Proyecciones Journal of Mathematics
Vol. 37, N^o 3, pp. 439-477, September 2018.
Universidad Católica del Norte
Antofagasta - Chile

A computer verification for the value of the Topological Entropy for some special subshifts in the Lexicographical Scenario

Solange Aranzubia
Universidad Central de Chile, Chile
Rubén Carvajal
Universidad de Santiago de Chile, Chile
and
Rafael Labarca
Universidad de Santiago de Chile, Chile
Received : July 2017. Accepted : August 2018

Abstract

The Lorenz Attractor has been a source for many mathematical studies. Most of them deal with lower dimensional representations of its first return map. An one dimensional scenario can be modeled by the standard two parameter family of contracting Lorenz maps. The dynamics, in this case, can be modeled by a subshift in the Lexicographical model. These subshifts are the maximal invariant set for the shift map in some interval. For some of them the extremes, of the interval, are a minimal periodic sequence and a maximal periodic sequence which is an iteration of the lower extreme (by the shift map). For some of these subshifts the topological entropy is zero. In this case the dynamics (of the respective Lorenz map) is simple. Associated to any of these subshifts (let call it Λ*) we consider an extension (let call it* Γ*) of it that contains* Λ *which also can be constructed by using an interval whose extremes can be defined by the extremes of* Λ*. For these extensions, we present here a computer verification of a result that compute its topological entropy. As a consequence, we can affirm: the longer the period of the periodic sequence is, then the lower complexity in the dynamics of the extension the associated map has.*

2000 AMS Classification : *Primary* $37B10$, $37B40$, *Secondary* $37E05$.

Key words and phrases : *Lexicographical world, topological entropy, maximal and minimal sequences, shift map, Lorenz Maps.*

1. Introduction

The topological theory of Dynamical Systems is a mathematical theory that started in the first quarter of the 20th century due, mainly, to the pioneer work by Poincaré which introduced the definition of the rotation number associated to an homeomorphism of the circle.
The main task of this topological theory is to study the dynamical behavior of a continuous map $f : X \to X$, where X is a compact topological space. In the 1960's and in a remarkable article by R. L. Adler, A. G. Konheim and M. H. McAndrew (see [1]) these authors introduced the topological entropy of a map $f : X \to X$.
Let us assume that (X, d) is a compact metric space. Let $C^0(X, X)$ be the set of continuous maps $f : X \to X$ endowed with the topology given by the metric $d_0(f, g) = Sup\{d(f(x), g(x)); x \in X\}$. The topological entropy is a map $h_{top} : C^0(X, X) \to [0, \infty[$ that is not continuous (see [30]). The number $h_{top}(f)$ indicates the complexity of the dynamic of the map $f : X \to X$. In fact, the number $e^{h_{top}(f)} = \lambda_f$ has the following property: Given a minimal open covering of the space X, say $\mathcal{A} = \{A_1, A_2, \ldots A_p\}$. Let $\mathcal{A}_n = \{A_{i_0} \cap f^{-1}(A_{i_1}) \cap \ldots \cap f^{-(n-1)}(A_{i_{n-1}}); A_i \in \mathcal{A}\}$ be the open covering $\mathcal{A} \vee f^{-1}(\mathcal{A}) \vee \ldots \vee f^{-(n-1)}(\mathcal{A})$. Let $N(\mathcal{A}_n, f)$ be the small cardinality of an open subcover of $\mathcal{A}_n$ then $N(\mathcal{A}_n, f) \sim \left(e^{h_{top}(f)}\right)^n$. That is: the number of different orbits of length $n, x_0, x_1, \ldots, x_n; x_{i+1} = f(x_i)$ such that $x_j \in A_{i_j}, j = 0, 1, \ldots, n-1$ is approximately equal to $\left(e^{h_{top}(f)}\right)^n$.

So, the greater entropy the map f has then the greater number of different orbits the map has. Hence, to compute the topological entropy defined by a map $f : X \to X$ is an interesting problem.

The present paper is related with the computation of the topological entropy associated to certain subshifts of the shift map $\sigma : \Sigma_2 \to \Sigma_2$ (here $\Sigma_2 = \{\theta : N \cup \{0\} \to \{0, 1\}\}$) and $\sigma(\theta_0, \theta_1, \theta_2, \ldots) = (\theta_1, \theta_2, \theta_3, \ldots)$. In fact, let us consider $a \in \mathcal{A}_\infty^\infty$ (see section 2 for the definition of the set $\mathcal{A}_\infty^\infty$) be a minimal periodic orbit. Accordingly with Aranzubia (see [4]) for $\Sigma[a_-\underline{b}, b_+\underline{a}] = \{\theta \in \Sigma_2; a_-\underline{b} \le \sigma^i(\theta) \le b_+\underline{a}, \forall i \in N_0\}$ and $\sigma|_{\Sigma[a_-\underline{b}, b_+\underline{a}]} : \Sigma[a_-\underline{b}, b_+\underline{a}] \to \Sigma[a_-\underline{b}, b_+\underline{a}]$, we have:

$$(*) h_{top}(\sigma|_{\Sigma[a_-\underline{b}, b_+\underline{a}]}) = \frac{1}{\#(a)} \ln(2);$$

$$\text{here}\, \#(a) = per(a)$$

$$= \text{period of the periodic sequence } \underline{a}.$$

In this paper we compute $h_{top}(\sigma|_{\Sigma[a_{-}\underline{b},b_{+}\underline{a}]})$ for $a \in \mathcal{A}_{\infty}^{\infty}$ such that $\#(a)$ is a prime number such that $2 \leq \#(a) \leq 10.000$ and we verify, in these cases, that the equation $(*)$ is true.

2. Preliminaries

2.1. General Theory

It is well known that one of the purposes of the topological theory of Dynamical Systems is to find universal models describing the topological dynamics of a large class of systems (see for instance [3], [9], [33]).

One of these universal models is the shift on n-symbols $\sigma : \Sigma_n \to \Sigma_n$ where Σ_n is the set of sequences $\{\theta : N_0 \to \{0, 1, 2, \ldots, n-1\}\}$ and σ is the shift map defined by $(\sigma(\theta))(i) = \theta(i+1)$. Here Σ_n is endowed with a certain topology and certain order.

In fact, several signed orders can be defined in Σ_n. Let us doing this here. Let $0 = x_0 < x_1 < x_2 < \ldots < x_{2n-1} = 1$ be $2n$ points in the unit interval $[0, 1]$. Let $I_j = [x_{2j}, x_{2j+1}]$ for $j = 0, 1, 2, \ldots, n-1$; and $T : \bigcup_{j=0}^{n-1} I_j \to [0, 1]$ be a map such that its restriction to I_j is linear and onto $[0, 1]$, for any $j = 0, 1, \ldots, n-1$. The restriction of the map T to any interval I_j can be either orientation preserving or orientation reversing. Hence, we may define 2^n piecewise linear maps $T : \bigcup_{j=0}^{n-1} I_j \to [0, 1]$ as before. Let $Lin(n)$ denote the set formed by these 2^n maps. Associated to any $T \in Lin(n)$ we have its maximal invariant set $\Lambda_T = \left\{ x \in \bigcup_{j=0}^{n-1} I_j ; T^i(x) \in \bigcup_{j=0}^{n-1} I_j, \text{ for all } i \in N \right\}$. In this set, we consider the topology induced by the euclidean topology of the interval $[0, 1]$.

It is not hard to see that the set Λ_T is bijective to Σ_n. In fact, the itinerary map $I_T : \Lambda_T \to \Sigma_n$ defined by $I_T(x)(i) = j$ if and only if $T^i(x) \in I_j$ is bijective. Its inverse map $I_T^{-1} : \Sigma_n \to \Lambda_T$ is given by $I_T^{-1}(\theta) = I_{\theta_0} \cap T^{-1}(I_{\theta_1}) \cap T^{-2}(I_{\theta_2}) \cap \ldots = \bigcap_{j=0}^{\infty} T^{-j}(I_{\theta_j})$, where, from now and on, we denote $\theta = (\theta_0\theta_1\theta_2\ldots)$. Hence, by using the bijective map $I_T : \Lambda_T \to \Sigma_n$ we can induce in Σ_n:

(a) A topology $\tau_T : U \subset \Sigma_n$ is open if and only if $I_T^{-1}(U) \subset \Lambda_T$ is open, and

(b) An order: $\theta \leq_T \beta$ in Σ_n if and only if $I_T^{-1}(\theta) \leq I_T^{-1}(\beta)$ in Λ_T.

Let us denote by $\Sigma_n(T)$ the ordered, compact topological space $(\Sigma_n, \tau_T, \leq_T)$. In this way, we have introduced 2^n of these ordered compact metric spaces.

These models has been extensively used to obtain a great amount of information about maps defined in an interval (see for instances [3, 8, 9, 14, 17, 18, 19, 20, 29, 25]); vector fields on three dimensional manifolds (see for instance [2, 7, 12, 13, 15, 21, 22, 32, 34]) among other kinds of dynamical systems.

In the special case of one dimensional dynamics, the shift of two symbols may be used to study increasing (decreasing) map with one discontinuity like the Lorenz maps, unimodal maps like the quadratic family or increasing-decreasing (decreasing-increasing) maps with one discontinuity. Namely, for $n = 2$ the ordered metric compact space $(\Sigma_2, \tau_T, \leq_T)$ corresponding to the increasing-increasing map T is known as the lexicographical space which generates the lexicographical world (see for instance [17, 18, 19, 20, 23, 24]) which is denoted LW.

In this work we deal with the lexicographical world. That is, here we consider the set Σ_2 with the topology induced by the map $T : I_0 \cup I_1 \to [0,1]$ such that $T\,|_{I_0}$ and $T\,|_{I_1}$ are increasing maps.

Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map $\sigma(\theta_0, \theta_1, \theta_2, \ldots) = (\theta_1, \theta_2, \ldots)$. Let Σ_0 and Σ_1 denote the sets $\{\theta \in \Sigma_2 \,;\, \theta_0 = 0\}$ and $\{\theta \in \Sigma_2 \,;\, \theta_0 = 1\}$ respectively. It is clear that $\Sigma_2 = \Sigma_0 \cup \Sigma_1$.

In Σ_2 the order induced by T, $\leq_T$, is the <u>*lexicographical order*</u>: $\theta < \alpha$ for any $\theta \in \Sigma_0$ and $\alpha \in \Sigma_1$ or $\theta < \alpha$ if there is $n \in N$ such that $\theta_i = \alpha_i$ for $i = 0, 1, 2, \ldots, n-1$ and $\theta_n = 0$ and $\alpha_n = 1$.

For $a \leq b$ in Σ_2 let $[a, b]$ denote the interval $\{\theta \in \Sigma_2 | \, a \leq \theta \leq b\}$. $\Sigma[a, b]$ will denote the invariant set $\bigcap_{n=0}^{\infty} \sigma^{-n}([a, b])$.

Let a denote the finite string $a = a_0 a_1 \ldots a_n$ and $\underline{a}$ be the infinite sequence $\underline{a} = \underline{a_0 a_1 \ldots a_n} = a_0 a_1 \ldots a_n, a_0 a_1 \ldots a_n, \ldots$. For example, if $a = 0011$ then $\underline{a} = \underline{0011} = 00110011001100110011\ldots$.

Let $Max_2 = \{\theta \in \Sigma_2; \sigma^i(\theta) \leq \theta, \forall i \in N_0\}$ and $Min_2 = \{\alpha \in \Sigma_2; \alpha \leq \sigma^i(\alpha), \forall i \in N_0\}$ denotes the sets of *maximal* and *minimal* sequences in the lexicographical order.

Definition 2.1. *The set* $LW = \{(a, b) \in Min_2 \times Max_2 \,; \{a, b\} \subset \Sigma[a, b]\}$ *will be called the* lexicographical world.

The lexicographical world may be used to model the standard two parameter family of Lorenz maps given by

$$F_{(\mu,\nu)}(x) = \begin{cases} -x^2+\nu & , \quad x<0 \\ x^2-\mu & , \quad x>0 \end{cases} \quad \mu,\nu \geq 0$$

In fact, let us consider the restriction of the dynamics of the map $F_{(\mu,\nu)}$ to the interval $I(\mu,\nu) = [-\mu,\nu]$. That is $F_{(\mu,\nu)} : I(\mu,\nu) \to I(\mu,\nu)$. Let $\Lambda(\mu,\nu) = \bigcap_{j=0}^{\infty} F_{(\mu,\nu)}^{-j}(I(\mu,\nu) \setminus \{0\})$. For any $x \in \Lambda(\mu,\nu)$ let us consider its itinerary $I(\mu,\nu)(x) \in \Sigma_2 = \{\theta : N_0 \to \{0,1\}\}$ given by $I(\mu,\nu)(x)(i) = j \Leftrightarrow F^i_{(\mu,\nu)}(x) \in I_j(\mu,\nu)$; where $I_0(\mu,\nu) = \{x \in \Lambda(\mu,\nu); x<0\}$ and $I_1(\mu,\nu) = \{x \in \Lambda(\mu,\nu); x>0\}$.
$I(\mu,\nu) : \Lambda(\mu,\nu) \to \Sigma_2$ is a continuous map whose image $I(\mu,\nu)(\Lambda(\mu,\nu)) = \{I(\mu,\nu)(x); x \in \Lambda(\mu,\nu)\} = \Sigma[\mu,\nu]$ satisfies $\overline{\Sigma[\mu,\nu]} = \bigcap_{n=0}^{\infty} \sigma^{-n}([a(\mu,\nu), b(\mu,\nu)])$ where $a(\mu,\nu) = \lim_{x\downarrow -\mu} I(\mu,\nu)(x)$ and $b(\mu,\nu) = \lim_{x\uparrow\nu} I(\mu,\nu)(x)$; here the limit is taken over elements $x \in \Lambda(\mu,\nu)$. The sequences $a(\mu,\nu)$ and $b(\mu,\nu)$ satisfies $\sigma^i(a(\mu,\nu)) \geq a(\mu,\nu)$ and $\sigma^i(b(\mu,\nu)) \leq b(\mu,\nu)$ for any $i \in N_0$.

