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Fuzzy normed linear sequence space $bv_p^F(X)$ *

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Abstract

In this article we introduce the notion of class of sequences $bv_p^F(X), 1 \leq p < \infty$ with the concept of fuzzy norm. We study some of its properties such as completeness, solidness, symmeticity and convergence free. Also, we establish some inclusion results.

Key Words : *Fuzzy real number, fuzzy normed linear space, monotone, solidness, convergence free and symmetricity.*

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1. Introduction

The concept of fuzzy set, a set whose boundary is not sharp or precise has been introduced by L.A. Zadeh in 1965. This notion originated a new theory of uncertainty, distinct from the notion of probability. After the introduction of fuzzy sets, the scope for studies in different branches of pure and applied mathematics increased widely. The notion of fuzzy sets has successfully been applied in studying sequence spaces with classical metric by Das ([1], [2]), Nanda [5], Nuray and Savas [6], Tripathy and Baruah [10], Tripathy et. al. [11], Tripathy and Debnath [12], Tripathy and Dutta [13], Tripathy and Sen [14] and many others. The works with the concept of fuzzy metric have been done by Kelava and Seikkala [4], Syau [8] and many others. Using the fuzzy norm, a few works in different field have been done by Felbin [3] and some others.

2. Definitions and preliminaries

A fuzzy real number X is a *fuzzy set* on R, i.e. a mapping $X : R \to I(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min (X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper-semi continuous* if, for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$. Throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

The *α-level set* $[X]^\alpha$ of the fuzzy real number X, for $0 < \alpha \leq 1$, defined as $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$. If $\alpha = 0$, then it is the closure of the strong 0-cut. (*The strong α-cut* of the fuzzy real number X, for $0 \leq \alpha \leq 1$ is the set $\{t \in R : X(t) > \alpha\}$).

For $X, Y \in R(I)$ consider a partial ordering $\leq$ by

$$X \leq Y \text{ if and only if } a_1^\alpha \leq a_2^\alpha \text{ and } b_1^\alpha \leq b_2^\alpha, \text{ for all } \alpha \in (0,1],$$

where $[X]^\alpha = [a_1^\alpha,\ b_1^\alpha]$ and $[Y]^\alpha = [a_2^\alpha,\ b_2^\alpha]$.

Let $X, Y \in R(I)$ and α-level sets be $[X]^\alpha = [a_1^\alpha,\ b_1^\alpha]$, $[Y]^\alpha = [a_2^\alpha,\ b_2^\alpha]$, $\alpha \in [0,1]$. Then the arithmetic operations on $R(I)$ in terms of α-level sets are defined by

$$[X \oplus Y]^\alpha = [a_1^\alpha + a_2^\alpha,\ b_1^\alpha + b_2^\alpha],$$

$$[X \ominus Y]^\alpha = [a_1^\alpha - b_2^\alpha,\ b_1^\alpha - a_2^\alpha],$$

$$[X \otimes Y]^\alpha = \left[\min_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha,\ \max_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha\right]$$

and $$[\bar{1} \div Y]^\alpha = \left[\frac{1}{b_2^\alpha},\ \frac{1}{a_2^\alpha}\right],\ 0 \notin Y.$$

The *absolute value*, $|X|$ of $X \in R(I)$ is defined by (one may refer to Kaleva and Seikkala [4])

$$|X|(t) = \begin{cases} \max(X(t), X(-t)), & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

A fuzzy real number X is called *non-negative* if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$.

Fuzzy Normed Linear Space

Let X be a linear space over R and the mapping $\|\cdot\| : X \to R^*(I)$ and the mappings L, $M : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0$ and $M(1,1) = 1$. Write

$[\|x\|]^\alpha = [|||x|||_1^\alpha, |||x|||_2^\alpha,]$, for $x \in X, 0 < \alpha \leq 1$ and suppose for all $x \in X, x \neq 0$, there exists $\alpha_0 \in (0,1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $|||x|||_2^\alpha < \infty$,

(B) $\inf_{\alpha\in(0,1]} |||x|||_1^\alpha > 0$.

The quadruple $(X, \|\cdot\|, L, M)$ is called a *fuzzy normed linear space* and $\|\cdot\|$ a *fuzzy norm* on the linear space X, if

i) $\|x\| = \bar{0}$ if and only if $x = 0$,

ii) $\|rx\| = |r|\|x\|, x \in X, r \in R$,

iii) for all $x, y \in X$,

(a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$,
whenever $s \leq |||x|||_1^1, t \leq |||y|||_1^1$ and $s+t \leq |||x+y|||_1^1$,

(b) $\|x+y\|(s+t) \geq M(\|x\|(s), \|y\|(t))$,
whenever $s \geq |||x|||_1^1, t \geq |||y|||_1^1$ and $s+t \geq |||x+y|||_1^1$.

