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Riemann-Liouville fractional trapezium-like inequalities via generalized (m, h_1, h_2)-preinvexity

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Abstract

In this paper, we derive a fractional integral identity concerning three times differentiable generalized preinvex mappings defined on m-invex set. By using of this identity, we obtain new estimates on generalization of trapezium-like inequalities for functions whose third order derivatives are generalized (m, h_1, h_2)-preinvex via Riemann-Liouville fractional integrals. Some interesting special cases of our main result are also considered and shown to be connected with certain known ones.

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1. Introduction

Throughout this article, let $I = [a, b] \subseteq \mathbf{R}$ be the real interval and I° be the interior of I unless otherwise specified.

In [31], Sarikaya et al. established the following interesting Hermite-Hadamard type inequalities by using Riemann-Liouville fractional integrals.

Theorem 1.1. *Let $f : [u, v] \to \mathbf{R}$ be a positive function with $0 \leq u < v$ and let $f \in L^1[u, v]$. Suppose f is a convex function on $[u, v]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}[J^\alpha_{u^+}f(v) + J^\alpha_{v^-}f(u)] \leq \frac{f(u)+f(v)}{2}, \tag{1.1}$$

where the symbol $J^\alpha_{u^+}f$ and $J^\alpha_{v^-}f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha \in \mathbf{R}^+$ defined by

$J^\alpha_{u^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_u^x (x-t)^{\alpha-1}f(t)dt, \quad u < x$

and

$J^\alpha_{v^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^v (t-x)^{\alpha-1}f(t)dt, \quad x < v.$

Here, $\Gamma(\alpha)$ is the gamma function and its definition is

$$\Gamma(\alpha) = \int_0^\infty e^{-\mu}\mu^{\alpha-1}d\mu.$$

We observe that, for $\alpha = 1$, the inequality (1.1) reduces to the following Hermite-Hadamard inequality

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v f(x)dx \leq \frac{f(u)+f(v)}{2}, \tag{1.2}$$

where $f : I \subseteq \mathbf{R} \to \mathbf{R}$ is a convex mapping on the interval I of real numbers and $u, v \in I$ with $u < v$. The inequality (1.2) is also known as trapezium inequality.

In recent years, many researchers have studied error estimations with respect to the inequality (1.2); for refinements, counterparts, generalization please refer to [2, 9, 19, 7, 16, 17, 18, 28, 37, 21, 38].

Due to the wide application of Riemann-Liouville fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities based on the original Hermite-Hadamard's inequality for functions of different classes. For example, refer to [4, 6, 11, 10, 12, 27, 32] for convex functions, to [34, 40] for m-convex functions, to [1] for (s, m)-convex functions, to [35] for r-convex functions, to [5, 14] for harmonically convex functions, to [13] for quasi-geometrically convex functions, to [20] for GA-s-convex functions, to [25, 30] for preinvex functions, to [8] for generalized (α, m)-preinvex functions, to [15] for MT_m-preinvex functions, to [3] for s-Godunova-Levin functions, to [22] for h-convex functions and see the references cited therein.

In [23], Noor et al. established the following integral identity.

Lemma 1.1. *Let $f : I \to \mathbf{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a, b]$, then*

$$\begin{aligned} L_f(a,b;n,\alpha) &= \tfrac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)} \\ &\times \textstyle\int_0^1 (1-t)^{\alpha+2}\left[-f'''\left(\tfrac{n+t}{n+1}a+\tfrac{1-t}{n+1}b\right)+f'''\left(\tfrac{1-t}{n+1}a+\tfrac{n+t}{n+1}b\right)\right]dt, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} &L_f(a,b;n,\alpha) \\ &= \tfrac{(n+1)^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{(\frac{n}{n+1}a+\frac{1}{n+1}b)^-}f(a)+J^\alpha_{(\frac{1}{n+1}a+\frac{n}{n+1}b)^+}f(b)\right] \\ &\quad -\tfrac{(b-a)^2}{(n+1)^3(\alpha+1)(\alpha+2)}\left[f''\left(\tfrac{n}{n+1}a+\tfrac{1}{n+1}b)+f''\left(\tfrac{1}{n+1}a+\tfrac{n}{n+1}b\right)\right] \\ &\quad +\tfrac{b-a}{(n+1)^2(\alpha+1)}\left[f'\left(\tfrac{n}{n+1}a+\tfrac{1}{n+1}b\right)-f'\left(\tfrac{1}{n+1}a+\tfrac{n}{n+1}b\right)\right] \\ &\quad -\tfrac{1}{n+1}\left[f\left(\tfrac{n}{n+1}a+\tfrac{1}{n+1}b\right)+f\left(\tfrac{1}{n+1}a+\tfrac{n}{n+1}b\right)\right]. \end{aligned}$$

On the basis of the above equality, they presented some fractional Hermite-Hadamard type inequalities through h-convex mappings.

In the present paper, we extend Lemma 1.1 in [23] to generalized preinvexity via Riemann-Liouville fractional integrals. That is, we establish the following lemma.