Now we can consider the map $\alpha : \{(\mu,\nu); \mu \geq 0, \nu \geq 0\} \to Min_2$ and $\beta : \{(\mu,\nu); \mu \geq 0, \nu \geq 0\} \to Max_2$ defined by $\alpha(\mu,\nu) = a(\mu,\nu)$ and $\beta(\mu,\nu) = b(\mu,\nu)$, which associate to any (μ,ν) its kneading sequences $a(\mu,\nu)$ and $b(\mu,\nu)$. These maps are not continuous (see [20]).
It is clear that we can parameterize the positive $(\mu,\nu)-$plane by using the map α or the map β. In this way we have an α and a $\beta-$decomposition of the positive $(\mu,\nu)-$plane.
Also, we have a map $L : \{(\mu,\nu); \mu \geq 0, \nu \geq 0\} \to LW$ given by $L(\mu,\nu) = (\alpha(\mu,\nu), \beta(\mu,\nu)) = (a(\mu,\nu), b(\mu,\nu))$. This map is not continuous (see [20]).
Let us consider $h_{top} : \{(\mu,\nu); \mu \geq 0, \nu \geq 0\} \to [0, \ln(2)]$ the topological entropy $h_{top}(\mu,\nu) = h_{top}(\sigma|_{\Sigma[a(\mu,\nu),b(\mu,\nu)]})$. It is unknown if this map is continuous or not. Also, it is unknown if the set $\{(\mu,\nu) : h_{top}(\mu,\nu) = 0\}$ is a connected set. For us this set (the entropy zero set) seems to be arc connected but not locally connected.

Observation 2.2. *Let us now establish some notations.*

1.- *If* $\underline{a} \in Min_2$, $\underline{a} \neq \underline{0}$ *then* $\underline{b(a)} = sup\{\sigma^i(\underline{a}); i \in N\} \in Max_2$ *and if* $\underline{d} \in Max_2$, $\underline{d} \neq \underline{1}$ *then* $\underline{a(d)} = inf\{\sigma^i(\underline{d}); i \in N\} \in Min_2$, *if* $\underline{b(a)} = \underline{b_1 b_2 \ldots b_n}$ *then we will denote* $b(a) = b_1 b_2 \ldots b_n$.

2.- If $a = 0a_1a_2\dots a_{n-1}1$ *then* $a_- = 0a_1a_2\dots a_{n-1}0$ *and if* $b = 1b_1b_2\dots b_{n-1}0$ *then* $b_+ = 1b_1b_2\dots b_{n-1}1$.

3.- If $\underline{a_1}, \underline{a_2}$ *are two sequences then we define* $m(a_1, a_2)$ *by* $m(a_1, a_2) = \underline{a_1a_2}$. *For instance for* $\underline{a_1} = \underline{001}$ *and* $\underline{a_2} = \underline{01}$ *we have* $m(a_1, a_2) = \underline{00101}$.

2.2. The set $\mathcal{A}_\infty^\infty$

Let us define the set $\mathcal{A}_\infty^\infty$

Definition 2.3. *Let* $A_0 = \{\underline{0^n1}, \underline{01^m};\ n, m \in N\}$, *that is:*

$$A_0 = \{\dots \underline{00001},\ \underline{0001},\ \underline{001},\ \underline{01},\ \underline{011},\ \underline{0111},\ \underline{01111}\dots\}.$$

Let $A_1 = \{m(a_1, a_2), \underline{a_1} < \underline{a_2}$ *are consecutive sequences in* $A_0\} \cup A_0$ *and*

$A_{n+1} = \{m(a_1, a_2); \underline{a_1} < \underline{a_2}$ *are consecutive sequences in* $A_n\} \cup A_n$. *So, we have:*

$$A_0 = \dots \underline{00001},\ \underline{0001},\ \underline{001},\ \underline{01},\ \underline{011},\ \underline{0111},\ \underline{01111}\dots$$

$$A_1 = \dots \underline{0001},\ \underline{0001001},\ \underline{001},\ \underline{00101},\ \underline{01},\ \underline{01011},\ \underline{011},\ \underline{0110111},\ \underline{0111}\dots$$

$m(\underline{0001}, \underline{001})$ $\quad m(\underline{001}, \underline{01})$ $\quad m(\underline{01}, \underline{011})$ $\quad m(\underline{011}, \underline{0111})$

$$A_2 = \dots \underline{001},\ \underline{00100101},\ \underline{00101},\ \underline{0010101},\ \underline{01},\ \underline{0101011},\ \underline{01011},\ \underline{01011011},\ \underline{011}\dots$$

Let $\mathcal{A}_\infty = \bigcup_{n=0}^{\infty} A_n$. *For* $\underline{a} \in \mathcal{A}_\infty$ *let*

$A_0(\underline{a}) = \{\underline{a_-(b(a))^{m-1}b(a)_+},\ \underline{a_-b(a)_+a^{n-1}};\ n, m \in N\}$;

$A_1(\underline{a}) = \{m(a_1, a_2); \underline{a_1} < \underline{a_2}$ *are consecutive sequences in* $A_0(\underline{a})\} \cup A_0(\underline{a})$; *and*

$A_{n+1}(\underline{a}) = \{m(a_1, a_2); \underline{a_1} < \underline{a_2}$ *are consecutive sequences in* $A_n(\underline{a})\} \cup A_n(\underline{a})$

and let $\mathcal{A}_\infty(\underline{a}) = \bigcup_{n=0}^{\infty} A_n(\underline{a})$.

Example: For $a = \underline{01} \in \mathcal{A}_\infty$, *we have:*

$A_0(a) = \{\dots \underline{00101011}, \underline{001011}, \underline{0011}, \underline{001101}, \underline{00110101}, \dots\}$

$A_1(a) = \{\dots \underline{001011}, \underline{0010110011}, \underline{0011}, \underline{0011001101}, \underline{001101}, \underline{00110100110101}, \dots\}$.

Let us consider $\mathcal{A}_\infty^0 = \mathcal{A}_\infty$ *and* $\mathcal{A}_\infty^1 = \bigcup_{\underline{a} \in \mathcal{A}_\infty} \mathcal{A}_\infty(\underline{a}) \cup \mathcal{A}_\infty^0$.

Now, associated to any $\underline{a} \in \mathcal{A}_\infty^1$ *we construct:*

$A_0^2(\underline{a}) = \{\underline{a_-(b(a))^{m-1}(b(a))_+},\ \underline{a_-b(a)_+a^{n-1}};\ n, m \in N\}$;

$A^2_{n+1}(\underline{a}) = \{m(a_1,a_2); \underline{a_1} < \underline{a_2}$ *are consecutive sequences in* $A^2_n(\underline{a})\} \cup A^2_n(\underline{a})$;

$\mathcal{A}^2_\infty(\underline{a}) = \bigcup_{n=0}^{\infty} A^2_n(\underline{a})$ *and* $\mathcal{A}^2_\infty = \bigcup_{\underline{a}\in\mathcal{A}^1_\infty} \mathcal{A}^2_\infty(\underline{a}) \cup \mathcal{A}^1_\infty$.

Similarly, for any $n \geq 2$*, we may define:* $\mathcal{A}^{n+1}_\infty$ *and* $\mathcal{A}^\infty_\infty = \bigcup_{n=0}^{\infty} \mathcal{A}^n_\infty$.

Observation 2.4.

- *We note that all the elements in* $\mathcal{A}^\infty_\infty$ *are minimal periodic sequences.*
- *There are minimal periodic sequences* $\underline{a} \in Min_2$ *such that* $\underline{a} \notin \mathcal{A}^\infty_\infty$*. For instance:* $\underline{a} = \underline{00111}, \underline{a} = \underline{000111}$ *and* $\underline{a} = \underline{001011011}$.

As part of his PH.D. Thesis Solange Aranzubia (see [4]) proved the following:

Theorem 2.5. *For any* $a \in \mathcal{A}^\infty_\infty$ *we have that ,* $h_{top}(a_-\underline{b}, b_+\underline{a}) = \frac{1}{Per(a)}\ln(2)$.

So, from this result we conclude that: *For elements in* $\mathcal{A}^\infty_\infty$ *the longer the period of the periodic sequence is then the lower complexity in the dynamics of the extension the associated map has.*
This fact is not true for any minimal periodic sequence $\underline{a} \in Min_2$. In fact, let us consider $\alpha = \underline{0_q 1}$, $\beta = \underline{1_p 0}$ and $\underline{a} = \underline{a(p,q)} = \underline{\alpha_-\beta_+} = \underline{0_{q+1}1_{p+1}} \in Min_2$. For this sequence and for $\underline{b} = \underline{b(a)} = \underline{b(p,q)} = \underline{1_{p+1}0_{q+1}}$ we have that $\lim_{p\to\infty}\lim_{q\to\infty} h_{top}(a_-\underline{b}, b_+\underline{a}) = \ln 2$ and $\lim_{p\to\infty}\lim_{q\to\infty} \frac{1}{((p+1)+(q+1))}\ln 2 = 0$.

In the present paper we compute the value of the topological entropy, for all the periodic sequences $a \in \mathcal{A}^\infty_\infty$ whose period is a prime number, p, between 3 and 9973, by using an algorithm associated to graphs. We also verify that the cost of the algorithm is $\mathcal{O}(p^3)$. These computations can be considered as a computer verification of this theorem for prime period in this range.

2.3. Computation of the topological entropy for the map $\sigma : \Sigma[a_-\underline{b(a)}, b_+\underline{a}] \to \Sigma[a_-\underline{b(a)}, b_+\underline{a}]$; for $a \in \mathcal{A}^\infty_\infty$ such that $Per(a) = p$ is a prime number.

Initially, let us consider $a_p = \underline{0_{p-1}1} = \underbrace{0\ldots0}_{p-1}1\underbrace{0\ldots0}_{p-1}1\ldots$ the sequence of period p which has $(p-1)-$zeroes and one one.

Associated to a_p we will construct a discontinuous map $f : [0,1]\backslash\{c_p\} \to [0,1]$ that satisfies:

i) For $\alpha_0 = 0$ we have $I_f(\alpha_0) = (a_p)_- \underline{b(a_p)}$

ii) For $\beta = 1$ we have $I_f(\beta) = (b(a_p))_+ \underline{a_p}$

iii) $f|_{[\alpha_0, c_p]}$ is increasing and continuous and $f(c_p^-) = \beta$

iv) $f|_{[c_p, \beta]}$ is increasing and continuous and $f(c_p^+) = \alpha_0$

To construct the map f let us consider numbers $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{p-1} < c_p < \alpha_p < \beta = 1$.

For the map f we have: $f(\alpha_0) = \alpha_1, f(\alpha_1) = \alpha_2, \ldots, f(\alpha_{p-1}) = \alpha_p, f(c_p) = 1, f(\alpha_p) = \alpha_1, f(\beta) = \alpha_2$ and, in between two of these numbers, the map f is linear. We have figure 1:

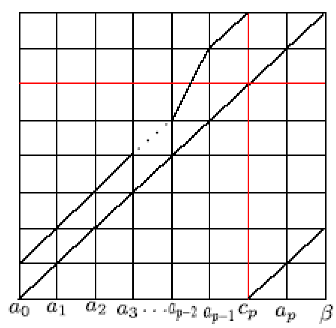


Figure 1: Function f.