In the sequel we take $L(x,y) = min(x,y)$ and $M(x,y) = max(x,y)$ for $x, y \in [0,1]$ and we denote $(X, \|\cdot\|, L, M)$ by $(X, \|\cdot\|)$ or simply by X in this case.

With these $L(x,y) = min(x,y)$ and $M(x,y) = max(x,y)$ for $x, y \in [0,1]$, we have (refer to Felbin [3]) in a fuzzy normed linear space $(X, \|\cdot\|)$, the triangle inequality (iii) of the definition of fuzzy normed linear space is equivalent to

$$\|x+y\| \leq \|x\| \oplus \|y\|.$$

The set $\omega(X)$ of all sequences in a vector space X is a vector space with respect to pointwise addition and scalar multiplication. Any subspace $\lambda(X)$ of $\omega(X)$ is called vector valued sequence space. When $(X, \|\cdot\|)$ is a fuzzy normed linear space, then $\lambda(X)$ is called a *fuzzy normed linear space-valued sequence space*. In short we denote a *fuzzy normed linear space* as fnls.

A fnls-valued sequence space $E^F(X)$ is said to be *normal* (or *solid*) if $(y_k) \in E^F(X)$, whenever $\|y_k\| \leq \|x_k\|$, for all $k \in N$ and $(x_k) \in E^F(X)$.

Let $K = k_1 < k_2 < k_3 \ldots \subseteq N$ and $E^F(X)$ be a fnls-valued sequence space. A *K-step space* of $E^F(X)$ is a space of sequences $\lambda_k^{E^F}(X) = \{(x_{k_n}) \in \omega^F(X) : (x_n) \in E^F(X)\}$.

A *canonical pre-image of a sequence* $(x_{k_n}) \in \lambda_k^{E^F}(X)$ is a sequence $(y_n) \in \omega^F(X)$, defined as follows:

$$y_n = \begin{cases} x_n, & \text{for } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A *canonical pre-image of a step space* $\lambda_k^{E^F}(X)$ is a space of canonical pre-images of all elements in $\lambda_k^{E^F}(X)$, i.e. y is in canonical pre-image $\lambda_k^{E^F}(X)$ if and only if y is canonical pre-image of some $x \in \lambda_k^{E^F}(X)$.

A fnls-valued sequence space $E^F(X)$ is said to be *monotone* if $E^F(X)$ contains the canonical pre-images of all its step spaces.

From the above definitions we have following remark.

Remark 2.1. A fnls-valued sequence space $E^F(X)$ is solid $\Rightarrow E^F(X)$ is monotone.

A fnls-valued sequence space $E^F(X)$ is said to be *symmetric* if $(x_{\pi(n)}) \in E^F(X)$, whenever $(x_k) \in E^F(X)$, where π is a permutation of N.

A fnls-valued sequence space $E^F(X)$ is said to be *convergence free* if $(y_k) \in E^F(X)$, whenever $(x_k) \in E^F(X)$ and $x_k = 0$ implies $y_k = 0$.

Let $(X, \|\cdot\|)$ be a fuzzy normed linear space. A sequence $(x_n) \in X$ is said to *converge* to $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if and only if $\lim_{n\to\infty} \|x_n - x\| = \bar{0}$.

i.e., $\lim_{n\to\infty} |||x_n - x|||_1^\alpha = \lim_{n\to\infty} |||x_n - x|||_2^\alpha = 0$, for $\alpha \in (0, 1]$.

Thus, $\lim_{n\to\infty} \|x_n - x\| = \bar{0}$ if and only if $\lim_{n\to\infty} |||x_n - x|||_2^\alpha = 0$, for $0 \in (0, 1]$.

A sequence (x_n) in a fuzzy normed linear space $(X, \|\cdot\|)$ is called *Cauchy* if

$\lim_{\substack{n\to\infty \\ m\to\infty}} \|x_m - x_n\| = \bar{0}$.

i.e., $\lim_{\substack{n\to\infty \\ m\to\infty}} |||x_n - x|||_2^\alpha = 0$, for $\alpha \in (0, 1]$.

A fuzzy normed linear space $(X, \|\cdot\|)$ is called *fuzzy complete* if every Cauchy sequence in X converges in X.