Lemma 1.2. *Let $I \subseteq \mathbf{R}$ be an open m-invex subset with respect to $\eta : I \times I \times (0,1] \to \mathbf{R}$ for some fixed $m \in (0,1]$ and let $a, b \in I$ with $\eta(b,a,m) > 0$. Suppose that $f : I \to \mathbf{R}$ be a three times differentiable function on I. If $f''' \in L[ma, ma + \eta(b,a,m)]$, then the following identity for Riemann-Liouville fractional integrals with $\alpha > 0$ and $n \in \mathbf{N}^+$ holds:*

$$R(\alpha; n, m, a, b)(f) = \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}$$

$$\times \int_0^1 (1-t)^{\alpha+2}\Bigg[- f'''\bigg(ma + \frac{1-t}{n+1}\eta(b,a,m)\bigg) + f'''\bigg(ma + \frac{n+t}{n+1}\eta(b,a,m)\bigg)\Bigg] dt, \tag{1.4}$$

where

$$R(\alpha; n, m, a, b)(f) = \frac{(n+1)^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a,m)}$$

$$\times \Bigg[J^{\alpha}_{(ma+\frac{1}{n+1}\eta(b,a,m))^-} f(ma) + J^{\alpha}_{(ma+\frac{n}{n+1}\eta(b,a,m))^+} f\bigg(ma + \eta(b,a,m)\bigg)\Bigg]$$

$$- \frac{\eta^2(b,a,m)}{(n+1)^3(\alpha+1)(\alpha+2)}\Bigg[f''\bigg(ma + \frac{1}{n+1}\eta(b,a,m)\bigg) + f''\bigg(ma + \frac{n}{n+1}\eta(b,a,m)\bigg)\Bigg]$$

$$+ \frac{\eta(b,a,m)}{(n+1)^2(\alpha+1)}\Bigg[f'\bigg(ma + \frac{1}{n+1}\eta(b,a,m)\bigg) - f'\bigg(ma + \frac{n}{n+1}\eta(b,a,m)\bigg)\Bigg]$$

$$- \frac{1}{n+1}\Bigg[f\bigg(ma + \frac{1}{n+1}\eta(b,a,m)\bigg) + f\bigg(ma + \frac{n}{n+1}\eta(b,a,m)\bigg)\Bigg].$$

Let us note that:
-if $\eta(b,a,m) = b - ma$ with $m = 1$, we obtain Lemma 2.1 in [23];
-if $\eta(b,a,m) = b - ma$ with $m = 1$ and $n = 1$, we obtain Lemma 3.1 in [24];
-if $\eta(b,a,m) = b - ma$ with $m = 1$, $n = 1$ and $\alpha = 1$, we obtain Lemma 2.1 in [39].

In this article, using the identity in Lemma 1.2 via Definition 1.5, we derive new left-sided Riemann-Liouville fractional integral inequalities involving the class of functions whose third derivatives in absolute values are generalized (m, h_1, h_2)-preinvex functions. These inequalities can be viewed as generalization of the results of [23], [24] and [39].

To end this section, we evoke some basic definitions and special functions as follows.

Definition 1.1. *([36]) A set $K \subseteq \mathbf{R}^n$ is said to be invex set with respect to the mapping $\eta : K \times K \to \mathbf{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$. The invex set K is also termed an $\eta-$connected set.*

Definition 1.2. *([7]) A set $K \subseteq \mathbf{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0, 1] \to \mathbf{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.*

Remark 1.1. *In Definition 1.2, under certain conditions, the mapping $\eta(y, x, m)$ with $m = 1$ could reduce to $\eta(y, x)$. In this case, the m-invex becomes to invex.*

Definition 1.3. *([33]) Let $f : K \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be a nonnegative function, we say that $f : K \longrightarrow \mathbf{R}$ is tgs-convex on K if the inequality*

$$f\Big(tx + (1 - t)y\Big) \leq t(1 - t)[f(x) + f(y)] \tag{1.5}$$

holds for all $x, y \in K$ and $t \in (0, 1)$.

Definition 1.4. *([26]) A function $f : I \subseteq \mathbf{R} \to \mathbf{R}$ is said to be m-MT-convex, if f is positive and for $\forall x, y \in I$, and $t \in (0, 1)$, with $m \in [0, 1]$, satisfies the following inequality*

$$f\Big(tx + m(1 - t)y\Big) \leq \frac{\sqrt{t}}{2\sqrt{1 - t}} f(x) + \frac{m\sqrt{1 - t}}{2\sqrt{t}} f(y). \tag{1.6}$$

Definition 1.5. *[29] Let $I \subseteq \mathbf{R}$ be an open m-invex set with respect to $\eta : I \times I \times (0, 1] \to \mathbf{R}$. A function $f : I \to \mathbf{R}$, $h_1, h_2 : [0, 1] \to \mathbf{R}_0$, if*

$$f\Big(mx + t\eta(y, x, m)\Big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{1.7}$$

is valid for all $x, y \in I$ and $t \in [0, 1]$, then we say that $f(x)$ is a generalized (m, h_1, h_2)-preinvex function with respect to η. If the inequality (1.7) reverses, then f is said to be (m, h_1, h_2)-preincave on I.

Remark 1.2. *Let us discuss some special cases in Definition 1.5 as follows.*

(I). If we take $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ *for* $s \in (0,1]$, *then we get generalized* (m,s)*-Breckner preinvex functions.*

(II). If we take $h_1(t) = h_2(t) = 1$, *then we get generalized* (m,P)*-preinvex functions.*

(III). If we take $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ *for* $s \in (0,1]$, *then we get generalized* (m,s)*-Godunova-Levin-Dragomir preinvex functions.*

(IV). If we take $h_1(t) = h(1-t)$ *and* $h_2(t) = h(t)$, *then we get generalized* (m,h)*-preinvex functions.*

(V). If we take $h_1(t) = h_2(t) = t(1-t)$, *then we get generalized* (m, tgs)*-preinvex functions.*

(VI). If we take $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$ *and* $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, *then we get generalized* m*-*MT*-preinvex functions.*

Let us consider the following special functions:
(1) The beta function:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0.$$

(2) The hypergeometric function :

$${}_2F_1(x,y;c;z) = \frac{1}{\beta(y,c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}dt$$

for $|z| < 1$, $c > y > 0$.