Concerning the itineraries associated to these points we have $I_f(\alpha_0) = 0_p1\underline{0_{p-1}1}, I_f(\alpha_1) = \underline{0_{p-1}1}, I_f(\alpha_2) = \underline{0_{p-2}10}, I_f(\alpha_3) = \underline{0_{p-3}100}, \ldots, I_f(\alpha_{p-1}) = \underline{010_{p-2}}, I_f(\alpha_p) = \underline{10_{p-1}}, I_f(\beta) = 10_{p-2}1\underline{0_{p-1}1}$.
Now, let us denote by $I_1 = [\alpha_{p-2}, \alpha_{p-1}], I_2 = [\alpha_{p-1}, c_p], I_3 = [c_p, \alpha_p], I_4 = [\alpha_p, \beta], I_5 = [\alpha_0, \alpha_1], I_6 = [\alpha_1, \alpha_2], I_7 = [\alpha_2, \alpha_3], \ldots, I_{p+2} = [\alpha_{p-1}, \alpha_{p-2}]$.
The map f satisfies: $f(I_1) = I_2 \cup I_3, f(I_2) = I_4, f(I_3) = I_5, f(I_4) = I_6, f(I_5) = I_6, f(I_6) = I_7, f(I_7) = I_8 \ldots, f(I_{p+2}) = I_1$. Hence we have the $A-$graph (Figure 2) see [4] or [16].

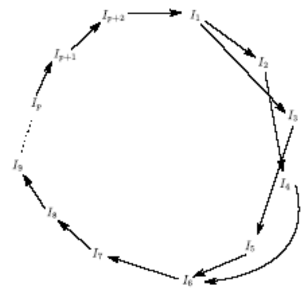


Figure 2: A-graph

Associated to these graph we have the incidence matrix M given by

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

This matrix has $\lambda = 0$ as eigenvalue and $h_{top}(\sigma|_{\Sigma[0_p 1\overline{0_{p-1}1}, 10_{p-2}1\overline{0_{p-1}1}]}) = \ln(\lambda_M)$ where λ_M is a greatest eigenvalue of M; $\lambda_M \in]1, 2[$ (see [4] or [16]).

Now, by using $a_p = \underline{0_{p-1}1}$ and an algorithm, we are able to construct $p-2$ sequences $a_1, a_2, \cdots, a_{p-2}$ such that $period(a_i) = p$ and $a_i \in \mathcal{A}^{\infty}_{\infty}$, for any $i = 1, 2, \cdots, p-2$.

Accordingly with Theorem 3.1.5 in [4], for any other minimal sequence $d \in \Sigma_0$ such that $period(d) = p$ we have that $d \notin \mathcal{A}^{\infty}_{\infty}$.

Hence, our algorithm give us all the minimal periodic sequences a with $period(a) = p$ and $a \in \mathcal{A}^{\infty}_{\infty}$.

The general construction of the algorithm may be found in section 3.1.1 in [4], pages 57-76, and will appear elsewhere.

Here, for the sake of completeness, we will show the algorithm for the cases $period(a) = 3, 5$, and 7.

Let us begin with the case $p = 3$.

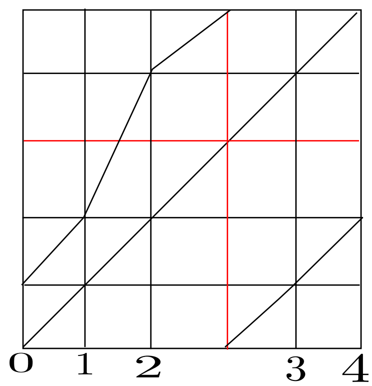
For $a = \underline{001}$, $b = \underline{100}$, we have $a_{-}\underline{b} = 000\underline{100} = 0\underline{001}$ and $b_{+}\underline{a} = 101\underline{001} = 1\underline{010}$, so $a_{-}\underline{b} = 0\underline{001} = 0 \rightarrow \sigma(0\underline{001}) = \underline{001} = 1 \rightarrow \sigma(\underline{001}) = \underline{010} = 2 \rightarrow$

$\sigma(\underline{010}) = \underline{100} = 3 \to \sigma(\underline{100}) = \underline{001} = 1;$
$b_+\underline{a} = 1\underline{010} = 4 \to \sigma(1\underline{010}) = \underline{010} = 2.$

Now we construct a permutation associated the iteration (under σ) of the elements
$a_-\underline{b} = 000\underline{100} = 0\underline{001}$ and $b_+\underline{a} = 101\underline{001} = 1\underline{010}$,

$$P = \begin{pmatrix} 0 & 1 & 2 & | & 3 & 4 \\ 1 & 2 & 3 & | & 1 & 2 \end{pmatrix}$$

This permutation allows us to define a discontinuous map of the interval (like the previous f).

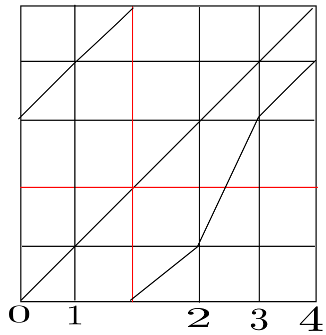


This map represents the dynamics of $\sigma\mid_{[a_-\underline{b}, b_+\underline{a}]}$ where $a_-\underline{b} = 0\underline{001}$ and $b_+\underline{a} = 1\underline{010}$
Now, multiplying the permutation P by itself we have $P \cdot P$

$$P \cdot P = \begin{pmatrix} 0 & 1 & 2 & | & 3 & 4 \\ 1 & 2 & 3 & | & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 & | & 3 & 4 \\ 1 & 2 & 3 & | & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & | & 2 & 3 & 4 \\ 2 & 3 & | & 1 & 2 & 3 \end{pmatrix} = P^2$$

Using P^2 we can define a discontinuous map of the interval:



This map represent the dynamics of $\sigma \mid_{[a_{-}\underline{b}, b_{+}\underline{a}]}$ where $a_{-}\underline{b} = 01\underline{011}$ and $b_{+}\underline{a} = 1\underline{110}$, therefore, for the sequence a, we have: $a = \underline{011}$ and $Per(a) = 3$.

Let us now consider the multiplication $P \cdot P^2$, we have:

$$P \cdot P^2 = \begin{pmatrix} 0 & 1 & 2 & | & 3 & 4 \\ 1 & 2 & 3 & | & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & | & 2 & 3 & 4 \\ 2 & 3 & | & 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & | & 1 & 2 & 3 & 4 \\ 3 & | & 1 & 2 & 3 & 1 \end{pmatrix} = P^3$$

Using this permutation we cannot construct a dynamics that represents some subshift of the lexicographical world.

As we known, there are only two minimal periodic orbits of period three and both of them are in $\mathcal{A}_{\infty}^{\infty}$. These sequences are: $\underline{001}$ and $\underline{011}$. So, the proof in the case $p = 3$ is complete.

• Let us work now with the case $p = 5$.

Let $a = \underline{00001}$, $b = \underline{10000}$, then $a_{-}\underline{b} = 00000\underline{10000} = 0\underline{00001}$ and $b_{+}\underline{a} = 10001\underline{00001} = 1\underline{00010}$. Therefore:

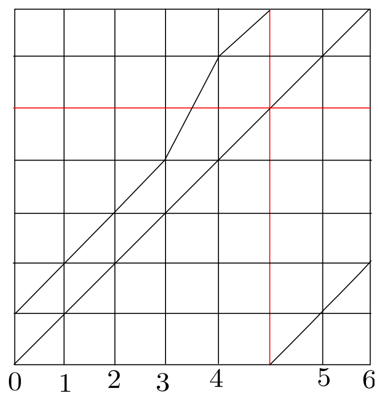
$a_{-}\underline{b} = 0\underline{00001} = 0 \rightarrow \sigma(0\underline{00001}) = \underline{00001} = 1 \rightarrow \sigma(\underline{00001}) = \underline{00010} = 2 \rightarrow \sigma(\underline{00010}) = \underline{00100} = 3 \rightarrow \sigma(\underline{00100}) = \underline{01000} = 4 \rightarrow \sigma(\underline{01000}) = \underline{10000} = 5 \rightarrow \sigma(\underline{10000}) = \underline{00001} = 1$;

$b_{+}\underline{a} = 1\underline{00010} = 6 \rightarrow \sigma(1\underline{00010}) = \underline{00010} = 2$.

So, associated to the iterations (by σ) of the sequences $a_{-}\underline{b}$ and $b_{+}\underline{a}$ we have the following permutation:

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & | & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & | & 1 & 2 \end{pmatrix}$$

This permutation allows us to define a discontinuous map of the interval (like in the previous case $p = 3$)

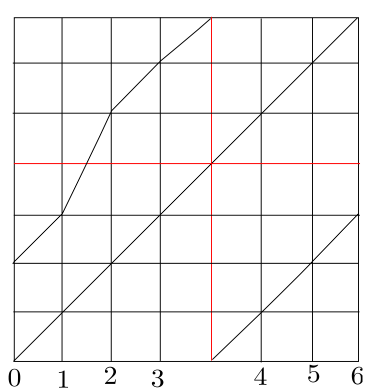


This map represent the dynamics of $\sigma\mid_{[a_-\underline{b}, b_+\underline{a}]}$, where $a_-\underline{b} = 0\underline{00001}$ and $b_+\underline{a} = 1\underline{00010}$. Therefore, for the sequence a, we have: $a = \underline{00001}$ and $Per(a) = 5$.

Now, let us compute $P \cdot P$

$$P \cdot P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \mid & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & \mid & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & a & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & a & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 2 & 3 & \mid & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & \mid & 1 & 2 & 3 \end{pmatrix} = P^2$$

Using P^2 we can define a discontinuous map of the interval:

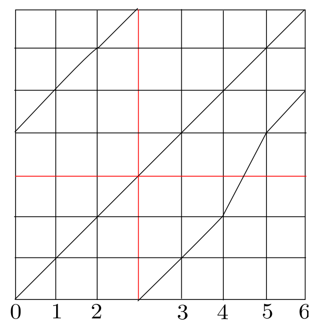


This map represent the dynamics of $\sigma\mid_{[a_-\underline{b}, b_+\underline{a}]}$, with $a_-\underline{b} = 0\underline{01001}$ and $b_+\underline{a} = 1\underline{01010}$, where $a = \underline{00101}$ and $per(a) = 5$.

Now, let us continuous with P^3:

$$P \cdot P^2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & | & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & | & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 & 3 & | & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & | & 1 & 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 2 & | & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & | & 1 & 2 & 3 & 4 \end{pmatrix} = P^3$$

Using P^3 we can define a discontinuous map of the interval:

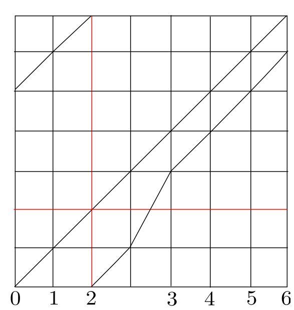


This map represent the dynamics of $\sigma \mid_{[a_-\underline{b}, b_+\underline{a}]}$ with $a_-\underline{b} = 01\underline{01011}$ and $b_+\underline{a} = 1\underline{10110}$, where $a = \underline{01011}$ and $Per(a) = 5$.

Now, let us continuous with P^4:

$$P \cdot P^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & | & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & | & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 & | & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & | & 1 & 2 & 3 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & | & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & | & 1 & 2 & 3 & 4 & 5 \end{pmatrix} = P^4$$

Using P^4 we can define a discontinuous map of the interval



This map represent the dynamics of $a_-\underline{b} = 0\underline{11101}$ and $b_+\underline{a} = 1\underline{11110}$,

where $a = \underline{01111}$ and $Per(a) = 5$.
Now, for P^5 we have:

$$P \cdot P^4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & | & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & | & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & | & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & | & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & | & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & | & 1 & 2 & 3 & 4 & 5 & 1 \end{pmatrix} = P^5$$

Using this permutation we cannot construct a map whose dynamics represents some subshift of the lexicographical world.

Now, these four sequences are minimal periodic sequences with period 5 in $\mathcal{A}_\infty^\infty$. These sequences are: $\underline{00001}$, $\underline{00101}$, $\underline{01011}$ and $\underline{01111}$.

There are another two minimal periodic sequences of period 5. These are: $\underline{00111} = \underline{a_- d_+}$ with $a = \underline{01}, d = \underline{110}$ and $\underline{00011} = \underline{a_- d_+}$ with $a = \underline{001}, d = \underline{10}$.

Accordingly with section 3.1.1 in [4] we have
$h_{top}(\underline{00111}, \underline{11100}) = h_{top}(\underline{00011}, \underline{11000}) = h_{top}(\underline{01}, \underline{110}) = h_{top}(\underline{001}, \underline{10}) > 0$.

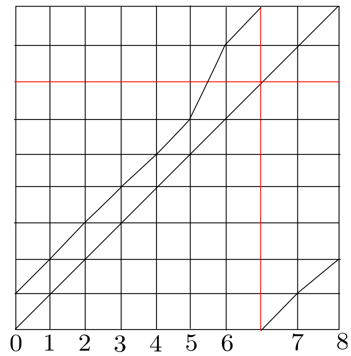
We observe, accordingly with section 3.1.1. in [4] that for any $a \in \mathcal{A}_\infty^\infty$ we have that $h_{top}(a, b(a)) = 0$. Hence, we conclude that $\underline{00111}$ and $\underline{00011}$ are minimal periodic sequences not in $\mathcal{A}_\infty^\infty$.