With the concept of fuzzy norm $\|\cdot\|$, the class of all p-bounded variation sequences, $bv_p^F(X)$ in $(X, \|\cdot\|)$ is defined by

$$bv_p^F(X) = \left\{ x = (x_k) \in \omega^F(X) : \sum_{k=1}^{\infty} \|x_k - x_{k+1}\|^p \leq \lambda, \text{ for some } \lambda \in R^*(I) \right\}.$$

Let $(X, \|\cdot\|)$ be a fuzzy normed linear space. Then for a sequence $x = (x_k) \in bv_p^F(X), 1 \leq p < \infty$, the class of all p-bounded varition sequences in $(X, \|\cdot\|)$, we define

$$\|x\| = \|x_1\| \oplus \left\{ \sum_{k=1}^{\infty} \|x_k - x_{k+1}\|^p \right\}^{\frac{1}{p}}.$$

Clearly, $\|x\|$ is a norm.

Throughout $\omega^F(X)$ and $bv_p^F(X)$ denote the spaces of all and p-bounded variation sequences in fuzzy normed linear space X respectively.

3. Main results

Theorem 3.1. In a fuzzy normed linear space $(X, \|\cdot\|)$, the class of p-bounded variation sequences, $bv_p^F(X), 1 \leq p < \infty$ is fuzzy normed linear space-valued sequence space.

Proof: Let $(X, \|\cdot\|)$ be a fnls and $x = (x_k),\ y = (y_k) \in bv_p^F(X)$. We have for $k \in N$,

$$\|(x_k + y_k) - (x_{k+1} + y_{k+1})\|^p = \|(x_k - x_{k+1}) + (y_k - y_{k+1})\|^p$$

$$\leq 2^p \max\{\|x_k - x_{k+1}\|^p, \|y_k - y_{k+1}\|^p\}$$
$$\leq 2^p\{\|x_k - x_{k+1}\|^p \oplus \|y_k - y_{k+1}\|^p\}.$$

It follows that $\sum\limits_{k=1}^{\infty} \|(x_k + y_k) - (x_{k+1} + y_{k+1})\|^p < \infty$. Thus $(x_k + y_k) \in bv_p^F(X)$.

Also, for any $r \in R$, we have $\sum\limits_{k=1}^{\infty} \|rx_k - rx_{k+1}\|^p = |r|^p \sum\limits_{k=1}^{\infty} \|x_k - rx_{k+1}\|^p < \infty$. Thus $(rx_k) \in \ell_p^F(X)$. So, $bv_p^F(X)$ is a subspace of $\omega^F(X)$ and hence it is fnls-valued sequence space.

Theorem 3.2. In a fuzzy normed linear space $(X, \|\cdot\|)$, the class of p-bounded variation sequences, $bv_p^F(X), 1 \leq p < \infty$ is complete with the fuzzy norm

$$\|x\| = \|x_1\| \oplus \left\{\sum_{k=1}^{\infty} \|x_k - x_{k+1}\|^p\right\}^{\frac{1}{p}},$$

where $x = (x_k)$ is in $bv_p^F(X)$ and X is complete.

Proof. Let $(x^{(n)})$ be a Cauchy sequence in $bv_p^F(X)$,
where $x^{(n)} = \left(x_k^{(n)}\right) = \left(x_1^{(n)},\ x_2^{(n)},\ x_3^{(n)}, \ldots\right) \in bv_p^F$, for each $n \in N$. Then for a given $\bar{\varepsilon} > 0$ there exists such $n_0 \in N$ that for each $m, n \geq n_0$ we have

$$\|x_k^{(n)} - x_k^{(m)}\| = \|x_1^{(n)} - x_1^{(m)}\| \oplus \left\{\sum \|(x_k^{(n)} - x_{k+1}^{(n)}) - (x_k^{(m)} - x_{k+1}^{(m)})\|^p\right\}^{\frac{1}{p}} < \bar{\varepsilon}$$

(3.1) $$\Rightarrow \|x_1^{(n)} - x_1^{(m)}\| < \bar{\varepsilon},\ \text{for all } m, n \geq n_0$$

and $\left\{\sum \|(x_k^{(n)} - x_{k+1}^{(n)}) - (x_k^{(m)} - x_{k+1}^{(m)})\|^p\right\}^{\frac{1}{p}} < \bar{\varepsilon}$, for all $m, n \geq n_0$

(3.2) $$\Rightarrow \|(x_k^{(n)} - x_{k+1}^{(n)}) - (x_k^{(m)} - x_{k+1}^{(m)})\| < \bar{\varepsilon},\ \text{for all } m, n \geq n_0.$$

Thus $(x_1^{(n)})$ and $(x_k^{(n)} - x_{k+1}^{(n)}) = (\triangle x_k^{(n)})$, for all $k \in N$ are Cauchy sequences in X. Since X is complete, so $(x_1^{(n)})$ and $(\triangle x_k^{(n)})$, for all $k \in N$ are convergent. This implies $x_1, z_k \in X$ such that

$$\|x_1^{(n)} - x_1\| \to \overline{0} \text{ as } n \to \infty \tag{3.3}$$

and

$$\|[x_k^{(n)} - x_{k+1}^{(n)}] - z_k\| \to \overline{0} \text{ as } n \to \infty, \text{ for all } k \in N. \tag{3.4}$$

From (3.3) and (3.4) we get, $\lim_{n\to\infty} \|x_k^{(n)} - x_k\| = \bar{0}$, for all $k \in N$.
$\Rightarrow \lim_{n\to\infty} |||x_k^{(n)} - x_k||| = 0$, for each $\alpha \in (0,1]$ and for all $k \in N$.