2. Proof of Lemma 1.2

Proof.
Let

$$
\begin{aligned}
I^* &= \int_0^1 (1-t)^{\alpha+2}\Big[-f'''\Big(ma+\tfrac{1-t}{n+1}\eta(b,a,m)\Big)+f'''\Big(ma+\tfrac{n+t}{n+1}\eta(b,a,m)\Big)\Big]dt\\
&= -\int_0^1 (1-t)^{\alpha+2} f'''\Big(ma+\tfrac{1-t}{n+1}\eta(b,a,m)\Big)dt\\
&+\int_0^1 (1-t)^{\alpha+2} f'''\Big(ma+\tfrac{n+t}{n+1}\eta(b,a,m)\Big)dt\\
&:= I_1+I_2.
\end{aligned}
\tag{2.1}
$$

Integrating I_1 on $[0,1]$ yields

$$
\begin{aligned}
I_1 &= -\int_0^1 (1-t)^{\alpha+2} f'''\Big(ma+\tfrac{1-t}{n+1}\eta(b,a,m)\Big)dt\\
&= -\tfrac{n+1}{\eta(b,a,m)} f''\Big(ma+\tfrac{1}{n+1}\eta(b,a,m)\Big)+\tfrac{(n+1)^2(\alpha+2)}{\eta^2(b,a,m)}\\
&\times f'\Big(ma+\tfrac{1}{n+1}\eta(b,a,m)\Big)-\tfrac{(n+1)^3(\alpha+1)(\alpha+2)}{\eta^3(b,a,m)} f\Big(ma+\tfrac{1}{n+1}\eta(b,a,m)\Big)\\
&+\tfrac{(n+1)^3\alpha(\alpha+1)(\alpha+2)}{\eta^3(b,a,m)}\int_0^1 (1-t)^{\alpha-1} f\Big(ma+\tfrac{1-t}{n+1}\eta(b,a,m)\Big)dt.
\end{aligned}
\tag{2.2}
$$

Analogously, integrating I_2 on $[0,1]$, we also have

$$
\begin{aligned}
I_2 &= \int_0^1 (1-t)^{\alpha+2} f'''\Big(ma+\tfrac{n+t}{n+1}\eta(b,a,m)\Big)dt\\
&= -\tfrac{n+1}{\eta(b,a,m)} f''\Big(ma+\tfrac{n}{n+1}\eta(b,a,m)\Big)-\tfrac{(n+1)^2(\alpha+2)}{\eta^2(b,a,m)}\\
&\times f'\Big(ma+\tfrac{n}{n+1}\eta(b,a,m)\Big)-\tfrac{(n+1)^3(\alpha+1)(\alpha+2)}{\eta^3(b,a,m)} f\Big(ma+\tfrac{n}{n+1}\eta(b,a,m)\Big)\\
&+\tfrac{(n+1)^3\alpha(\alpha+1)(\alpha+2)}{\eta^3(b,a,m)}\int_0^1 (1-t)^{\alpha-1} f\Big(ma+\tfrac{n+t}{n+1}\eta(b,a,m)\Big)dt.
\end{aligned}
\tag{2.3}
$$

Using the reduction formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ $(\alpha > 0)$ for Euler gamma function, we get

$$
\begin{aligned}
&\int_0^1 (1-t)^{\alpha-1} f\left(ma + \tfrac{1-t}{n+1}\eta(b,a,m)\right)dt \\
&= \tfrac{(n+1)^\alpha \Gamma(\alpha)}{\eta^\alpha(b,a,m)} J^\alpha_{(ma+\frac{1}{n+1}\eta(b,a,m))^-} f(ma)
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
&\int_0^1 (1-t)^{\alpha-1} f\left(ma + \tfrac{n+t}{n+1}\eta(b,a,m)\right)dt \\
&= \tfrac{(n+1)^\alpha \Gamma(\alpha)}{\eta^\alpha(b,a,m)} J^\alpha_{(ma+\frac{n}{n+1}\eta(b,a,m))^+} f\left(ma + \eta(b,a,m)\right).
\end{aligned}
\tag{2.5}
$$

Putting (2.4) and (2.5) in (2.2) and (2.3), respectively, and applying (2.2) and (2.3) to (2.1), then multiplying both sides by

$$\frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}$$

completes the proof.

3. Main result

Using Lemma 1.2, we now state the following theorem.

Theorem 3.1. *Let $I \subseteq \mathbf{R}$ be an open m-invex subset with respect to $\eta : I \times I \times (0,1] \to \mathbf{R}$ for some fixed $m \in (0,1]$ and let $a, b \in I$, $a < b$ with $\eta(b,a,m) > 0$. Suppose that $f : I \to \mathbf{R}$ be a three times differentiable function on the interior I° of I, $h_1, h_2 : [0,1] \to \mathbf{R}_0$, $f''' \in L[ma, ma+\eta(b,a,m)]$ and $|f'''|^q$ for $q \geq 1$ is (m, h_1, h_2)-preinvex on $[ma, ma+\eta(b,a,m)]$, then the following inequality for Riemann-Liouville fractional integrals with $\alpha > 0$ and $n \in \mathbf{N}^+$ holds*

$$\left|R(\alpha;n,m,a,b)(f)\right| \leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$

$$(3.1)\quad \times\left\{\left[\int_0^1(1-t)^{\alpha+2}\left(mh_1\left(\frac{1-t}{n+1}\right)\left|f'''(a)\right|^q + h_2\left(\frac{1-t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1(1-t)^{\alpha+2}\left(mh_1\left(\frac{n+t}{n+1}\right)\left|f'''(a)\right|^q + h_2\left(\frac{n+t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right\}.$$

Proof. Using given hypotheses, Lemma 1.2 and power mean inequality, we have

$$\left|R(\alpha;n,m,a,b)(f)\right|$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(ma+\frac{1-t}{n+1}\eta(b,a,m)\right)\right|dt$$

$$+\frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(ma+\frac{n+t}{n+1}\eta(b,a,m)\right)\right|dt$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left[\int_0^1(1-t)^{\alpha+2}\left(mh_1\left(\frac{1-t}{n+1}\right)\left|f'''(a)\right|^q + h_2\left(\frac{1-t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1(1-t)^{\alpha+2}\left(mh_1\left(\frac{n+t}{n+1}\right)\left|f'''(a)\right|^q + h_2\left(\frac{n+t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right\},$$

which completes the proof.