• Let us now consider minimal periodic sequences with period $p = 7$. Let $a = \underline{0000001} = \underline{0_6 1}$, $b = \underline{1000000} = \underline{10_6}$. We have: $a_- \underline{b} = 0\underline{0_6 1}$ and $b_+ \underline{a} = 1\underline{0_5 10}$. Hence:
$a_- \underline{b} = 0\underline{0_6 1} = 0 \to \sigma(0\underline{0_6 1}) = \underline{0_6 1} = 1 \to \sigma(\underline{0_6 1}) = \underline{0_5 10} = 2 \to \sigma(\underline{0_5 10}) = \underline{0_4 100} = 3 \to \sigma(\underline{0_4 100}) = \underline{0_3 10_3} = 4 \to \sigma(\underline{0_3 10_3}) = \underline{0_2 10_4} = 5 \to \sigma(\underline{0_2 10_4}) = \underline{010_5} = 6 \to \sigma(\underline{010_5}) = \underline{10_6} = 7$;
$b_+ \underline{a} = 1\underline{0_5 10} = 8 \to \sigma(1\underline{0_5 10}) = \underline{0_5 10} = 1$.

So, associated to the iterations (by σ) of the sequences $a_- \underline{b}$ and $b_+ \underline{a}$ we have the following permutation:

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & | & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & | & 1 & 2 \end{pmatrix}$$

This permutation allows us to define a discontinuous map of the interval (like in the previous cases $p = 3, 5$)

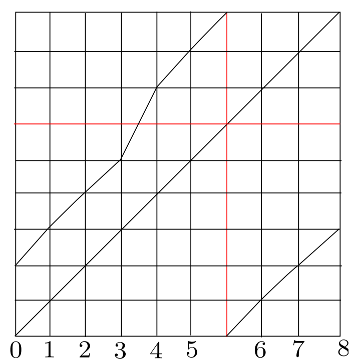


This map represent the dynamics of $\sigma\mid_{[a_-\underline{b},b_+\underline{a}]}$, with $a_-\underline{b} = 0\underline{0_61}$ and $b_+\underline{a} = 1\underline{0_510}$, where $a = \underline{0000001}$ and $Per(a) = 7$.

Let us now consider P^2:

$$P^2 = P \cdot P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & \mid & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & \mid & 1 & 2 & 3 \end{pmatrix}$$

This permutation allows us to define a discontinuous map of the interval (like in the previous case $p = 3, 5$)

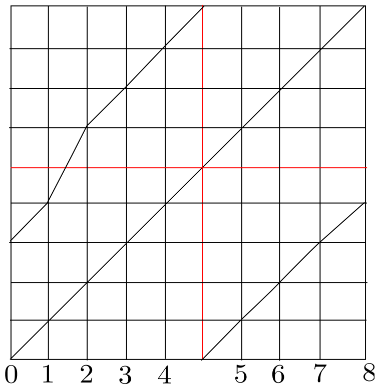


This map represent the dynamics of $\sigma\mid_{[a_-\underline{b},b_+\underline{a}]}$ with $a_-\underline{b} = 0\underline{0010001}$ and $b_+\underline{a} = 1\underline{0010010}$, where $a = \underline{0001001}$ and $Per(a) = 7$.

Let us now consider P^3:

$$P^3 = P \cdot P^2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \mid & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & \mid & 1 & 2 & 3 & 4 \end{pmatrix}$$

This permutation allows us to define a discontinuous map of the interval (like in the previous cases $p = 3, 5$)

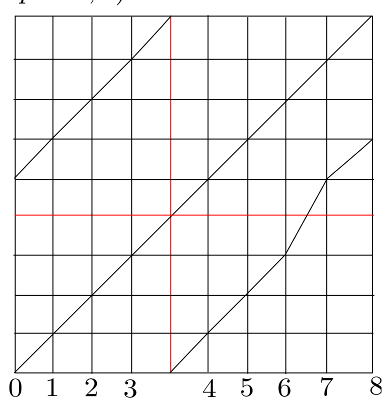


This map represents the dynamics of $\sigma\mid_{[a_-\underline{b},b_+\underline{a}]}$ with $a_-\underline{b} = 0\underline{0101001}$ and $b_+\underline{a} = 1\underline{0101010}$, for $a = \underline{0010101}$ and $Per(a) = 7$.

Let us consider P^4:

$$P^4 = P\cdot P^3 = \left(\begin{array}{cccc|ccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 & 5 \end{array}\right)$$

This permutation allows us to define a discontinuous map of the interval (like in the previous cases $p = 3, 5$)

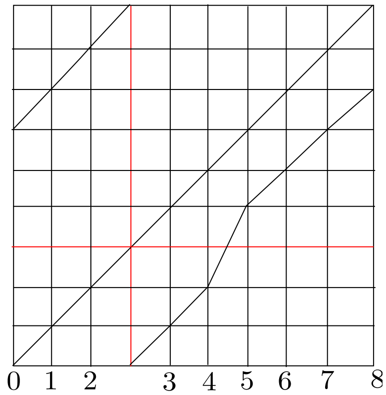


This map represents the dynamics of $\sigma\mid_{[a_-\underline{b},b_+\underline{a}]}$ with $a_-\underline{b} = 0\underline{1010101}$ and $b_+\underline{a} = 1\underline{1010110}$, where $a = \underline{0101011}$ and $Per(a) = 7$.

Let us now consider P^5:

$$P^5 = P\cdot P^4 = \left(\begin{array}{ccc|cccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}\right)$$

This permutation allows us to define a discontinuous map of the interval (like in the previous cases $p = 3, 5$)

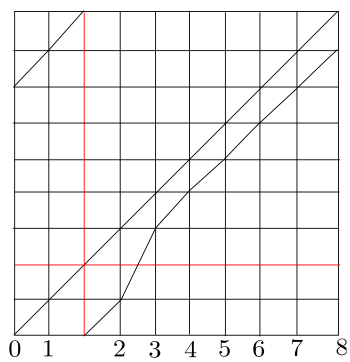


This map represents the dynamics of $\sigma\mid_{[a_-\underline{b},b_+\underline{a}]}$ with $a_-\underline{b} = 0\underline{1101101}$ and $b_+\underline{a} = 1\underline{1101110}$, where $a = \underline{0110111}$ and $Per(a) = 7$.

Let us now consider P^6:

$$P^6 = P \cdot P^5 = \begin{pmatrix} 0 & 1 & | & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & | & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

This permutation allows us to define a discontinuous map of the interval (like in the previous cases $p = 3, 5$)



This map represents the dynamics of $\sigma\mid_{[a_-\underline{b},b_+\underline{a}]}$ with $a_-\underline{b} = 0\underline{1111101}$ and $b_+\underline{a} = 1\underline{1_6 0}$,
where $a = \underline{01_6}$ and $Per(a) = 7$.

Let us now consider P^7:

$$P^7 = P \cdot P^6 = \begin{pmatrix} 0 & | & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & | & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix}$$

Using this permutation we cannot construct a dynamics that represents some subshift of the lexicographical world.

Now, these six sequences are minimal periodic sequences with period 7 in A^∞_∞. These are: $\underline{0_6 1}$, $\underline{0001001}$, $\underline{0010101}$, $\underline{0101011}$, $\underline{0110111}$ and $\underline{01_6}$.

We observe that the other minimal sequences of period seven are:

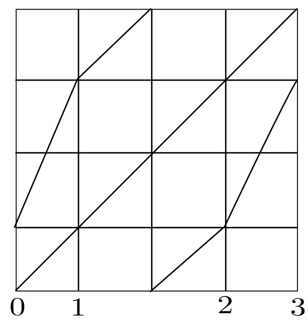
1. $\underline{0000111} = \underline{a_- d_+}$ with $a = \underline{0001}, d = \underline{110}$
2. $\underline{0001111} = \underline{a_- d_+}$ with $a = \underline{001}, d = \underline{1110}$
3. $\underline{0000101} = \underline{a_- d_+}$ with $a = \underline{0001}, d = \underline{100}$
4. $\underline{0011101} = \underline{a_- d_+ a}$ with $a = \underline{01}, d = \underline{110}$
5. $\underline{0001011} = \underline{a_- d d_+}$ with $a = \underline{001}, d = \underline{10}$
6. $\underline{0001101} = \underline{a_- d_+}$ with $a = \underline{001}, d = \underline{1100}$
7. $\underline{0010111} = \underline{a_- d_+}$ with $a = \underline{0011}, d = \underline{110}$
8. $\underline{0011111} = \underline{a_- d_+}$ with $a = \underline{01}, d = \underline{11110}$
9. $\underline{0011011} = \underline{a_- d_+}$ with $a = \underline{01}, d = \underline{11010}$
10. $\underline{0000011} = \underline{a_- d_+}$ with $a = \underline{00001}, d = \underline{10}$
11. $\underline{0010011} = \underline{a_- d_+}$ with $a = \underline{00101}, d = \underline{10}$
12. $\underline{0101111} = \underline{a_- d_+}$ with $a = \underline{011}, d = \underline{1110}$

Accordingly with theorem A in [4] all of these sequences satisfies $h_{top}(\alpha, b(\alpha)) = h_{top}(\underline{a}, \underline{d})$ and, from observation 2.0.10 at [4] we have: $h_{top}(\underline{a}, \underline{d}) > 0$.

We also observe that for any $\alpha \in \mathcal{A}^\infty_\infty$ we have $h_{top}(\alpha, b(\alpha)) = 0$.

Now, let us consider a sequence $a \in \mathcal{A}^\infty_\infty$ such that $per(a) = p$. Accordingly with the proof of Theorem C (in [4]) for any $a \in \mathcal{A}^\infty_\infty$ as before, we can construct a discontinuous map $f_a : I \setminus \{c_a\} \to I, I = [0,1]$ (like the previous f) such that $I_{f_a}(0) = a_- \underline{b(a)}$ and $I_{f_a}(1) = (b(a))_+ \underline{a}$. For this map, f_a, we have a decomposition of the interval $I = \bigcup_{i=1}^{p+2} I_i$, such that the associated graph to the dynamics of f_a under this decomposition of the interval, $G(f_a)$, is equal to the graph associated to the previous f (=$G(f)$ associated to $\underline{0_{p-1}1}$). This, in particular, implies that $h_{top}(\sigma|_{\Sigma[a_- \underline{b(a)}, (b(a))_+ \underline{a}]}) = h_{top}(\sigma|_{\Sigma[0_p 1 \underline{0_{p-1}1}, 10_{p-2}1 \underline{0_{p-1}1}]}) = \ln(\lambda_M)$.

Observation 2.6. *The result is also true for* $p = 2$. *In this case* $a = \underline{01} \in \mathcal{A}^{\infty}_{\infty}$.
In fact, for $a = \underline{01}$, $b = b(a)\underline{10}$, *we have* $a_{-}\underline{b} = 00\underline{10} = 0\underline{01}$ *and* $b_{+}\underline{a} = 1\underline{10}$, *so* $0 = a_{-}\underline{b} = 0\underline{01} \rightarrow \sigma(0\underline{01}) = \underline{01} = 1 \rightarrow \sigma(\underline{01}) = \underline{10} = 2 \rightarrow \sigma(\underline{10}) = \underline{01} = 1$; $b_{+}\underline{a} = 1\underline{10} = 3 \rightarrow \sigma(1\underline{10}) = \underline{10} = 2$.
Then we can define a discontinuous map of the interval



The characteristic polynomial associated to the respective graph is $p(\lambda) = \lambda^4 - 2\lambda^2$ *then* $p(\lambda) = 0$ *for* $\lambda = 0$ *or* $\lambda = \sqrt{2}$, *hence* $h_{top}(\sigma|_{\Sigma[a_{-}\underline{b}, b_{+}\underline{a}]}) = \log(\sqrt{2}) = \dfrac{1}{2}\log(2)$

3. The Algorithm

In this section the algorithm that computes the entropy is described. The figure 3 shows the instructions of the algorithm. The first step consist in to generate the permutations $p^0, p^1, \ldots, p^{n-1}$, each permutation has $n+2$ elements indexed between $0 \ldots n+1$. The generation of $p^0, p^1, \ldots, p^{n-1}$ is performed by the instructions between lines 1 and 11.

In the second step (line 12), the algorithm generates the graph associated to the permutation p^1 and then computes the maximum eigenvalue associated to it (line 13). The algorithm returns its natural logarithm (line 15).

Algorithm Entropy

Input: $n \in \mathcal{N},\ n$ a prime number
Output: $e \in \mathcal{R},\ e$ entropy value if exists
-1 if not exists such entropy value

1.- $p_j^i \leftarrow 0 \ \ \forall i,\ 0 \leq i \leq n, \ \ \forall j,\ 0 \leq j \leq n+1$
2.- **For** $i \leftarrow 0$ **to** $n-1$ **do**
3.- $p_i^0 \leftarrow i+1$
4.- $p_i^1 \leftarrow i+1$
EndFor
5.- $p_n^0 \leftarrow 1$
6.- $p_{n+1}^0 \leftarrow 2$
7.- $p_n^1 \leftarrow 1$
8.- $p_{n+1}^1 \leftarrow 2$
9.- **For** $i \leftarrow 2$ **to** n **do**
10.- **For** $j \leftarrow 0$ **to** $n+1$ **do**
11.- $p_j^i \leftarrow p_{p_j^0}^{i+1}$
EndFor
EndFor
12.- $g \leftarrow$ GenerateGraphFrom(p^1)
13.- $u \leftarrow$ MaxEigenValue(g)
14.- $e \leftarrow log(u)$
15.- **Return** e

Figure : The Algorithm.