Next, for each $\alpha \in (0,1]$ and $m, n \geq n_0$ we have from (3.1) and (3.2),
$|||x_1^{(n)} - x_1^{(m)}|||_2^{\alpha} < \varepsilon$
and $\left[\left\{\sum_{k=1}^{\infty} |||(x_k^{(n)} - x_{k+1}^{(n)}) - (x_k^{(m)} - x_{k+1}^{(m)})|||_2^{\alpha}\right\}^p\right]^{\frac{1}{p}} < \varepsilon.$
Now fix $n \geq n_0$ and let $m \to \infty$, we have for each $\alpha \in (0,1]$,
$|||x_1^{(n)} - x_1|||_2^{\alpha} < \varepsilon$
and $\left[\left\{\sum_{k=1}^{\infty} |||(x_k^{(n)} - x_k) - (x_{k+1}^{(n)} - x_{k+1})|||_2^{\alpha}\right\}^p\right]^{\frac{1}{p}} < \varepsilon$, for all $n \geq n_0$
$\Rightarrow \|x_1^{(n)} - x_1\| < \overline{\varepsilon}$

and

$$\left\{\sum_{k=1}^{\infty} \|(x_k^{(n)} - x_k) - (x_{k+1}^{(n)} - x_{k+1})\|^p\right\}^{\frac{1}{p}} < \overline{\varepsilon}, \text{ for all } n \geq n_0 \tag{3.5}$$

$\Rightarrow \|x^{(n)} - x\| < \overline{\varepsilon}$, for all $n \geq n_0$, where $x = (x_k)$.
Hence $x^{(n)} \to x$.

Now we establish that $x = (x_k) \in bv_p^F(X)$.
We have for all $n \geq n_0$ and for each $\alpha \in (0,1]$,
$\sum_{k=1}^{\infty} [\ |||x_k - x_{k+1}|||_2^{\alpha}]^p = \sum_{k=1}^{\infty} \left[|||x_k - x_k^{(n)} + x_k^{(n)} - x_{k+1}^{(n)} + x_{k+1}^{(n)} - x_{k+1}|||_2^{\alpha}\right]^p$
$\leq 2^p\left[\sum_{k=1}^{\infty} \left\{|||x_k^{(n)} - x_{k+1}^{(n)}|||_2^{\alpha}\right\}^p + \sum_{k=1}^{\infty} \left\{|||(x_k^{(n)} - x_k) - (x_{k+1}^{(n)} - x_{k+1})|||_2^{\alpha}\right\}^p\right]$
$\Rightarrow \sum_{k=1}^{\infty} \|x_k - x_{k+1}\|^p < \infty$, [Since $(x^{(n)}) \in bv_p^F(X)$ and using (3.5)].

Hence $x = (x_k) \in bv_p^F(X)$.

Theorem 3.3. In a fuzzy normed linear space $(X, \|\cdot\|)$, the space of p-bounded variation sequences, $bv_p^F(X), 1 \le p < \infty$ is neither monotone nor solid.

Proof. The result follows from the following example.

Example 3.1. Let X be fuzzy normed linear space. For any sequence $z = (z_k)$ in X , let us consider $\|z_x\|$, defined as follows.

$$\text{(3.6)} \quad \text{For } k \in N, z_k \neq 0, \|x_k\|(t) = \begin{cases} \frac{2t}{|x_k|} - 1, & \text{for } \frac{|x_k|}{2} \le t \le |x_k|, \\ 0, & \text{otherwise} \end{cases}$$

and for $z_k = 0, \|z_k\|(t) = \begin{cases} 1, & \text{for } t = 0, \\ 0, & \text{otherwise.} \end{cases}$

Consider the sequence $x = (x_k)$ defined by

$x_1 = 2$

and for $k \ge 2,\ x_k = 2 + \sum\limits_{r=1}^{k-1} r^{-\frac{2}{p}},\ 1 \le p < \infty.$

Using (3.6), we have for $k \in N, x_k \neq 0$,

$\|x_k\|(t) = \begin{cases} \frac{2t}{|x_k|} - 1, & \text{for } \frac{|x_k|}{2} \le t \le |x_k|, \\ 0, & \text{otherwise} \end{cases}$

and for $x_k = 0, \|x_k\|(t) = \begin{cases} 1, & \text{for } t = 0, \\ 0, & \text{otherwise.} \end{cases}$

For $k \in N$ we have, $\triangle x_k = x_k - x_{k+1} = -k^{-\frac{2}{p}}$.