We point out, now, some special cases of Theorem 3.1.

Corollary 3.1. *In Theorem 3.1, putting $q = 1$, we have*

$$\left|R(\alpha;n,m,a,b)(f)\right| \le \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)} \tag{3.2}$$
$$\left[m\Psi(h_1;n;t)\left|f'''(a)\right| + \Psi(h_2;n;t)\left|f'''(b)\right|\right],$$

where

$$\Psi(h_i;n;t) = \int_0^1 (1-t)^{\alpha+2}\left[h_i\left(\frac{1-t}{n+1}\right) + h_i\left(\frac{n+t}{n+1}\right)\right]dt,\ (i=1,2),$$

specially, putting $\eta(b,a,m) = b - ma$ with $m = 1$, $h_1(t) = h_2(t) = 1$ in (3.2), we obtain Corollary 2.7 in [23]. Further, if we put $n = 1$ and $\alpha = 1$, then we have Corollary 3.1.1 in [39].

Corollary 3.2. *In Theorem 3.1, if we take $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$ for $s \in (0,1]$, then we have the following inequality for generalized (m,s)-Breckner preinvex functions*

$$\left|R(\alpha;n,m,a,b)(f)\right| \le \frac{\eta^3(b,a,m)}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\left(\frac{mn^s{}_2F_1\left[-s,1;\alpha+4;-\frac{1}{n}\right]}{\alpha+3}\right)\left|f'''(a)\right|^q + \frac{1}{\alpha+s+3}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right. \tag{3.3}$$
$$\left.+\left[\frac{m}{\alpha+s+3}\left|f'''(a)\right|^q + \left(\frac{n^s{}_2F_1\left[-s,1;\alpha+4;-\frac{1}{n}\right]}{\alpha+3}\right)\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Corollary 3.3. *In Theorem 3.1, taking $h_1(t) = h_2(t) = 1$, we have the following inequality for generalized (m,P)-preinvex functions*

$$\left|R(\alpha;n,m,a,b)(f)\right| \le \frac{2\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)(\alpha+3)} \tag{3.4}$$
$$\times\left(m\left|f'''(a)\right|^q + \left|f'''(b)\right|^q\right)^{\frac{1}{q}},$$

specially, if we put $\eta(b,a,m) = b - ma$ with $m = 1$ in (3.4), and take $n = \alpha = 1$, then we have Theorem 3.1 in [39].

Corollary 3.4. *In Theorem 3.1, if we take* $h_1(t) = (1-t)^{-s}$ *and* $h_2(t) = t^{-s}$ *for* $s \in (0,1]$*, then we have the following inequality for generalized* (m,s)*-Godunova-Levin-Dragomir preinvex functions*

$$\begin{aligned}
&\left|R(\alpha;n,m,a,b)(f)\right| \leq \frac{\eta^3(b,a,m)}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\\
&\times\Bigg\{\left[\left(\frac{mn^{-s}{}_2F_1\left[s,1;\alpha+4;-\frac{1}{n}\right]}{\alpha+3}\right)\left|f'''(a)\right|^q+\frac{1}{\alpha-s+3}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\\
&+\left[\frac{m}{\alpha-s+3}\left|f'''(a)\right|^q+\left(\frac{n^{-s}{}_2F_1\left[s,1;\alpha+4;-\frac{1}{n}\right]}{\alpha+3}\right)\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned} \tag{3.5}$$

Corollary 3.5. *In Theorem 3.1, if we take* $h_1(t) = h(1-t)$ *and* $h_2(t) = h(t)$*, then we have the following inequality for generalized* (m,h)*-preinvex functions*

$$\begin{aligned}
&\left|R(\alpha;n,m,a,b)(f)\right| \leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\\
&\times\Bigg\{\left[\int_0^1(1-t)^{\alpha+2}\left(mh\left(\frac{n+t}{n+1}\right)\left|f'''(a)\right|^q+h\left(\frac{1-t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\\
&+\left[\int_0^1(1-t)^{\alpha+2}\left(mh\left(\frac{1-t}{n+1}\right)\left|f'''(a)\right|^q+h\left(\frac{n+t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\Bigg\},
\end{aligned} \tag{3.6}$$

specially, putting $\eta(b,a,m) = b - ma$ *with* $m = 1$ *in (3.6), we obtain Theorem 2.4 in [23]. Further, if we put* $h(t) = t$ *and* $n = 1$*, then we have the following inequality for convex functions*

$$
\begin{aligned}
&\Big|R(\alpha;1,1,a,b)(f)\Big| \\
&= \Bigg|\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})^{-}}f(a)+J^{\alpha}_{(\frac{a+b}{2})^{+}}f(b)\right] \\
&\quad -\tfrac{(b-a)^2}{4(\alpha+1)(\alpha+2)}f''\left(\tfrac{a+b}{2}\right)-f\left(\tfrac{a+b}{2}\right)\Bigg| \\
&\le \tfrac{(b-a)^3}{2^{4+\frac{1}{q}}(\alpha+1)(\alpha+2)}\left(\tfrac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\left(\tfrac{1}{\alpha+4}\right)^{\frac{1}{q}} \\
&\quad \times\left\{\left[\tfrac{\alpha+5}{\alpha+3}\Big|f'''(a)\Big|^{q}+\Big|f'''(b)\Big|^{q}\right]^{\frac{1}{q}}+\left[\Big|f'''(a)\Big|^{q}+\tfrac{\alpha+5}{\alpha+3}\Big|f'''(b)\Big|^{q}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Corollary 3.6. *In Theorem 3.1, taking $h_1(t)=h_2(t)=t(1-t)$, we have the following inequality for generalized (m,tgs)-preinvex functions*