3.1. Cost of the Algorithm

We analyze the efficiency of the algorithm in terms of the memory space and the number of operations. Both parameters are related with the value of n. In terms of space it is necessary to store the permutations, the graphs and the auxiliary vectors to compute the maximum eigenvalue.

Steps	**Required Space**	**Number of Operations**
Permutations	$(n+1)\times(n+2)$	$\mathcal{O}(n^2)$
Graphs	$(n-1)\times((n+2)\times(n+2))$	$\mathcal{O}(n^2)$
Eigenvalues	$(n-1)\times n\times(n\times n)$	$\mathcal{O}(n^3)$

Figure 3.1: Cost of the Algorithm

Figure 4 shows the required space and the number of operations of the different sections of the algorithm. The maximal number of operations correspond to compute the maximum eigenvalue that uses the method QR [10] (line 13) with a cost $\mathcal{O}(n^3)$.

4. Experimental Results

The algorithm was implemented in C language under Linux OS. The computation of the maximum eigenvalues was performed with the GNU Scientific Library (GSL) [11]. The hardware we used is a Intel 3.60GHz with 4 Gib of RAM. Because the GSL function that computes the maximum eigenvalue of a matrix require 8 bytes (*double*) per element. The maximum value of n is $20,123$ because the space needed is $20,123\times 20,123\times 8 = 3,239,481,032$ bytes. In the appendix A are reported the entropy values for prime numbers from $p=3$ and up to $p=9973$.

5. Discussion

The main motivation for our studies, concerning the standard quadratic family of Lorenz maps, is to prove the *isentrope conjecture* for this family. This conjecture were established by John Milnor ([28]) in a different context. Essentially it consists in to prove that the set $H_0 = \{(\mu,\nu); h_{top}(\mu,\nu) = 0\}$ is a connected set.

The main result, that we have verify here, allows us to advance a step further in to find the boundary of the set H_0 in the region, B, bounded by the curves $\{(0,\nu); 0 \leq \nu \leq 1\}$, $\{(\mu,\nu); \nu = \frac{1+\sqrt{1+4\mu}}{2}, 0 \leq \mu \leq 2\}$, $\{(\mu,0); 0 \leq \mu \leq 1\}$, $\{(\mu,\nu); \mu = \frac{1+\sqrt{1+4\nu}}{2}, 0 \leq \nu \leq 2\}$. We note that for any $(\mu,\nu) \in \{(\mu,\nu); \nu = \frac{1+\sqrt{1+4\mu}}{2}, 0 \leq \mu \leq 2\}$, we have $b(\mu,\nu) = \underline{1}$ and for any $(\mu,\nu) \in \{(\mu,\nu); \mu = \frac{1+\sqrt{1+4\nu}}{2}, 0 \leq \nu \leq 2\}$ we have $a(\mu,\nu) = \underline{0}$.

In fact, accordingly with [5] we have the following results in the quadratic family.

Let us consider $a \in \mathcal{A}_\infty^\infty$ and $b = b(a) = \sup\{\sigma^j(a); j \in N\}$.

Let $B_1(\underline{0}, a_-\underline{b}, b_+\underline{a}, \underline{1}) = \{(\mu,\nu) \in B; \underline{0} \leq I(-\mu) \leq a_-\underline{b}, b_+\underline{a} \leq I(\nu) \leq \underline{1}\}$, $B_2(a_-\underline{b}, \underline{a}, b_+\underline{a}, \underline{1}) = \{(\mu,\nu) \in B; a_-\underline{b} \leq I(-\mu) \leq \underline{a}, b_+\underline{a} < I(\nu) \leq \underline{1}\}$, $B_3(\underline{0}, a_-\underline{b}, \underline{b}, b_+\underline{a}) = \{(\mu,\nu) \in B; \underline{0} \leq I(-\mu) < a_-\underline{b}, \underline{b} \leq I(\nu) \leq b_+\underline{a}\}$. For any $(\mu,\nu) \in B_1 \cup B_2 \cup B_3$ let $\Lambda(\mu,\nu) = \bigcap_{j=0}^{\infty} F_{\mu,\nu}^{-j}([-\mu,\nu])$

Proposition 5.1. *For $(\mu,\nu) \in B_1 \cup B_2 \cup B_3$ we have that* $h_{top}\left(F_{\mu,\nu}\mid_{\Lambda(\mu,\nu)}\right) > 0$

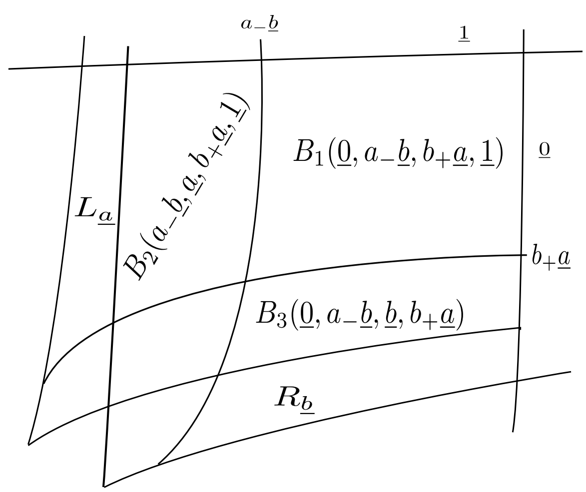


Figure 5: Regions $B_1 \cup B_2 \cup B_3$

Observation 5.2. *1. The set $B_2(a_{-}\underline{b}, \underline{a}, b_{+}\underline{a}, \underline{1})$ include the region $L_{\underline{a}} = \{(\mu,\nu); I(-\mu) = \underline{a}, b_{+}\underline{a} < I(\nu) \leq \underline{1}\}$ which has a non-empty interior (see [20]).*

2. The set $B_3(\underline{0}, a_{-}\underline{b}, \underline{b}, b_{+}\underline{a})$ include the region $R_{\underline{b}} = \{(\mu,\nu); I(\nu) = \underline{b}, \underline{0} \leq I(-\mu) < a_{-}\underline{b}\}$ which has a non-empty interior.

3. The explanation of the figures in this section were given at [20] and [31]

For $a \in A_{\infty}^{\infty}$ let $C_1(\underline{a}, b_{+}\underline{a}) = \{(\mu,\nu) \in B; I(-\mu) = \underline{a}, I(\nu) = b_{+}\underline{a}\}$, $C_2(\underline{a}, \underline{b}) = \{(\mu,\nu) \in B; I(-\mu) = \underline{a}, I(\nu) = \underline{b}\}$, $C_3(a_{-}\underline{b}, \underline{b}) = \{(\mu,\nu) \in B; I(-\mu) = a_{-}\underline{b}, I(\nu) = \underline{b}\}$

Proposition 5.3. *For any $(\mu,\nu) \in C_1 \cup C_2 \cup C_3$, we have that* $h_{top}(F_{\mu,\nu} \mid_{\Lambda(\mu,\nu)}) = 0$

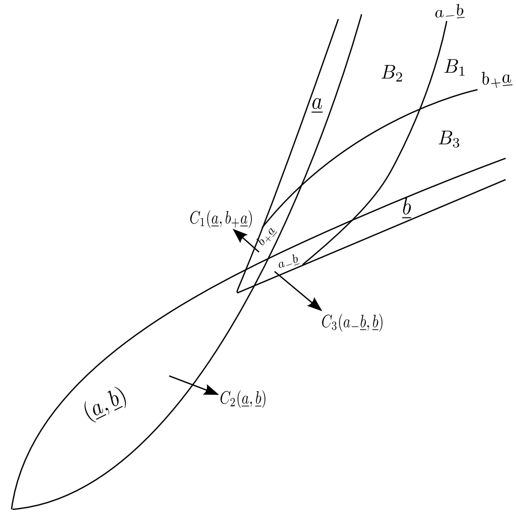


Figure 6: Region $C_1 \cup C_2 \cup C_3$

Now, let us denote by $B(a,b)$, for $a \in \mathcal{A}_{\infty}^{\infty}$ and $b = b(a)$ the region bounded by the curves $\gamma_{a,b_{+}\underline{a}} = \{(\mu,\nu); a(\mu,\nu) = a; b(\mu,\nu) = b_{+}\underline{a}\}$, $\gamma_{a_{-}\underline{b},\underline{b}} = \{(\mu,\nu); a(\mu,\nu) = a_{-}\underline{b}; b(\mu,\nu) = \underline{b}\}$, $\gamma_{b_{+}\underline{a}} = \{(\mu,\nu); a_{-}\underline{b} \leq a(\mu,\nu) \leq \underline{a}\}$, and $\gamma_{a_{-}\underline{b}} = \{(\mu,\nu); \underline{b} \leq b(\mu,\nu) \leq b_{+}\underline{a}\}$.

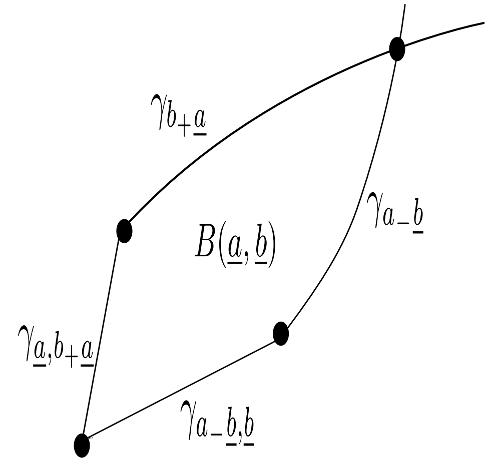


Figure 7: Region $B(a,b)$

The maximal value for the topological entropy in the region $B(a,b)$ is obtained at the intersection of the curves $\gamma_{b_+\underline{a}}$ and $\gamma_{a_-\underline{b}}$ and this value is exactly $h_{top}(a_-\underline{b}, b_+\underline{a}) = \frac{1}{Per(a)}\ln(2)$. That is, our values are an upper bound for the values for the topological entropy $h_{top}(\mu,\nu)$ for $(\mu,\nu) \in B(a,b)$.

Now, let us consider the doubling period sequences $\underline{a_1} = \overline{a_-b_+}$ and $\underline{b_1} = \overline{b_+a_-} = \underline{b(a_1)}$. Let us define the corresponding regions $C_1(\underline{a_1}, \underline{(b_1)_+a_1})$, $C_2(\underline{a_1}, \underline{b_1})$, $C_3(\underline{(a_1)_-b_1}, \underline{b_1})$, $B_1(\underline{0}, \underline{(a_1)_-b_1}, \underline{(b_1)_+a_1}, \underline{1})$, $B_2(\underline{(a_1)_-b_1}, \underline{a_1}, \underline{(b_1)_+a_1}, \underline{1})$, $B_3(\underline{0}, \underline{(a_1)_-b_1}, \underline{b_1}, \underline{(b_1)_+a_1})$ and $B(\underline{a_1}, \underline{b_1})$. All of these regions are included in $B(\underline{a}, \underline{b})$ (see figure 8)

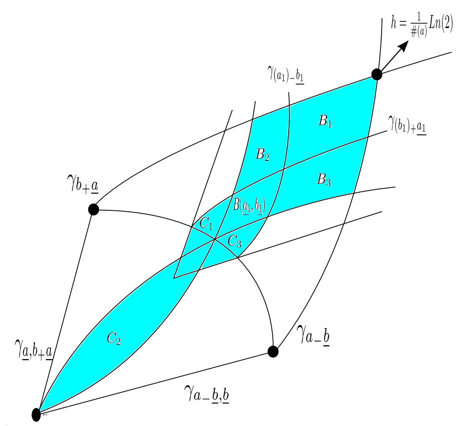


Figure 8: Region $B(a_1, b_1)$

In this case for the parameter value $(\overline{\mu},\overline{\nu}) = \gamma_{a_-\underline{b}} \cap \gamma_{b_+\underline{a}}$ and $(\overline{\mu_1},\overline{\nu_1}) = \gamma_{(a_1)_-\underline{b_1}} \cap \gamma_{(b_1)_+\underline{a_1}}$ we have that $h_{top}(\overline{\mu_1},\overline{\nu_1}) = \dfrac{1}{2\#(a)}Ln(2) = \dfrac{1}{2}h_{top}(\overline{\mu},\overline{\nu})$.