Now, $\|\triangle x_k\|(t) = \begin{cases} \frac{2t}{|\triangle x_k|} - 1, & \text{for } \frac{|\triangle x_k|}{2} \le t \le |\triangle x_k| = k^{-\frac{2}{p}}, \\ 0, & \text{otherwise.} \end{cases}$

$\Rightarrow [\|\triangle x_k\|]^\alpha = \left[\frac{k^{-\frac{2}{p}}}{2}(\alpha + 1),\ k^{-\frac{2}{p}}\right]$, for each $\alpha \in (0, 1]$.

Hence for each $\alpha \in (0, 1]$ we have,

$\sum\limits_{k=1}^{\infty} [|||\triangle x_k|||_2^\alpha]^p = \sum\limits_{k=1}^{\infty} \left[k^{-\frac{2}{p}}\right]^p < \infty$

$\Rightarrow \sum\limits_{k=1}^{\infty} \|\triangle x_k\|^p < \infty$. Thus $x = (x_k) \in bv_p^F(X)$.

Let $J = \{k \in N : k = 2i-1, i \in N\}$ be a subset of N and let $\overline{\overline{(bv_p^F(X))}}_J$ be the canonical pre-image of the J-step space $(bv_p^F(X))_J$ of $bv_p^F(X)$, defined as follows:

$(y_k) \in \overline{\overline{(bv_p^F(X))}}_J$ is the canonical pre-image of $(x_k) \in bv_p^F(X)$ implies

$$y_k = \begin{cases} x_k, & \text{for } k \notin J, \\ 0 & \text{for } k \in J. \end{cases}$$

Now, for $k \notin J, y_k \neq 0$ and using (3.6) we have

$$\|y_k\|(t) = \|x_k\|(t) = \begin{cases} \frac{2t}{|x_k|} - 1, & \text{for } \frac{|x_k|}{2} \leq t \leq |x_k| = 2 + \sum_{r=1}^{k-1} r^{-\frac{2}{p}}, \\ 0, & \text{otherwise} \end{cases}$$

and for $k \in J, \|y_k\|(t) = \overline{0}$.
Again, for $k \in J$ and using (3.6)we have

$$\|\triangle y_k\|(t) = \begin{cases} \frac{2t}{|\triangle y_k|} - 1, & \text{for } \frac{|\triangle y_k|}{2} \leq t \leq |\triangle y_k| = |-(2 + \sum_{r=1}^{k} r^{-\frac{2}{p}})|, \\ 0, & \text{otherwise.} \end{cases}$$

$$\Rightarrow \|\triangle y_k\|^\alpha = \left[\frac{1}{2}(2 + \sum_{r=1}^{k-1} r^{-\frac{2}{p}})(1+\alpha),\ 2 + \sum_{r=1}^{k-1} r^{-\frac{2}{p}}\right], \text{ for each } \alpha \in (0,1].$$

Hence for each $\alpha \in (0,1]$ we have,

$$\sum_{k=1}^{\infty} [|||\triangle y_k|||_2^\alpha]^p = \sum_{k \in J}^{\infty} \left[2 + \sum_{r=1}^{k} r^{-\frac{2}{p}}\right]^p + \sum_{k \notin J}^{\infty} \left[2 + \sum_{r=1}^{k-1} r^{-\frac{2}{p}}\right]^p$$

$$= \sum_{k=1}^{\infty} \left[2 + \sum_{r=1}^{k} r^{-\frac{2}{p}}\right]^p, \text{ which is unbounded.}$$

$$\Rightarrow \sum_{k=1}^{\infty} \|\triangle y_k\|^p \text{ is unbounded, } 1 \leq p < \infty.$$

Thus the space $bv_p^F(X)$ is not monotone. Also, the space $bv_p^F(X)$ is not solid follows from the Remark 2.1.

Theorem 3.4. In a fuzzy normed linear space $(X, \|\cdot\|)$, the space of p-bounded variation sequences, $bv_p^F(X), p > 1$ is not symmetric.

Proof. The result follows from the following example.

Example 3.2. Consider the sequence $x = (x_k) \in bv_p^F(X)$ defined as follows.

$$x_1 = -\frac{1}{2}$$

and for $k \geq 2$, $x_k = -\sum_{r=1}^{k-1} \frac{1}{r}$.

