v_{10})\geq 9$ and hence we can move one pebble to v_{13} from the vertices v_6, v_8 and v_{10}. Thus we can move one pebble to v_2 using the pebbles at the three vertices of $S_1-\{v_1,v_3\}$.

Subcase 3.2: Two vertices of $S_1-\{v_1,v_3\}$ have two or more pebbles each. Clearly, we are done if $C(v_1)=1$ or $C(v_3)=1$ or $C(v_4)\geq 2$ or $C(v_5)\geq 2$ or $C(v_{11})\geq 2$ or $C(v_{12})\geq 2$ or $C(v_{13})\geq 2$. Let $C(v_{13})=1$ and so we can move three pebbles to v_{13} from the two vertices of $S_1-\{v_1,v_3\}$ and v_6, v_8 and v_{10}. Assume $C(v_{13})=0$ and so $C(v_6)+C(v_8)+C(v_{10})\geq 11$. Thus we can move two pebbles to v_{13} from the vertices v_6, v_8 and v_{10} and then we move two more pebbles to v_{13} from the two vertices of $S_1-\{v_1,v_3\}$ and hence we are done.

Subcase 3.3: One vertex of $S_1-\{v_1,v_3\}$ has two or more pebbles. Clearly, we are done if $C(v_1)=1$ or $C(v_3)=1$ or $C(v_4)\geq 2$ or $C(v_5)\geq 2$ or $C(v_{11})\geq 2$ or $C(v_{12})\geq 2$ or $C(v_{13})\geq 2$. Let $C(v_{13})=1$ and so $C(v_6)+C(v_8)+C(v_{10})\geq 12$. Thus we can move three pebbles to v_{13} from the vertex of $S_1-\{v_1,v_3\}$ and the vertices v_6, v_8 and v_{10}. Assume $C(v_{13})=0$ and let v_5 is the vertex of $S_1-\{v_1,v_3\}$ contains more than one pebble on it. So $C(v_6)+C(v_8)+C(v_{10})\geq 13$. If $C(v_7)=1$ or $C(v_9)=1$ then we can move three pebbles to v_{13} from v_6, v_8 and v_{10} and hence we are done since $C(v_5)\geq 2$. Assume $C(v_7)=C(v_9)=0$ and so we can move three pebbles to v_{13} from v_6, v_8 and v_{10} and hence we are done. In a similar way, we can move one pebble to v_2 if $C(v_{11})\geq 2$, $C(v_7)\geq 2$ and $C(v_9)\geq 2$.

Subcase 3.4: No vertex of $S_1-\{v_1,v_3\}$ has two or more pebbles. Clearly, we are done if $C(v_1)=1$ or $C(v_3)=1$ or $C(v_4)\geq 2$ or $C(v_5)\geq 2$ or $C(v_{11})\geq 2$ or $C(v_{12})\geq 2$ or $C(v_{13})\geq 2$. Thus we have $C(v_6)+C(v_8)+C(v_{10})\geq 14$. Let $C(v_{13})=1$. Clearly we can move three pebbles to v_{13} if $C(v_7)=1$ or $C(v_9)=1$. Assume $C(v_7)=C(v_9)=0$ and so we can move three pebbles to v_{13} since $C(v_6)+C(v_8)+C(v_{10}) \geq 15$ and hence we are done. Assume $C(v_{13}) = 0$. Without loss of generality, we let $C(v_6) \geq 5$. If $C(v_4) = 1$ or $C(v_5) = 1$ or $C(v_7) = 1$ then we can move two pebbles to v_3 and hence we are done. Assume $C(v_4) = C(v_5) = C(v_7) = 0$. Let $C(v_8) \geq 2$. If $C(v_9) = 1$ then we move one pebble to v_{13} and then we move another three pebbles to v_{13} from v_6, v_8 and v_{10} since $C(v_6)+C(v_8)+C(v_{10})-2 \geq 16$ and hence we are done. Assume $C(v_9) = 0$ and so $C(v_6)+C(v_8)+C(v_{10}) \geq 20$. Clearly we can move one pebble to v_2 from v_6, v_8 and v_{10}. So, the result is true for $t = 1$. Assume the result is true for $t' \geq 2$. Now consider the distribution of $16(t-1)+20$ pebbles on the vertices of L_6. Clearly, we can move one pebble to any target vertex v_i at a cost of at most sixteen pebbles, since $C(L_6) \geq 36$ and $f(L_6) = 20$. After moving one pebble to v_i, the remaining number of pebbles on the vertices of L_6 is at least $16(t-2)+20$. Hence we can move the additional $t-1$ pebbles to v_i, by induction. Thus $f_t(L_6) \leq 16(t-1)+20$. □