That is for any element $(\mu,\nu)\in B(\underline{a_1},\underline{b_1})$ we have $h_{top}(\mu,\nu)\le\dfrac{1}{2}h_{top}(\overline{\mu},\overline{\nu})$.
Moreover, for $(\mu,\nu)\in B_1(\underline{0},(a_1)_-\underline{b_1},(b_1)_+\underline{a_1},\underline{1})\cup B_2((a_1)_-\underline{b_1},\underline{a_1},(b_1)_+\underline{a_1},\underline{1})\cup B_3(\underline{0},(a_1)_-\underline{b_1},\underline{b_1},(b_1)_+\underline{a_1})$ we have $h_{top}(\mu,\nu)>0$.
Also for any $(\mu,\nu)\in C_1(\underline{a_1},(b_1)_+\underline{a_1})\cup C_2(\underline{a_1},\underline{b_1})\cup C_3((a_1)_-\underline{b_1},\underline{b_1})$ we have $h_{top}(\mu,\nu)=0$.

Therefore, we can connect any point of $C_2(\underline{a},\underline{b})$ with any point of $C_1(\underline{a_1},(b_1)_+\underline{a_1})\cup C_2(\underline{a_1},\underline{b_1})\cup C_3((a_1)_-\underline{b_1},\underline{b_1})$ with a path completely contained in the region $\{(\mu,\nu);h_{top}(\mu,\nu)=0\}$.

In a similar way we can consider $\underline{a_2}=\underline{(a_1)_-(b_1)_+}$ and $\underline{b_2}=\underline{(b_1)_+(a_1)_-}=b(a_2)$ and to proceed to construct regions $C_1,C_2,C_3,B_1,B_2,B_3,B$ associated to $(\underline{a_2},\underline{b_2})$ and so on. We observe that for elements $(\mu,\nu)\in B(\underline{a_2},\underline{b_2})$ we will have $h_{top}(\mu,\nu)\le\dfrac{1}{2\#(a_1)}Ln(2)=\dfrac{1}{2^2\#(a)}Ln(2)$; that is we have a Lower bound for the topological entropy than in $B(\underline{a_1},\underline{b_1})$.

Successively, for $\underline{a_{n+1}}=\underline{(a_n)_-(b_n)_+}$, $\underline{b_{n+1}}=\underline{(b_n)_+(a_n)_-}=b(a_{n+1})$ we construct similar regions $C_1,C_2,C_3,B_1,B_2,B_3,B$.
Moreover, any element in
$C_1(\underline{a_{n+1}},(b_{n+1})_+\underline{a_{n+1}})\cup C_2(\underline{a_{n+1}},\underline{b_{n+1}})\cup C_3((a_{n+1})_-\underline{b_{n+1}},\underline{b_{n+1}})$ can be connected to any element of $C_2(\underline{a},\underline{b})$ by an arc γ completely contained in the region $\{(\mu,\nu);h_{top}(\mu,\nu)=0\}$. In this way we observe that there is a good chance to obtain a proof that the isentrope $\{(\mu,\nu);h_{top}(\mu,\nu)=0\}$ is a arc connected set.

6. Materials and Methods

The algorithm for the computation of the topological entropy is know, see for instance [6] or [16].

Acknowledgments: Part of this paper is an outgrowth of research of S. Aranzubia and R. Labarca during a visit to IMPA Brasil. Also, R. Labarca acknowledge the kind hospitality of the Imperial College while preparing part of this work and the finnantial support of project USA1656(CHILE).

A. Entropy Values for Prime Numbers Between 3 and 9923

Entropy Values for Prime Numbers Between 3 and 751

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
3	0.231048000	5	0.138631000	7	0.099021500	11	0.063012400
13	0.053322700	17	0.040777200	19	0.036486200	23	0.030141200
29	0.023902100	31	0.022358200	37	0.018733400	41	0.016906300
43	0.016119400	47	0.014750700	53	0.013074200	59	0.011750700
61	0.011365200	67	0.010346300	71	0.009762190	73	0.009494780
79	0.008771420	83	0.008355000	89	0.007789580	97	0.007144420
101	0.006866370	103	0.006727320	107	0.006478970	109	0.006359730
113	0.006131170	127	0.005455090	131	0.005295950	137	0.005057190
139	0.004987540	149	0.004649180	151	0.004589450	157	0.004410260
163	0.004250950	167	0.004151370	173	0.004001980	179	0.003872490
181	0.003832650	191	0.003633390	193	0.003593540	197	0.003513820
199	0.003483920	211	0.003284600	223	0.003105170	227	0.003055330
229	0.003025420	233	0.002975570	239	0.002895800	241	0.002875860
251	0.002766170	257	0.002696360	263	0.002636520	269	0.002576680
271	0.002556730	277	0.002506860	281	0.002466950	283	0.002447000
293	0.002367200	307	0.002257450	311	0.002227520	313	0.002217540
317	0.002187610	331	0.002097800	337	0.002057880	347	0.001998000
349	0.001988020	353	0.001968060	359	0.001928140	367	0.001888220
373	0.001858270	379	0.001828330	383	0.001808360	389	0.001778420
397	0.001748470	401	0.001728510	409	0.001698560	419	0.001658620
421	0.001648640	431	0.001608710	433	0.001598720	439	0.001578750
443	0.001568770	449	0.001538820	457	0.001518850	461	0.001498880
463	0.001498880	467	0.001488890	479	0.001448950	487	0.001418990
491	0.001409010	499	0.001389030	503	0.001379050	509	0.001359080
521	0.001329120	523	0.001329120	541	0.001279180	547	0.001269190
557	0.001249220	563	0.001229240	569	0.001219260	571	0.001209270
577	0.001199280	587	0.001179300	593	0.001169320	599	0.001159330
601	0.001149340	607	0.001139350	613	0.001129360	617	0.001119370
619	0.001119370	631	0.001099400	641	0.001079420	643	0.001079420
647	0.001069430	653	0.001059440	659	0.001049450	661	0.001049450
673	0.001029470	677	0.001019480	683	0.001019480	691	0.000999500
701	0.000989510	709	0.000979520	719	0.000959539	727	0.000949549
733	0.000949549	739	0.000939558	743	0.000929568	751	0.000919577

Entropy Values for Prime Numbers Between 757 and 1697

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
757	0.000919577	761	0.000909586	769	0.000899595	773	0.000899595
787	0.000879613	797	0.000869622	809	0.000859630	811	0.000859630
821	0.000839647	823	0.000839647	827	0.000839647	829	0.000839647
839	0.000829656	853	0.000809672	857	0.000809672	859	0.000809672
863	0.000799680	877	0.000789688	881	0.000789688	883	0.000789688
887	0.000779696	907	0.000759711	911	0.000759711	919	0.000749719
929	0.000749719	937	0.000739726	941	0.000739726	947	0.000729734
953	0.000729734	967	0.000719741	971	0.000709748	977	0.000709748
983	0.000709748	991	0.000699755	997	0.000699755	1009	0.000689762
1013	0.000679769	1019	0.000679769	1021	0.000679769	1031	0.000669776
1033	0.000669776	1039	0.000669776	1049	0.000659782	1051	0.000659782
1061	0.000649789	1063	0.000649789	1069	0.000649789	1087	0.000639795
1091	0.000639795	1093	0.000629802	1097	0.000629802	1103	0.000629802
1109	0.000629802	1117	0.000619808	1123	0.000619808	1129	0.000609814
1151	0.000599820	1153	0.000599820	1163	0.000599820	1171	0.000589826
1181	0.000589826	1187	0.000579832	1193	0.000579832	1201	0.000579832
1213	0.000569838	1217	0.000569838	1223	0.000569838	1229	0.000559843
1231	0.000559843	1237	0.000559843	1249	0.000559843	1259	0.000549849
1277	0.000539854	1279	0.000539854	1283	0.000539854	1289	0.000539854
1291	0.000539854	1297	0.000529860	1301	0.000529860	1303	0.000529860
1307	0.000529860	1319	0.000529860	1321	0.000519865	1327	0.000519865
1361	0.000509870	1367	0.000509870	1373	0.000499875	1381	0.000499875
1399	0.000499875	1409	0.000489880	1423	0.000489880	1427	0.000489880
1429	0.000489880	1433	0.000479885	1439	0.000479885	1447	0.000479885
1451	0.000479885	1453	0.000479885	1459	0.000479885	1471	0.000469890
1481	0.000469890	1483	0.000469890	1487	0.000469890	1489	0.000469890
1493	0.000459894	1499	0.000459894	1511	0.000459894	1523	0.000459894
1531	0.000449899	1543	0.000449899	1549	0.000449899	1553	0.000449899
1559	0.000439903	1567	0.000439903	1571	0.000439903	1579	0.000439903
1583	0.000439903	1597	0.000429908	1601	0.000429908	1607	0.000429908
1609	0.000429908	1613	0.000429908	1619	0.000429908	1621	0.000429908
1627	0.000429908	1637	0.000419912	1657	0.000419912	1663	0.000419912
1667	0.000419912	1669	0.000419912	1693	0.000409916	1697	0.000409916

Entropy Values for Prime Numbers Between 1699 and 2719

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
1699	0.000409916	1709	0.000409916	1721	0.000399920	1723	0.000399920
1733	0.000399920	1741	0.000399920	1747	0.000399920	1753	0.000399920
1759	0.000389924	1777	0.000389924	1783	0.000389924	1787	0.000389924
1789	0.000389924	1801	0.000379928	1811	0.000379928	1823	0.000379928
1831	0.000379928	1847	0.000379928	1861	0.000369932	1867	0.000369932
1871	0.000369932	1873	0.000369932	1877	0.000369932	1879	0.000369932
1889	0.000369932	1901	0.000359935	1907	0.000359935	1913	0.000359935
1931	0.000359935	1933	0.000359935	1949	0.000359935	1951	0.000359935
1973	0.000349939	1979	0.000349939	1987	0.000349939	1993	0.000349939
1997	0.000349939	1999	0.000349939	2003	0.000349939	2011	0.000339942
2017	0.000339942	2027	0.000339942	2029	0.000339942	2039	0.000339942
2053	0.000339942	2063	0.000339942	2069	0.000339942	2081	0.000329946
2083	0.000329946	2087	0.000329946	2089	0.000329946	2099	0.000329946
2111	0.000329946	2113	0.000329946	2129	0.000329946	2131	0.000329946
2137	0.000319949	2141	0.000319949	2143	0.000319949	2153	0.000319949
2161	0.000319949	2179	0.000319949	2203	0.000309952	2207	0.000309952
2213	0.000309952	2221	0.000309952	2237	0.000309952	2239	0.000309952
2243	0.000309952	2251	0.000309952	2267	0.000309952	2269	0.000309952
2273	0.000299955	2281	0.000299955	2287	0.000299955	2293	0.000299955
2297	0.000299955	2309	0.000299955	2311	0.000299955	2333	0.000299955
2339	0.000299955	2341	0.000299955	2347	0.000299955	2351	0.000289958
2357	0.000289958	2371	0.000289958	2377	0.000289958	2381	0.000289958
2383	0.000289958	2389	0.000289958	2393	0.000289958	2399	0.000289958
2411	0.000289958	2417	0.000289958	2423	0.000289958	2437	0.000279961
2441	0.000279961	2447	0.000279961	2459	0.000279961	2467	0.000279961
2473	0.000279961	2477	0.000279961	2503	0.000279961	2521	0.000269964
2531	0.000269964	2539	0.000269964	2543	0.000269964	2549	0.000269964
2551	0.000269964	2557	0.000269964	2579	0.000269964	2591	0.000269964
2593	0.000269964	2609	0.000269964	2617	0.000259966	2621	0.000259966
2633	0.000259966	2647	0.000259966	2657	0.000259966	2659	0.000259966
2663	0.000259966	2671	0.000259966	2677	0.000259966	2683	0.000259966
2687	0.000259966	2689	0.000259966	2693	0.000259966	2699	0.000259966
2707	0.000259966	2711	0.000259966	2713	0.000259966	2719	0.000249969