Using (3.6) of example 3.1, we have for $x_k \neq 0$,

$$\|x_k\|(t) = \begin{cases} \frac{2t}{|x_k|} - 1, & \text{for } \frac{|x_k|}{2} \leq t \leq |x_k|, \\ 0, & \text{otherwise} \end{cases}$$

and $\|0\|(t) = \begin{cases} 1, & \text{for } t = 0, \\ 0, & \text{otherwise.} \end{cases}$

For $k \in N$ we have, $\triangle x_k = x_k - x_{k+1} = \frac{1}{k}$.

Now, $\|\triangle x_k\|(t) = \begin{cases} \frac{2t}{|\triangle x_k|} - 1, & \text{for } \frac{|\triangle x_k|}{2} \leq t \leq |\triangle x_k| = \frac{1}{k}, \\ 0, & \text{otherwise.} \end{cases}$

$\Rightarrow [\|\triangle x_k\|]^\alpha = \left[\frac{(\alpha+1)}{2}k^{-1},\ k^{-1}\right]$, for each $\alpha \in (0,1]$

Hence for $p > 1$ and $\alpha \in (0,1]$ we have,

$\sum_{k=1}^{\infty} [|||\triangle x_k|||_2^\alpha]^p = \sum_{k=1}^{\infty} \left[k^{-1}\right]^p < \infty$

$\Rightarrow \sum_{k=1}^{\infty} \|\triangle x_k\|^p < \infty$. Thus $x = (x_k) \in bv_p^F(X)$.

Let (y_k) be a rearrangement of the sequence (x_k), defined by

$$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7 \ldots).$$

i.e.,

$$(y_k) = \begin{cases} x_{\left(\frac{k+1}{2}\right)^2}, & \text{for } k \text{ odd}, \\ x_{\left(n+\frac{k}{2}\right)}, & \text{for } k \text{ even and } n \in N, \text{ satisfies } n(n-1) < \frac{k}{2} \leq n(n+1). \end{cases}$$

Thus for $k = 1$ we have,

$$\begin{aligned} \|y_k\|(t) &= \|x_{\left(\frac{k+1}{2}\right)^2}\|(t) \\ &= \begin{cases} \frac{2t}{|x_{\left(\frac{k+1}{2}\right)^2}|} - 1, & \text{for } \frac{|x_{\left(\frac{k+1}{2}\right)^2}|}{2} \leq t \leq |x_{\left(\frac{k+1}{2}\right)^2}| = \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Next, for k odd with $k > 1$ we have,

$$\|y_k\|(t) = \|x_{\left(\frac{k+1}{2}\right)^2}\|(t) = \begin{cases} \frac{2t}{|x_{\left(\frac{k+1}{2}\right)^2}|} - 1, & \text{for } \frac{|x_{\left(\frac{k+1}{2}\right)^2}|}{2} \le t \le |x_{\left(\frac{k+1}{2}\right)^2}| = |-\sum_{r=1}^{\left(\frac{k+1}{2}\right)^2-1} \frac{1}{r}|, \\ 0, & \text{otherwise} \end{cases}$$

and for k even and $n \in N$, satisfying $n(n-1) < \frac{k}{2} \le n(n+1)$,

$$\|y_k\|(t) = \|x_{\left(n+\frac{k}{2}\right)}\|(t) = \begin{cases} \frac{2t}{|x_{\left(n+\frac{k}{2}\right)}|} - 1, & \text{for } \frac{|x_{\left(n+\frac{k}{2}\right)}|}{2} \le t \le |x_{\left(n+\frac{k}{2}\right)}| = |-\sum_{r=1}^{\left(n+\frac{k}{2}\right)-1} \frac{1}{r}|, \\ 0, & \text{otherwise.} \end{cases}$$

Again, for k odd with $k > 1$ and $n \in N$, satisfying $n(n-1) < \frac{k+1}{2} \le n(n+1)$,

$$\|\triangle y_k\|(t) = \|x_{\left(\frac{k+1}{2}\right)^2} - x_{\left(n+\frac{k+1}{2}\right)}\|(t)$$

$$= \begin{cases} \frac{2t}{|\triangle y_k|} - 1, & \text{for } \frac{|\triangle y_k|}{2} \le t \le |\triangle y_k| = |-\sum_{r=n+\frac{k+1}{2}}^{\left(\frac{k+1}{2}\right)^2-1} \frac{1}{r}|, \\ 0, & \text{otherwise} \end{cases}$$

and for k even and $n \in N$, satisfying $n(n-1) < \frac{k}{2} \le n(n+1)$,

$$\|\triangle y_k\|(t) = \begin{cases} \frac{2t}{|\triangle y_k|} - 1, & \text{for } \frac{|\triangle y_k|}{2} \le t \le |\triangle y_k| = |-\sum_{r=n+\frac{k}{2}}^{\left(\frac{k+1}{2}\right)^2-1} \frac{1}{r}|, \\ 0, & \text{otherwise.} \end{cases}$$

For $k = 1$ we have, $\|\triangle y_k\|^\alpha = \|x_1 - x_2\|^\alpha = \left[\frac{(\alpha+1)}{2}, 1\right]$, for each $\alpha \in (0,1]$.