$$
\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\le \tfrac{2\eta^3(b,a,m)}{(n+1)^{4+\frac{2}{q}}(\alpha+1)(\alpha+2)}\left(\tfrac{1}{\alpha+3}\right)^{1-\frac{1}{q}} \\
&\times\left[\tfrac{n}{\alpha+4}+\tfrac{1}{(\alpha+4)(\alpha+5)}\right]^{\frac{1}{q}}\left[m|f'''(a)|^{q}+|f'''(b)|^{q}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.7. *In Theorem 3.1, if we take $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, then we have the following inequality for generalized m-MT-preinvex functions*

$$
\begin{aligned}
&\left|R(\alpha;n,m,a,b)(f)\right| \\
&\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}} \\
&\quad\times\left\{\left[\left(\frac{mn^{\frac{1}{2}}\,{}_2F_1\left[-\frac{1}{2},1;\alpha+\frac{7}{2};-\frac{1}{n}\right]}{2\alpha+5}\right)\left|f'''(a)\right|^q\right.\right. \\
&\quad\left.+\frac{n^{-\frac{1}{2}}\,{}_2F_1\left[\frac{1}{2},1;\alpha+\frac{9}{2};-\frac{1}{n}\right]}{2\alpha+7}\left|f'''(b)\right|^q\right]^{\frac{1}{q}} \\
&\quad+\left[\frac{mn^{-\frac{1}{2}}\,{}_2F_1\left[\frac{1}{2},1;\alpha+\frac{9}{2};-\frac{1}{n}\right]}{2\alpha+7}\left|f'''(a)\right|^q\right. \\
&\quad\left.\left.+\left(\frac{n^{\frac{1}{2}}\,{}_2F_1\left[-\frac{1}{2},1;\alpha+\frac{7}{2};-\frac{1}{n}\right]}{2\alpha+5}\right)\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Now, we are ready to state the second theorem in this section.

Theorem 3.2. *Let $I \subseteq \mathbf{R}$ be an open m-invex subset with respect to $\eta : I \times I \times (0,1] \to \mathbf{R}$ for some fixed $m \in (0,1]$ and let $a, b \in I$, $a < b$ with $\eta(b,a,m) > 0$. Suppose that $f : I \to \mathbf{R}$ be a three times differentiable function on the interior I° of I, $h_1, h_2 : [0,1] \to \mathbf{R}_0$, $f''' \in L[ma, ma+\eta(b,a,m)]$ and $|f'''|^q$ for $q > 1$ is (m,h_1,h_2)-preinvex on $[ma, ma+\eta(b,a,m)]$ with $\frac{1}{q}+\frac{1}{p}=1$, then the following inequality for Riemann-Liouville fractional integrals with $\alpha > 0$ and $n \in \mathbf{N}^+$ holds*

$$
\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\le \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Big(\tfrac{1}{p(\alpha+2)+1}\Big)^{\frac{1}{p}} \\
&\quad\times\Bigg\{\Big[\textstyle\int_0^1 mh_1\Big(\tfrac{1-t}{n+1}\Big)dt\Big|f'''(a)\Big|^q+\int_0^1 h_2\Big(\tfrac{1-t}{n+1}\Big)dt\Big|f'''(b)\Big|^q\Big]^{\frac{1}{q}} \\
&\quad+\Big[\textstyle\int_0^1 mh_1\Big(\tfrac{n+t}{n+1}\Big)dt\Big|f'''(a)\Big|^q+\int_0^1 h_2\Big(\tfrac{n+t}{n+1}\Big)dt\Big|f'''(b)\Big|^q\Big]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
\tag{3.7}
$$

Proof. Using given hypotheses, Lemma 1.2 and the Hölder's inequality, we have

$$
\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\le \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\textstyle\int_0^1(1-t)^{\alpha+2}\Big|f'''\Big(ma+\tfrac{1-t}{n+1}\eta(b,a,m)\Big)\Big|dt \\
&\quad+\tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\textstyle\int_0^1(1-t)^{\alpha+2}\Big|f'''\Big(ma+\tfrac{n+t}{n+1}\eta(b,a,m)\Big)\Big|dt \\
&\le \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Big(\textstyle\int_0^1(1-t)^{(\alpha+2)p}dt\Big)^{\frac{1}{p}} \\
&\quad\times\Big(\textstyle\int_0^1\Big|f'''\Big(ma+\tfrac{1-t}{n+1}\eta(b,a,m)\Big)\Big|^q dt\Big)^{\frac{1}{q}} \\
&\quad+\tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Big(\textstyle\int_0^1(1-t)^{(\alpha+2)p}dt\Big)^{\frac{1}{p}} \\
&\quad\times\Big(\textstyle\int_0^1\Big|f'''\Big(ma+\tfrac{n+t}{n+1}\eta(b,a,m)\Big)\Big|^q dt\Big)^{\frac{1}{q}} \\
&\le \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Big(\tfrac{1}{p(\alpha+2)+1}\Big)^{\frac{1}{p}} \\
&\quad\times\Bigg\{\Bigg[\int_0^1 mh_1\Big(\frac{1-t}{n+1}\Big)dt\Big|f'''(a)\Big|^q+\int_0^1 h_2\Big(\frac{1-t}{n+1}\Big)dt\Big|f'''(b)\Big|^q\Bigg]^{\frac{1}{q}} \\
&\quad+\Bigg[\int_0^1 mh_1\Big(\frac{n+t}{n+1}\Big)dt\Big|f'''(a)\Big|^q+\int_0^1 h_2\Big(\frac{n+t}{n+1}\Big)dt\Big|f'''(b)\Big|^q\Bigg]^{\frac{1}{q}}\Bigg\},
\end{aligned}
$$

which completes the proof.