Entropy Values for Prime Numbers Between 2729 and 3803

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
2729	0.000249969	2731	0.000249969	2741	0.000249969	2749	0.000249969
2753	0.000249969	2767	0.000249969	2777	0.000249969	2789	0.000249969
2791	0.000249969	2797	0.000249969	2801	0.000249969	2803	0.000249969
2819	0.000249969	2833	0.000239971	2837	0.000239971	2843	0.000239971
2851	0.000239971	2857	0.000239971	2861	0.000239971	2879	0.000239971
2887	0.000239971	2897	0.000239971	2903	0.000239971	2909	0.000239971
2917	0.000239971	2927	0.000239971	2939	0.000239971	2953	0.000229974
2957	0.000229974	2963	0.000229974	2969	0.000229974	2971	0.000229974
2999	0.000229974	3001	0.000229974	3011	0.000229974	3019	0.000229974
3023	0.000229974	3037	0.000229974	3041	0.000229974	3049	0.000229974
3061	0.000229974	3067	0.000229974	3079	0.000229974	3083	0.000219976
3089	0.000219976	3109	0.000219976	3119	0.000219976	3121	0.000219976
3137	0.000219976	3163	0.000219976	3167	0.000219976	3169	0.000219976
3181	0.000219976	3187	0.000219976	3191	0.000219976	3203	0.000219976
3209	0.000219976	3217	0.000219976	3221	0.000219976	3229	0.000209978
3251	0.000209978	3253	0.000209978	3257	0.000209978	3259	0.000209978
3271	0.000209978	3299	0.000209978	3301	0.000209978	3307	0.000209978
3313	0.000209978	3319	0.000209978	3323	0.000209978	3329	0.000209978
3331	0.000209978	3343	0.000209978	3347	0.000209978	3359	0.000209978
3361	0.000209978	3371	0.000209978	3373	0.000209978	3389	0.000199980
3391	0.000199980	3407	0.000199980	3413	0.000199980	3433	0.000199980
3449	0.000199980	3457	0.000199980	3461	0.000199980	3463	0.000199980
3467	0.000199980	3469	0.000199980	3491	0.000199980	3499	0.000199980
3511	0.000199980	3517	0.000199980	3527	0.000199980	3529	0.000199980
3533	0.000199980	3539	0.000199980	3541	0.000199980	3547	0.000199980
3557	0.000189982	3559	0.000189982	3571	0.000189982	3581	0.000189982
3583	0.000189982	3593	0.000189982	3607	0.000189982	3613	0.000189982
3617	0.000189982	3623	0.000189982	3631	0.000189982	3637	0.000189982
3643	0.000189982	3659	0.000189982	3671	0.000189982	3673	0.000189982
3677	0.000189982	3691	0.000189982	3697	0.000189982	3701	0.000189982
3709	0.000189982	3719	0.000189982	3727	0.000189982	3733	0.000189982
3739	0.000189982	3761	0.000179984	3767	0.000179984	3769	0.000179984
3779	0.000179984	3793	0.000179984	3797	0.000179984	3803	0.000179984

Entropy Values for Prime Numbers Between 3821 and 4943

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
3821	0.000179984	3823	0.000179984	3833	0.000179984	3847	0.000179984
3851	0.000179984	3853	0.000179984	3863	0.000179984	3877	0.000179984
3881	0.000179984	3889	0.000179984	3907	0.000179984	3911	0.000179984
3917	0.000179984	3919	0.000179984	3923	0.000179984	3929	0.000179984
3931	0.000179984	3943	0.000179984	3947	0.000179984	3967	0.000169986
3989	0.000169986	4001	0.000169986	4003	0.000169986	4007	0.000169986
4013	0.000169986	4019	0.000169986	4021	0.000169986	4027	0.000169986
4049	0.000169986	4051	0.000169986	4057	0.000169986	4073	0.000169986
4079	0.000169986	4091	0.000169986	4093	0.000169986	4099	0.000169986
4111	0.000169986	4127	0.000169986	4129	0.000169986	4133	0.000169986
4139	0.000169986	4153	0.000169986	4157	0.000169986	4159	0.000169986
4177	0.000169986	4201	0.000169986	4211	0.000159987	4217	0.000159987
4219	0.000159987	4229	0.000159987	4231	0.000159987	4241	0.000159987
4243	0.000159987	4253	0.000159987	4259	0.000159987	4261	0.000159987
4271	0.000159987	4273	0.000159987	4283	0.000159987	4289	0.000159987
4297	0.000159987	4327	0.000159987	4337	0.000159987	4339	0.000159987
4349	0.000159987	4357	0.000159987	4363	0.000159987	4373	0.000159987
4391	0.000159987	4397	0.000159987	4409	0.000159987	4421	0.000159987
4423	0.000159987	4441	0.000159987	4447	0.000159987	4451	0.000159987
4457	0.000159987	4463	0.000159987	4481	0.000149989	4483	0.000149989
4493	0.000149989	4507	0.000149989	4513	0.000149989	4517	0.000149989
4519	0.000149989	4523	0.000149989	4547	0.000149989	4549	0.000149989
4561	0.000149989	4567	0.000149989	4583	0.000149989	4591	0.000149989
4597	0.000149989	4603	0.000149989	4621	0.000149989	4637	0.000149989
4639	0.000149989	4643	0.000149989	4649	0.000149989	4651	0.000149989
4657	0.000149989	4663	0.000149989	4673	0.000149989	4679	0.000149989
4691	0.000149989	4703	0.000149989	4721	0.000149989	4723	0.000149989
4729	0.000149989	4733	0.000149989	4751	0.000149989	4759	0.000149989
4783	0.000139990	4787	0.000139990	4789	0.000139990	4793	0.000139990
4799	0.000139990	4801	0.000139990	4813	0.000139990	4817	0.000139990
4831	0.000139990	4861	0.000139990	4871	0.000139990	4877	0.000139990
4889	0.000139990	4903	0.000139990	4909	0.000139990	4919	0.000139990
4931	0.000139990	4933	0.000139990	4937	0.000139990	4943	0.000139990

Entropy Values for Prime Numbers Between 4951 and 6079

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
4951	0.000139990	4957	0.000139990	4967	0.000139990	4969	0.000139990
4973	0.000139990	4987	0.000139990	4993	0.000139990	4999	0.000139990
5003	0.000139990	5009	0.000139990	5011	0.000139990	5021	0.000139990
5023	0.000139990	5039	0.000139990	5051	0.000139990	5059	0.000139990
5077	0.000139990	5081	0.000139990	5087	0.000139990	5099	0.000139990
5101	0.000139990	5107	0.000139990	5113	0.000139990	5119	0.000139990
5147	0.000129992	5153	0.000129992	5167	0.000129992	5171	0.000129992
5179	0.000129992	5189	0.000129992	5197	0.000129992	5209	0.000129992
5227	0.000129992	5231	0.000129992	5233	0.000129992	5237	0.000129992
5261	0.000129992	5273	0.000129992	5279	0.000129992	5281	0.000129992
5297	0.000129992	5303	0.000129992	5309	0.000129992	5323	0.000129992
5333	0.000129992	5347	0.000129992	5351	0.000129992	5381	0.000129992
5387	0.000129992	5393	0.000129992	5399	0.000129992	5407	0.000129992
5413	0.000129992	5417	0.000129992	5419	0.000129992	5431	0.000129992
5437	0.000129992	5441	0.000129992	5443	0.000129992	5449	0.000129992
5471	0.000129992	5477	0.000129992	5479	0.000129992	5483	0.000129992
5501	0.000129992	5503	0.000129992	5507	0.000129992	5519	0.000129992
5521	0.000129992	5527	0.000129992	5531	0.000129992	5557	0.000119993
5563	0.000119993	5569	0.000119993	5573	0.000119993	5581	0.000119993
5591	0.000119993	5623	0.000119993	5639	0.000119993	5641	0.000119993
5647	0.000119993	5651	0.000119993	5653	0.000119993	5657	0.000119993
5659	0.000119993	5669	0.000119993	5683	0.000119993	5689	0.000119993
5693	0.000119993	5701	0.000119993	5711	0.000119993	5717	0.000119993
5737	0.000119993	5741	0.000119993	5743	0.000119993	5749	0.000119993
5779	0.000119993	5783	0.000119993	5791	0.000119993	5801	0.000119993
5807	0.000119993	5813	0.000119993	5821	0.000119993	5827	0.000119993
5839	0.000119993	5843	0.000119993	5849	0.000119993	5851	0.000119993
5857	0.000119993	5861	0.000119993	5867	0.000119993	5869	0.000119993
5879	0.000119993	5881	0.000119993	5897	0.000119993	5903	0.000119993
5923	0.000119993	5927	0.000119993	5939	0.000119993	5953	0.000119993
5981	0.000119993	5987	0.000119993	6007	0.000119993	6011	0.000119993
6029	0.000109994	6037	0.000109994	6043	0.000109994	6047	0.000109994
6053	0.000109994	6067	0.000109994	6073	0.000109994	6079	0.000109994

Entropy Values for Prime Numbers Between 6089 and 7237

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
6089	0.000109994	6091	0.000109994	6101	0.000109994	6113	0.000109994
6121	0.000109994	6131	0.000109994	6133	0.000109994	6143	0.000109994
6151	0.000109994	6163	0.000109994	6173	0.000109994	6197	0.000109994
6199	0.000109994	6203	0.000109994	6211	0.000109994	6217	0.000109994
6221	0.000109994	6229	0.000109994	6247	0.000109994	6257	0.000109994
6263	0.000109994	6269	0.000109994	6271	0.000109994	6277	0.000109994
6287	0.000109994	6299	0.000109994	6301	0.000109994	6311	0.000109994
6317	0.000109994	6323	0.000109994	6329	0.000109994	6337	0.000109994
6343	0.000109994	6353	0.000109994	6359	0.000109994	6361	0.000109994
6367	0.000109994	6373	0.000109994	6379	0.000109994	6389	0.000109994
6397	0.000109994	6421	0.000109994	6427	0.000109994	6449	0.000109994
6451	0.000109994	6469	0.000109994	6473	0.000109994	6481	0.000109994
6491	0.000109994	6521	0.000109994	6529	0.000109994	6547	0.000109994
6551	0.000109994	6553	0.000109994	6563	0.000109994	6569	0.000109994
6571	0.000109994	6577	0.000109994	6581	0.000109994	6599	0.000109994
6607	1.00E-004	6619	1.00E-004	6637	1.00E-004	6653	1.00E-004
6659	1.00E-004	6661	1.00E-004	6673	1.00E-004	6679	1.00E-004
6689	1.00E-004	6691	1.00E-004	6701	1.00E-004	6703	1.00E-004
6709	1.00E-004	6719	1.00E-004	6733	1.00E-004	6737	1.00E-004
6761	1.00E-004	6763	1.00E-004	6779	1.00E-004	6781	1.00E-004
6791	1.00E-004	6793	1.00E-004	6803	1.00E-004	6823	1.00E-004
6827	1.00E-004	6829	1.00E-004	6833	1.00E-004	6841	1.00E-004
6857	1.00E-004	6863	1.00E-004	6869	1.00E-004	6871	1.00E-004
6883	1.00E-004	6899	1.00E-004	6907	1.00E-004	6911	1.00E-004
6917	1.00E-004	6947	1.00E-004	6949	1.00E-004	6959	1.00E-004
6961	1.00E-004	6967	1.00E-004	6971	1.00E-004	6977	1.00E-004
6983	1.00E-004	6991	1.00E-004	6997	1.00E-004	7001	1.00E-004
7013	1.00E-004	7019	1.00E-004	7027	1.00E-004	7039	1.00E-004
7043	1.00E-004	7057	1.00E-004	7069	1.00E-004	7079	1.00E-004
7103	1.00E-004	7109	1.00E-004	7121	1.00E-004	7127	1.00E-004
7129	1.00E-004	7151	1.00E-004	7159	1.00E-004	7177	1.00E-004
7187	1.00E-004	7193	1.00E-004	7207	1.00E-004	7211	1.00E-004
7213	1.00E-004	7219	1.00E-004	7229	1.00E-004	7237	1.00E-004

Entropy Values for Prime Numbers Between 7243 and 8447

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
7243	1.00E-004	7247	1.00E-004	7253	1.00E-004	7283	1.00E-004
7297	9.00E-005	7307	9.00E-005	7309	9.00E-005	7321	9.00E-005
7331	9.00E-005	7333	9.00E-005	7349	9.00E-005	7351	9.00E-005
7369	9.00E-005	7393	9.00E-005	7411	9.00E-005	7417	9.00E-005
7433	9.00E-005	7451	9.00E-005	7457	9.00E-005	7459	9.00E-005
7477	9.00E-005	7481	9.00E-005	7487	9.00E-005	7489	9.00E-005
7499	9.00E-005	7507	9.00E-005	7517	9.00E-005	7523	9.00E-005
7529	9.00E-005	7537	9.00E-005	7541	9.00E-005	7547	9.00E-005
7549	9.00E-005	7559	9.00E-005	7561	9.00E-005	7573	9.00E-005
7577	9.00E-005	7583	9.00E-005	7589	9.00E-005	7591	9.00E-005
7603	9.00E-005	7607	9.00E-005	7621	9.00E-005	7639	9.00E-005
7643	9.00E-005	7649	9.00E-005	7669	9.00E-005	7673	9.00E-005
7681	9.00E-005	7687	9.00E-005	7691	9.00E-005	7699	9.00E-005
7703	9.00E-005	7717	9.00E-005	7723	9.00E-005	7727	9.00E-005
7741	9.00E-005	7753	9.00E-005	7757	9.00E-005	7759	9.00E-005
7789	9.00E-005	7793	9.00E-005	7817	9.00E-005	7823	9.00E-005
7829	9.00E-005	7841	9.00E-005	7853	9.00E-005	7867	9.00E-005
7873	9.00E-005	7877	9.00E-005	7879	9.00E-005	7883	9.00E-005
7901	9.00E-005	7907	9.00E-005	7919	9.00E-005	7927	9.00E-005
7933	9.00E-005	7937	9.00E-005	7949	9.00E-005	7951	9.00E-005
7963	9.00E-005	7993	9.00E-005	8009	9.00E-005	8011	9.00E-005
8017	9.00E-005	8039	9.00E-005	8053	9.00E-005	8059	9.00E-005
8069	9.00E-005	8081	9.00E-005	8087	9.00E-005	8089	9.00E-005
8093	9.00E-005	8101	9.00E-005	8111	9.00E-005	8117	9.00E-005
8123	9.00E-005	8147	9.00E-005	8161	8.00E-005	8167	8.00E-005
8171	8.00E-005	8179	8.00E-005	8191	8.00E-005	8209	8.00E-005
8219	8.00E-005	8221	8.00E-005	8231	8.00E-005	8233	8.00E-005
8237	8.00E-005	8243	8.00E-005	8263	8.00E-005	8269	8.00E-005
8273	8.00E-005	8287	8.00E-005	8291	8.00E-005	8293	8.00E-005
8297	8.00E-005	8311	8.00E-005	8317	8.00E-005	8329	8.00E-005
8353	8.00E-005	8363	8.00E-005	8369	8.00E-005	8377	8.00E-005
8387	8.00E-005	8389	8.00E-005	8419	8.00E-005	8423	8.00E-005
8429	8.00E-005	8431	8.00E-005	8443	8.00E-005	8447	8.00E-005