Thus for k odd with $k > 1$ and $n \in N$, satisfying $n(n-1) < \frac{k+1}{2} \le n(n+1)$,

$$(3.7)\ [\|\triangle y_k\|]^\alpha = \left[\frac{(\alpha+1)}{2} \sum_{r=n+\frac{k+1}{2}}^{\left(\frac{k+1}{2}\right)^2-1} \frac{1}{r},\ \sum_{r=n+\frac{k+1}{2}}^{\left(\frac{k+1}{2}\right)^2-1} \frac{1}{r} \right], \text{ for each } \alpha \in (0,1]$$

and for k even and $n \in N$, satisfying $n(n-1) < \frac{k}{2} \le n(n+1)$,

$$(3.8)\,[\|\triangle y_k\|]^{\alpha} = \left[\frac{(\alpha+1)}{2}\sum_{r=n+\frac{k}{2}}^{\left(\frac{k+2}{2}\right)^2-1}\frac{1}{r},\ \sum_{r=n+\frac{k}{2}}^{\left(\frac{k+2}{2}\right)^2-1}\frac{1}{r}\right],\ \text{for each } \alpha \in (0,1].$$

From (3.7) and (3.8) it is observed that
$\sum\limits_{k=1}^{\infty} |||\triangle y_k|||_2^{\alpha}$ is unbounded, for each $\alpha \in (0,1]$.
$\Rightarrow \sum\limits_{k=1}^{\infty} [|||\triangle y_k|||_2^{\alpha}]^p$ is unbounded for $p > 1$. Hence $\sum\limits_{k=1}^{\infty} \|\triangle y_k\|^p$ is unbounded.
Thus $(y_k) \notin bv_p^F(X), p > 1$. Hence $bv_p^F(X), p > 1$ is not symmetric.

Theorem 3.5. In a fuzzy normed linear space $(X, \|\cdot\|)$, the space of p-bounded variation sequences, $bv_p^F(X), 1 \le p < \infty$ is not convergence free.

Proof. The result follows from the following example.

Example 3.3. Consider the sequence (x_k) defined by

$$x_k = \begin{cases} k^{-2}, & \text{for } k \text{ even}, \\ 0, & \text{for } k \text{ odd}. \end{cases}$$

Using (3.6) of Example 3.1, we have for k even, $x_k \neq 0$,

$\|x_k\|(t) = \begin{cases} \frac{2t}{|x_k|} - 1, & \text{for } \frac{|x_k|}{2} \le t \le |x_k|, \\ 0, & \text{otherwise} \end{cases}$

and for k odd, $\|x_k\|(t) = \begin{cases} 1, & \text{for } t = 0, \\ 0, & \text{otherwise}. \end{cases}$

We have for all $k \in N, \triangle x_k = x_k - x_{k+1} = \begin{cases} k^{-2}, & \text{for } k \text{ even}, \\ -(k+1)^{-2}, & \text{for } k \text{ odd}. \end{cases}$

Now for k even, $\|\triangle x_k\|(t) = \begin{cases} \frac{2t}{|\triangle x_k|} - 1, & \text{for } \frac{|\triangle x_k|}{2} \le t \le |\triangle x_k| = k^{-2}, \\ 0, & \text{otherwise} \end{cases}$

and for k odd, $\|\triangle x_k\|(t) = \begin{cases} \frac{2t}{|\triangle x_k|} - 1, & \text{for } \frac{|\triangle x_k|}{2} \le t \le |\triangle x_k| = (k+1)^{-2}, \\ 0, & \text{otherwise}. \end{cases}$

Next, for each $\alpha \in (0,1]$ we have,

$$[\|\triangle x_k\|]^\alpha = \begin{cases} \left[\frac{(\alpha+1)}{2}k^{-2},\ k^{-2}\right], & \text{for } k \text{ even},\\ \left[\frac{(\alpha+1)}{2}(k+1)^{-2},\ (k+1)^{-2}\right] & \text{for } k \text{ odd}.\end{cases}$$

Hence for each $\alpha \in (0,1]$,

$\sum\limits_{k=1}^{\infty}[|||\triangle x_k|||_2^\alpha]^p = 2\sum\limits_{k=1}^{\infty}\left\{\frac{1}{(2k)^2}\right\}^p < \infty$

$\Rightarrow \sum\limits_{k=1}^{\infty}\|\triangle x_k\|^p < \infty$. Thus $x = (x_k) \in bv_p^F(X)$.