We point out, now, some special cases of Theorem 3.2.

Corollary 3.8. *In Theorem 3.2, if we take $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$ for $s \in (0,1]$, then we have the following inequality for generalized (m,s)-Breckner preinvex functions*

$$\begin{aligned}
&\left|R(\alpha;n,m,a,b)(f)\right| \\
&\leq \frac{\eta^3(b,a,m)}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}\left\{\left[\frac{m((n+1)^{s+1}-n^{s+1})}{s+1}\left|f'''(a)\right|^q+\frac{1}{s+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{m}{s+1}\left|f'''(a)\right|^q+\frac{(n+1)^{s+1}-n^{s+1}}{s+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{3.8}$$

specially, putting $\eta(b,a,m) = b - ma$ with $m = 1$ in (3.8), we obtain Corollary 2.8 in [23].

Corollary 3.9. *In Theorem 3.2, taking $h_1(t) = h_2(t) = 1$, we have the following inequality for generalized (m,P)-preinvex functions*

$$\begin{aligned}
\left|R(\alpha;n,m,a,b)(f)\right| &\leq \frac{2\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}} \\
&\times\left(m\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{3.9}$$

specially, putting $\eta(b,a,m) = b - ma$ with $m = 1$ in (3.9), we obtain Corollary 2.10 in [23].

Corollary 3.10. *In Theorem 3.2, if we take $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ for $s \in (0,1)$, then we have the following inequality for generalized (m,s)-Godunova-Levin-Dragomir preinvex functions*

$$\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\le \frac{\eta^3(b,a,m)}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\frac{m((n+1)^{1-s}-n^{1-s})}{1-s}\Big|f'''(a)\Big|^q+\frac{1}{1-s}\Big|f'''(b)\Big|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{m}{1-s}\Big|f'''(a)\Big|^q+\frac{(n+1)^{1-s}-n^{1-s}}{1-s}\Big|f'''(b)\Big|^q\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{3.10}$$

specially, putting $\eta(b,a,m) = b - ma$ with $m = 1$ in (3.10), we obtain Corollary 2.9 in [23].

Corollary 3.11. *In Theorem 3.2, if we take $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, then we have the following inequality for generalized (m,h)-preinvex functions*

$$\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\le \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\int_0^1 mh\left(\frac{n+t}{n+1}\right)dt\Big|f'''(a)\Big|^q+\int_0^1 h\left(\frac{1-t}{n+1}\right)dt\Big|f'''(b)\Big|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 mh\left(\frac{1-t}{n+1}\right)dt\Big|f'''(a)\Big|^q+\int_0^1 h\left(\frac{n+t}{n+1}\right)dt\Big|f'''(b)\Big|^q\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{3.11}$$

specially, putting $\eta(b,a,m) = b - ma$ with $m = 1$ in (3.11), we obtain Theorem 2.3 in [23]. Further, if we put $h(t) = t$ and $n = 1$, then we have Theorem 3.3 in [24].

Corollary 3.12. *In Theorem 3.2, taking $h_1(t) = h_2(t) = t(1-t)$, we have the following inequality for generalized (m, tgs)-preinvex functions*

$$\begin{aligned}&\Big|R(\alpha; n, m, a, b)(f)\Big|\\ &\leq \tfrac{2\eta^3(b,a,m)}{(n+1)^{4+\frac{2}{q}}(\alpha+1)(\alpha+2)}\left(\tfrac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}\left(\tfrac{3n+1}{6}\right)^{\frac{1}{q}}\\ &\quad\times\left(m|f'''(a)|^q + |f'''(b)|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 3.13. *In Theorem 3.2, if we take $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we have the following inequality for generalized m-MT-preinvex functions*

$$\begin{aligned}&\Big|R(\alpha; n, m, a, b)(f)\Big|\\ &\leq \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\tfrac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}\\ &\quad\times\Bigg\{\left(mn^{\frac{1}{2}}{}_2F_1\left[-\tfrac{1}{2}, 1; \tfrac{3}{2}; -\tfrac{1}{n}\right]\Big|f'''(a)\Big|^q + \tfrac{1}{3}n^{-\frac{1}{2}}{}_2F_1\left[\tfrac{1}{2}, 1; \tfrac{5}{2}; -\tfrac{1}{n}\right]\Big|f'''(b)\Big|^q\right)^{\frac{1}{q}}\\ &\quad+\left(\tfrac{1}{3}mn^{-\frac{1}{2}}{}_2F_1\left[\tfrac{1}{2}, 1; \tfrac{5}{2}; -\tfrac{1}{n}\right]\Big|f'''(a)\Big|^q + n^{\frac{1}{2}}{}_2F_1\left[-\tfrac{1}{2}, 1; \tfrac{3}{2}; -\tfrac{1}{n}\right]\Big|f'''(b)\Big|^q\right)^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

Now, we are ready to state the third theorem in this section.