Entropy Values for Prime Numbers Between 8461 and 9697

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
8461	8.00E-005	8467	8.00E-005	8501	8.00E-005	8513	8.00E-005
8521	8.00E-005	8527	8.00E-005	8537	8.00E-005	8539	8.00E-005
8543	8.00E-005	8563	8.00E-005	8573	8.00E-005	8581	8.00E-005
8597	8.00E-005	8599	8.00E-005	8609	8.00E-005	8623	8.00E-005
8627	8.00E-005	8629	8.00E-005	8641	8.00E-005	8647	8.00E-005
8663	8.00E-005	8669	8.00E-005	8677	8.00E-005	8681	8.00E-005
8689	8.00E-005	8693	8.00E-005	8699	8.00E-005	8707	8.00E-005
8713	8.00E-005	8719	8.00E-005	8731	8.00E-005	8737	8.00E-005
8741	8.00E-005	8747	8.00E-005	8753	8.00E-005	8761	8.00E-005
8779	8.00E-005	8783	8.00E-005	8803	8.00E-005	8807	8.00E-005
8819	8.00E-005	8821	8.00E-005	8831	8.00E-005	8837	8.00E-005
8839	8.00E-005	8849	8.00E-005	8861	8.00E-005	8863	8.00E-005
8867	8.00E-005	8887	8.00E-005	8893	8.00E-005	8923	8.00E-005
8929	8.00E-005	8933	8.00E-005	8941	8.00E-005	8951	8.00E-005
8963	8.00E-005	8969	8.00E-005	8971	8.00E-005	8999	8.00E-005
9001	8.00E-005	9007	8.00E-005	9011	8.00E-005	9013	8.00E-005
9029	8.00E-005	9041	8.00E-005	9043	8.00E-005	9049	8.00E-005
9059	8.00E-005	9067	8.00E-005	9091	8.00E-005	9103	8.00E-005
9109	8.00E-005	9127	8.00E-005	9133	8.00E-005	9137	8.00E-005
9151	8.00E-005	9157	8.00E-005	9161	8.00E-005	9173	8.00E-005
9181	8.00E-005	9187	8.00E-005	9199	8.00E-005	9203	8.00E-005
9209	8.00E-005	9221	8.00E-005	9227	8.00E-005	9239	8.00E-005
9241	8.00E-005	9257	7.00E-005	9277	7.00E-005	9281	7.00E-005
9283	7.00E-005	9293	7.00E-005	9311	7.00E-005	9319	7.00E-005
9323	7.00E-005	9337	7.00E-005	9341	7.00E-005	9343	7.00E-005
9349	7.00E-005	9371	7.00E-005	9377	7.00E-005	9391	7.00E-005
9397	7.00E-005	9403	7.00E-005	9413	7.00E-005	9419	7.00E-005
9421	7.00E-005	9431	7.00E-005	9433	7.00E-005	9437	7.00E-005
9439	7.00E-005	9461	7.00E-005	9463	7.00E-005	9467	7.00E-005
9473	7.00E-005	9479	7.00E-005	9491	7.00E-005	9497	7.00E-005
9511	7.00E-005	9521	7.00E-005	9533	7.00E-005	9539	7.00E-005
9547	7.00E-005	9551	7.00E-005	9587	7.00E-005	9601	7.00E-005
9613	7.00E-005	9619	7.00E-005	9623	7.00E-005	9629	7.00E-005
9631	7.00E-005	9643	7.00E-005	9649	7.00E-005	9661	7.00E-005
9677	7.00E-005	9679	7.00E-005	9689	7.00E-005	9697	7.00E-005

Entropy Values for Prime Numbers Between 9719 and 9973

Value	Entropy	Value	Entropy	Value	Entropy	Value	Entropy
9719	7.00E-005	9721	7.00E-005	9733	7.00E-005	9739	7.00E-005
9743	7.00E-005	9749	7.00E-005	9767	7.00E-005	9769	7.00E-005
9781	7.00E-005	9787	7.00E-005	9791	7.00E-005	9803	7.00E-005
9811	7.00E-005	9817	7.00E-005	9829	7.00E-005	9833	7.00E-005
9839	7.00E-005	9851	7.00E-005	9857	7.00E-005	9859	7.00E-005
9871	7.00E-005	9883	7.00E-005	9887	7.00E-005	9901	7.00E-005
9907	7.00E-005	9923	7.00E-005	9929	7.00E-005	9931	7.00E-005
9941	7.00E-005	9949	7.00E-005	9967	7.00E-005	9973	7.00E-005

References

[1] Adler, R. L.; Konheim, A.G.; Mc Andrew, M.H. *Topological Entropy* Transactiones of the A.M.S. vol 144, pp. 309-319, (1965).

[2] Alekseev, V. M. *Quasirandom Dynamical Systems* Math. USSR Sbornik - New Series Vol. 5 N^o 1, (1968).

[3] Alseda, Lluís; Llibre, Jaume; Misiurewicz, Michal *Combinatorial dynamics and entropy in dimension one.* Second edition. Advenced Series in Nonlinear Dynamics, 5. World Scientific Publishing Co., Inc., River Edge, NJ, xvi+15, pp. ISBN: 981-02-4053-8, (2000).

[4] Aranzubia, S.*Hacia una demostración de la arco conexidad de la isentropa de entropía cero de la familia cuadrática de Lorenz lexicográfica: burbujas de entropía constante y cotas superiores para burbujas las de entropía cero* Tesis de doctorado Universidad de Santiago de Chile, (2015)

[5] Aranzubia, S.; Labarca R. *A Formula for the Boundary of Chaos in the Lexicographical Scenario and Applications to the Bifurcation Diagram of the Standard Two Parameter Family of Quadratic Increasing-Increasing Lorenz maps.* Discrete and Continuous Dynamical Systems-Series A., Vol. 38, N^o 4, pp. 1745-1776, (2018).

[6] Block,L; Guckenheimer, J.; Misiurewicz, M; Young, L.S. *Periodic points and topological entropy.* Lectures Notes in Mathematics. N^o 819, Springer Verlag, (1978).

[7] Bamón,R.; Labarca, R.; Mañé R.; Pacifico, M.J.*The explosion of singular cycles* Publ. Math. IHES Vol 78, pp 207-232, (1993).

[8] Bruin, H.; Van Strien, S. *Monotonicity of Entropy for Real Multimodal Maps* Journal of the American Mathematical Society, Vol 28, No 1, pp. 1-61, (2015).

[9] de Melo W., Van Strien S. *Lectures on One Dimensional Dynamics* Springer Verlag, (1993).

[10] Francis, J. G. F.; The QR Transformation, I. *The Computer Journal*, *4(3)*, pp. 265-271, (1961).

[11] The GNU Scientific Library (GSL); http://www.gnu.org/software/gsl/

[12] Guckenheimer J., Williams R. F. *Structural Stability of Lorenz Attractors* Publ. Math. IHES **50**, pp 59-72, (1979).

[13] Labarca, R. *Bifurcation of Contracting Singular Cycles* Ann. Scient. Ec. Norm. Sup. 4^eserie, t. 28, pp 705-745, (1995).

[14] Labarca, R. *A note on the topological classification of Lorenz maps on the interval.* Topics in symbolic dynamics an applications, London Math. Soc. Lect. Not. Ser. 279, Cambridge Univ. Press. pp. 229-245, (2000).

[15] Labarca, R. *Unfolding singular cycles.* Notas Soc. Mat. Chile (N5) No 1, pp. 38-71, (2001).

[16] Labarca, R,*La Entropía topológica, propiedades generales y algunos cálculos en el caso del shift de Milnor-Thurston.*Con la colaboración de Solange Aranzubia V. y Erick Inda R. Libro de 150 páginas publicado en la serie de las Escuelas Venezolanas de Matemáticas, con comité editorial, de Ediciones IVIC. Instituto Venezolano de Investigaciones Científicas, (2011)

[17] Labarca,R.; Moreira C. *Bifurcation of the Essential Dynamics of Lorenz Maps of the real Line and the Bifurcation Scenario for the Linear family* Scientia Ser A Math. Sci (N-S), Vol. 7, pp. 13-29, (2001).

[18] Labarca, R.; Moreira C. *Bifurcation of the Essential Dynamics of Lorenz Maps and applications to Lorenz Like Flows: Contributions to the study of the Expanding Case.* Bol. Soc. Bros. Mat. (N-S), Vol. 32, pp. 107-144, (2001).

[19] Labarca, R.; Moreira C. *Essential Dynamics for Lorenz Maps on the real line and Lexicographical World.* Ann. de L'Institut H. Poincaré Anal. non Linneaire, Vol. 23, pp. 683-694, (2006).

[20] Labarca, R.; Moreira, C.*Bifurcation of the essential dynamics of Lorenz Maps on the real line and the bifurcation scenario for Lorenz like flows: The Contracting Case* Proyecciones. Journal of Mathematics, Vol 29, N^o. 3, pp. 241-289 (2010).

[21] Labarca R., Plaza S. *Bifurcations of discontinuous maps of the interval and palindromic numbers.* Boletín de la Sociedad Matemática Mexicana(3), Vol 5., pp. 87-104, (2001)

[22] Labarca, R.; San Martín, B. *Prevalence of hyperbolicity for complex singular cycles.* Bol. Soc. Brasil. Mat. (N5) 28, N^o 2 pp. 343-362, (1997).

[23] Labarca, R.; Vásquez, L. *On the Characterization of the kneading sequences associated to Lorenz maps of the interval and to orientation preserving homeomorphism of the circle* Boletín de la Sociedad Matemática Méxicana (3), Vol. 16 N^o. 2, pp. 103-118, (2010).

[24] Labarca R., Vásquez L. *On the Characterization of the Kneading Sequences Associated to Lorenz Maps of the Interval.* Bol. Soc. Bras. Mat. (NS), Vol. 43 N^o 2, pp. 221-245, (2012).

[25] Moreira, C. *Maximal invariant sets for restriction of tent and unimodal maps.* Qual. Thory Dyn. Syst, 2, N^o2, pp. 385-398, (2001).

[26] Metropolis, N.; Stein, M. L. ; Stein, P.R. *"Stabe States of a nonlinear transformation."*. Numerisch Mathematik 10, pp. 1-19, (1967).

[27] Metropolis, N.; Stein, M. L.; Stein, P. R. *"On finite limit sets for transformations on the unit interval."*. Journal of combinatorial Theory (A) 15, pp. 25-44, (1973).

[28] Milnor, J. *Remarks on iterated cubic maps* Experiment. Math. 1, N^o 1, pp. 5-24 MR1181083, (1992).

[29] Milnor, J. ; Thurston, W. *On iterated maps on the interval.* Lect. Notes in Math 1342 pp 465-563 Springer Verlag, (1988).

[30] Misiurewicz, M. *On non continuity of topological entropy* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 19, pp. 319-320, (1971).

[31] Sait Pierre, Matthias *Topological and measurable dynamics of Lorenz Maps.* Dissertationes Mathematicae (ROZPRAWY MATEMATYCZNE) Polska Akademie Nauk, Institut Matematiczny. CCCLXXXII, (1999).

[32] Silnikov, L.P.*A Contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type*, Math. USSR Sbornik, Vol. 10 N^o1, (1970).

[33] Smale, S. *Differentiable dynamical systems.* Bull. Amer. Math. Soc, 73, pp. 747-819, (1967).

[34] Zacks, Mikhail A. *Scaling Properties and renormalization invariants for the "homoclinic quasiperiodicity".* Physyca D, Vol 92, pp. 300-316, (1993).

S. Aranzubia
Escuela de Matemática y Estadística
Universidad Central de Chile
Santiago
Chile
e-mail : solange.aranzubia@gmail.com

R. Carvajal
Departamento de Matematica y Ciencia de la Computacin
Universidad de Santiago de Chile
Casilla 307 Correo 2
Santiago
Chile
e-mail : ruben.carvajal@usach.cl

and

R. Labarca
Departamento de Matematica y Ciencia de la Computacin
Universidad de Santiago de Chile
Casilla 307 Correo 2
Santiago
Chile
e-mail : rafael.labarca@usach.cl