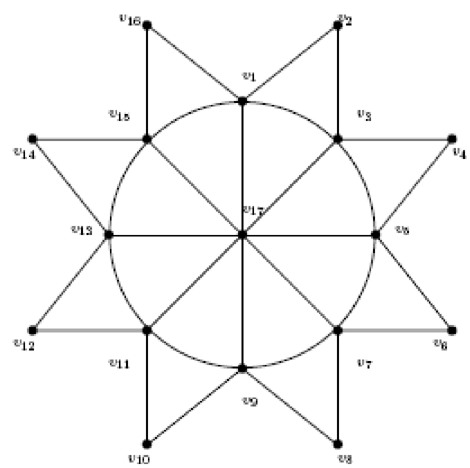

FIGURE 2. The Lamp graph L_8.

Theorem 2.8. *For L_7, the t-pebbling number is $f_t(L_7) = 16(t-1)+23$.*

Proof. Let $C(v_8) = 16(t-1)+15$, $C(v_4) = C(v_6) = C(v_{10}) = C(v_{14}) = 1$, $C(v_{12}) = 3$ and $C(v_i) = 0$ for all $i \neq 4, 6, 8, 10, 12, 14$. Then we cannot move t pebbles to v_2. Thus $f_t(L_7) \geq 16(t-1)+23$. Since $J_{2,7}$ is a spanning subgraph of L_7 and by Lemma 1, we have $f(L_7) \leq f(J_{2,7}) = 23$ (Theorem

1.7). So the result is true for $t = 1$. Assume the result is true for $t' \geq 2$. Now consider the distribution of $16(t-1) + 23$ pebbles on the vertices of L_7. Clearly, we can move one pebble to any target vertex v_i at a cost of at most sixteen pebbles, since $C(L_7) \geq 39$ and $f(L_7) = 23$. After moving one pebble to v_i, the remaining number of pebbles on the vertices of L_7 is at least $16(t-2) + 23$. Hence we can move the additional $t-1$ pebbles to v_i, by induction. Thus $f_t(L_7) \leq 16(t-1) + 23$. □

Theorem 2.9. *For L_n ($n \geq 8$), the t-pebbling number is*
$f_t(L_n) = 16(t-1) + 2n + 10.$

Proof. Consider the following distribution for L_n ($n \geq 8$):
If n is odd, let $C(v_{n+1}) = 16(t-1) + 15$, $C(v_{n-3}) = 3$, $C(v_{n+5}) = 3$, $C(x) = 1$ where $x \notin N[v_2]$, $x \notin N[v_{n+1}]$, $x \notin N[v_{n-3}]$, and $x \notin N[v_{n+5}]$, and $C(y) = 0$ where $y \in N[v_2]$, $y \in N(v_{n+1})$, $y \in N(v_{n-3})$, and $y \in N(v_{n+5})$.
If n is even, let $C(v_{n+2}) = 16(t-1) + 15$, $C(v_{n-2}) = 3$, $C(v_{n+6}) = 3$, $C(x) = 1$ where $x \notin N[v_2]$, $x \notin N[v_{n+2}]$, $x \notin N[v_{n-2}]$, and $x \notin N[v_{n+6}]$, and $C(y) = 0$ where $y \in N[v_2]$, $y \in N(v_{n+2})$, $y \in N(v_{n-2})$, and $y \in N(v_{n+6})$.
Then, we cannot move a pebble to v_2. The total number of pebbles placed in both configurations are $16(t-1) + 15 + 2(3) + (n-4)(1) + (n-8)(1) = 16(t-1) + 2n + 9$. Therefore, $f_t(L_n) \geq 16(t-1) + 2n + 10$.

Since $J_{2,n}$ is a spanning subgraph of L_n ($n \geq 8$) and by Lemma 1, we have $f(L_n) \leq f(J_{2,n}) = 2n + 10$ (Theorem **??**). So the result is true for $t = 1$. Assume the result is true for $t' \geq 2$. Now consider the distribution of $16(t-1) + 2n + 10$ pebbles on the vertices of L_n. Clearly, we can move one pebble to any target vertex v_i at a cost of at most sixteen pebbles, since $C(L_n) \geq 2n + 26$ and $f(L_n) = 2n + 10$. After moving one pebble to v_i, the remaining number of pebbles on the vertices of L_n is at least $16(t-2) + 2n + 10$. Hence we can move the additional $t-1$ pebbles to v_i, by induction. Thus $f_t(L_n) \leq 16(t-1) + 2n + 10$. □

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