Theorem 3.3. *Under the assumptions of Thereom 3.2, then the following inequality for Riemann-Liouville fractional integrals with $\alpha > 0$ and $n \in \mathbf{N}^+$ holds:*

$$\left|R(\alpha;n,m,a,b)(f)\right|$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+2)(q-p)+q-1}\right]^{\frac{q-1}{q}}$$

$$\times\left\{\left[\int_0^1(1-t)^{p(\alpha+2)}\left(mh_1\left(\frac{1-t}{n+1}\right)\left|f'''(a)\right|^q+h_2\left(\frac{1-t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1(1-t)^{p(\alpha+2)}\left(mh_1\left(\frac{n+t}{n+1}\right)\left|f'''(a)\right|^q+h_2\left(\frac{n+t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right\}.$$

Proof. Using given hypotheses, Lemma 1.2 and the Hölder's inequality, we have

$$\left|R(\alpha;n,m,a,b)(f)\right|$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(ma+\frac{1-t}{n+1}\eta(b,a,m)\right)\right|dt$$

$$+\frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(ma+\frac{n+t}{n+1}\eta(b,a,m)\right)\right|dt$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1[(1-t)^{(\alpha+2)}]^{\frac{q-p}{q-1}}dt\right)^{\frac{q-1}{q}}$$

$$\times\left[\left(\int_0^1[(1-t)^{\alpha+2}]^p\left|f'''\left(ma+\frac{1-t}{n+1}\eta(b,a,m)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1[(1-t)^{\alpha+2}]^p\left|f'''\left(ma+\frac{n+t}{n+1}\eta(b,a,m)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+2)(q-p)+q-1}\right]^{\frac{q-1}{q}}$$

$$\times\left\{\left[\int_0^1(1-t)^{p(\alpha+2)}\left(mh_1\left(\frac{1-t}{n+1}\right)\left|f'''(a)\right|^q+h_2\left(\frac{1-t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1(1-t)^{p(\alpha+2)}\left(mh_1\left(\frac{n+t}{n+1}\right)\left|f'''(a)\right|^q+h_2\left(\frac{n+t}{n+1}\right)\left|f'''(b)\right|^q\right)dt\right]^{\frac{1}{q}}\right\},$$

which completes the proof.

We point out, now, some special cases of Theorem 3.3.

Corollary 3.14. *In Theorem 3.3, if we take $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$ for $s\in(0,1]$, then we have the following inequality for generalized (m,s)-Breckner preinvex functions*

$$\left|R(\alpha;n,m,a,b)(f)\right|$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{q-1}{(q-p)(\alpha+2)+q-1}\right)^{\frac{q-1}{q}}$$

$$\times\left\{\left[m\int_0^1(1-t)^{p(\alpha+2)}(n+t)^s dt\left|f'''(a)\right|^q+\frac{1}{p(\alpha+2)+s+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\frac{m}{p(\alpha+2)+s+1}\left|f'''(a)\right|^q+\int_0^1(1-t)^{p(\alpha+2)}(n+t)^s dt\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\}$$

$$=\frac{\eta^3(b,a,m)}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{q-1}{(q-p)(\alpha+2)+q-1}\right)^{\frac{q-1}{q}}$$

$$\times\left\{\left[\frac{mn^s{}_2F_1\left[-s,1;p(\alpha+2)+2;-\frac{1}{n}\right]}{p(\alpha+2)+1}\left|f'''(a)\right|^q+\frac{1}{p(\alpha+2)+s+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\frac{m}{p(\alpha+2)+s+1}\left|f'''(a)\right|^q+\frac{n^s{}_2F_1\left[-s,1;p(\alpha+2)+2;-\frac{1}{n}\right]}{p(\alpha+2)+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Corollary 3.15. *In Theorem 3.3, if we take $h_1(t) = h_2(t) = 1$ and $p = q = 2$, then we have the following inequality for generalized (m, P)-preinvex functions*

$$\left|R(\alpha; n, m, a, b)(f)\right|$$

$$\leq \frac{2\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)} \left(\frac{1}{2\alpha+5}\right)^{\frac{1}{2}} \left(m\left|f'''(a)\right|^2 + \left|f'''(b)\right|^2\right)^{\frac{1}{2}}.$$

Corollary 3.16. *In Theorem 3.3, if we take $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ for $s \in (0, 1]$, then we have the following inequality for generalized (m, s)-Godunova-Levin-Dragomir preinvex functions*

$$\left|R(\alpha; n, m, a, b)(f)\right|$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)} \left[\frac{q-1}{(q-p)(\alpha+2)+q-1}\right]^{\frac{q-1}{q}}$$

$$\times \left\{ \left[m\int_0^1 (1-t)^{p(\alpha+2)}(n+t)^{-s}dt \left|f'''(a)\right|^q + \frac{1}{p(\alpha+2)-s+1}\left|f'''(b)\right|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\frac{m}{p(\alpha+2)-s+1}\left|f'''(a)\right|^q + \int_0^1 (1-t)^{p(\alpha+2)}(n+t)^{-s}dt \left|f'''(b)\right|^q \right]^{\frac{1}{q}} \right\}$$

$$= \frac{\eta^3(b,a,m)}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)} \left[\frac{q-1}{(q-p)(\alpha+2)+q-1}\right]^{\frac{q-1}{q}}$$

$$\times \left\{ \left[\frac{mn^{-s}\,{}_2F_1\left[s,1;p(\alpha+2)+2;-\frac{1}{n}\right]}{p(\alpha+2)+1} \left|f'''(a)\right|^q + \frac{1}{p(\alpha+2)-s+1}\left|f'''(b)\right|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\frac{m}{p(\alpha+2)-s+1}\left|f'''(a)\right|^q + \frac{n^{-s}\,{}_2F_1\left[s,1;p(\alpha+2)+2;-\frac{1}{n}\right]}{p(\alpha+2)+1}\left|f'''(b)\right|^q \right]^{\frac{1}{q}} \right\}.$$