Let the sequence (y_k) be defined as follows.

$$y_k = \begin{cases} k^{-\frac{1}{p}}, & \text{for } k \text{ even},\\ 0, & \text{for } k \text{ odd}.\end{cases}$$

Using (3.6) of Example 3.1, we have for k even, $y_k \neq 0$,

$$\|y_k\|(t) = \begin{cases} \frac{2t}{|y_k|} - 1, & \text{for } \frac{|y_k|}{2} \le t \le |y_k|,\\ 0, & \text{otherwise}\end{cases}$$

and for k odd, $y_k = 0$, $\|y_k\|(t) = \begin{cases} 1, & \text{for } t = 0,\\ 0, & \text{otherwise}.\end{cases}$

We have for all $k \in N, \triangle y_k = y_k - y_{k+1} = \begin{cases} k^{-\frac{1}{p}}, & \text{for } k \text{ even},\\ -(k+1)^{-\frac{1}{p}}, & \text{for } k \text{ odd}.\end{cases}$

Now for k even, $\|\triangle y_k\|(t) = \begin{cases} \frac{2t}{|\triangle y_k|} - 1, & \text{for } \frac{|\triangle y_k|}{2} \le t \le |\triangle y_k| = k^{-\frac{1}{p}},\\ 0, & \text{otherwise}\end{cases}$

and for k odd, $\|\triangle y_k\|(t) = \begin{cases} \frac{2t}{|\triangle y_k|} - 1, & \text{for } \frac{|\triangle y_k|}{2} \le t \le |\triangle y_k| = (k+1)^{-\frac{1}{p}},\\ 0, & \text{otherwise}.\end{cases}$

Next, for each $\alpha \in (0,1]$ we have,

$$[\|\triangle y_k\|]^\alpha = \begin{cases} \left[\frac{(\alpha+1)}{2}k^{-\frac{1}{p}},\ k^{-\frac{1}{p}}\right], & \text{for } k \text{ even},\\ \left[\frac{(\alpha+1)}{2}(k+1)^{-\frac{1}{p}},\ (k+1)^{-\frac{1}{p}}\right], & \text{for } k \text{ odd}.\end{cases}$$

Hence for each $\alpha \in (0,1]$,

$\sum\limits_{k=1}^{\infty}[|||\triangle y_k|||_2^\alpha]^p = \sum\limits_{k=1}^{\infty}\left\{2(2k)^{-\frac{1}{p}}\right\}^p = \sum\limits_{k=1}^{\infty}k^{-1}$, which is unbounded.

$\Rightarrow \sum\limits_{k=1}^{\infty}\|\triangle y_k\|^p$ is unbounded, $1 \le p < \infty$. Thus $(y_k) \notin bv_p^F(X)$. Hence $bv_p^F(X), 1 \le p < \infty$ is not convergence free.

Theorem 3.6. In a fuzzy normed linear space $(X, \|\cdot\|)$, $bv_q^F(X) \subset bv_p^F(X)$, for $1 \leq q < p < \infty$.

Proof. Let $(x_k) \in bv_q^F(X)$. Then $\sum_{k=1}^{\infty} \|\triangle x_k\|^q < \infty$.

Since $\|\triangle x_k\| \to \overline{0}$, as $k \to \infty$, so there exists a positive integer n_0 such that
$\|\triangle x_k\| \leq \overline{1}$, for all $k > n_0$.

We have

$$\sum_{k=1}^{\infty} \|\triangle x_k\|^p = \sum_{k=1}^{n_0-1} \|\triangle x_k\|^p \oplus \sum_{k=n_0}^{\infty} \|\triangle x_k\|^p. \tag{3.9}$$

Clearly, $\sum_{k=n_0}^{\infty} \|\triangle x_k\|^p \leq \sum_{k=n_0}^{\infty} \|\triangle x_k\|^q < \infty$, for $p > q$ and $\sum_{k=1}^{n_0-1} \|\triangle x_k\|^p$ is finite sum. Hence (3.9) implies $\sum_{k=1}^{\infty} \|\triangle x_k\|^p < \infty$. Thus $(x_k) \in bv_p^F(X)$ and $bv_q^F(X) \subset bv_p^F(X)$, for $1 \leq q < p < \infty$.

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