Corollary 3.17. *In Theorem 3.3, if we take $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, then we have the following inequality for generalized (m,h)-preinvex functions*

$$\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\leq \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\tfrac{q-1}{(q-p)(\alpha+2)+q-1}\right)^{\frac{q-1}{q}} \\
&\quad\times\Bigg\{\left[\int_0^1(1-t)^{p(\alpha+2)}\left(mh\left(\tfrac{n+t}{n+1}\right)\Big|f'''(a)\Big|^q+h\left(\tfrac{1-t}{n+1}\right)\Big|f'''(b)\Big|^q\right)dt\right]^{\frac{1}{q}} \\
&\quad+\left[\int_0^1(1-t)^{p(\alpha+2)}\left(mh\left(\tfrac{1-t}{n+1}\right)\Big|f'''(a)\Big|^q+h\left(\tfrac{n+t}{n+1}\right)\Big|f'''(b)\Big|^q\right)dt\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Corollary 3.18. *In Theorem 3.3, if we take $h_1(t) = h_2(t) = t(1-t)$ with $p = q = 2$ and $n = 1$, then we have the following inequality for generalized (m,tgs)-preinvex functions*

$$\begin{aligned}
&\Big|R(\alpha;1,m,a,b)(f)\Big| \\
&\leq \tfrac{\eta^3(b,a,m)}{16(\alpha+1)(\alpha+2)}\left[\tfrac{\alpha+4}{(\alpha+3)(2\alpha+7)}\right]^{\frac{1}{2}}\left(m\Big|f'''(a)\Big|^2+\Big|f'''(b)\Big|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Corollary 3.19. *In Theorem 3.3, if we take $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we have the following inequality for generalized m-MT-preinvex functions*

$$
\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big|\\
&\le \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+2)(q-p)+q-1}\right]^{\frac{q-1}{q}}\\
&\quad\times\Bigg\{\Bigg[\left(\frac{mn^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2},1;p(\alpha+2)+\frac{3}{2};-\frac{1}{n}\right]}{2p(\alpha+2)+1}\right)\Big|f'''(a)\Big|^q\\
&\quad+\frac{n^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2},1;p(\alpha+2)+\frac{5}{2};-\frac{1}{n}\right]}{2p(\alpha+2)+3}\Big|f'''(b)\Big|^q\Bigg]^{\frac{1}{q}}\\
&\quad+\Bigg[\frac{mn^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2},1;p(\alpha+2)+\frac{5}{2};-\frac{1}{n}\right]}{2p(\alpha+2)+3}\Big|f'''(a)\Big|^q\\
&\quad+\left(\frac{n^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2},1;p(\alpha+2)+\frac{3}{2};-\frac{1}{n}\right]}{2p(\alpha+2)+1}\right)\Big|f'''(b)\Big|^q\Bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

Finally, we shall prove the following result.

Theorem 3.4. *Let $I \subseteq \mathbf{R}$ be an open m-invex subset with respect to $\eta : I \times I \times (0,1] \to \mathbf{R}$ for some fixed $m \in (0,1]$ and let $a, b \in I$, $a < b$ with $\eta(b,a,m) > 0$. Suppose that $f : I \to \mathbf{R}$ be a three times differentiable function on the interior I° of I, $h_1, h_2 : [0,1] \to \mathbf{R}_0$, $f''' \in L[ma, ma+\eta(b,a,m)]$ and $|f'''|^q$ for $q > 1$ is (m, h_1, h_2)-preinvex on $[ma, ma+\eta(b,a,m)]$, then the following inequality for Riemann-Liouville fractional integrals with $\alpha > 0$, $n \in \mathbf{N}^+$ and $0 < \mu, \lambda < \alpha + 2$ holds*

$$\Big|R(\alpha;n,m,a,b)(f)\Big|$$

$$\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Bigg\{\bigg(\frac{q-1}{(\alpha+3-\mu)q-1}\bigg)^{1-\frac{1}{q}}$$

$$\times\bigg[\int_0^1(1-t)^{\mu q}\bigg(mh_1\Big(\frac{1-t}{n+1}\Big)\Big|f'''(a)\Big|^q+h_2\Big(\frac{1-t}{n+1}\Big)\Big|f'''(b)\Big|^q\bigg)dt\bigg]^{\frac{1}{q}} \tag{3.12}$$

$$+\bigg(\frac{q-1}{(\alpha+3-\lambda)q-1}\bigg)^{1-\frac{1}{q}}\bigg[\int_0^1(1-t)^{\lambda q}\bigg(mh_1\Big(\frac{n+t}{n+1}\Big)\Big|f'''(a)\Big|^q$$

$$+h_2\Big(\frac{n+t}{n+1}\Big)\Big|f'''(b)\Big|^q\bigg)dt\bigg]^{\frac{1}{q}}\Bigg\}.$$

Proof. Using given hypotheses, Lemma 1.2 and the Hölder's inequality, we have

$$
\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\le \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\Big|f'''\Big(ma+\tfrac{1-t}{n+1}\eta(b,a,m)\Big)\Big|dt \\
&\quad+\tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\Big|f'''\Big(ma+\tfrac{n+t}{n+1}\eta(b,a,m)\Big)\Big|dt \\
&\le \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Bigg[\int_0^1(1-t)^{(\alpha+2-\mu)\frac{q}{q-1}}dt\Bigg]^{1-\frac{1}{q}} \\
&\quad\times\Bigg[\int_0^1(1-t)^{\mu q}\Big|f'''\Big(ma+\tfrac{1-t}{n+1}\eta(b,a,m)\Big)\Big|^q dt\Bigg]^{\frac{1}{q}} \\
&\quad+\tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Bigg[\int_0^1(1-t)^{(\alpha+2-\lambda)\frac{q}{q-1}}dt\Bigg]^{1-\frac{1}{q}} \\
&\quad\times\Bigg[\int_0^1(1-t)^{\lambda q}\Big|f'''\Big(ma+\tfrac{n+t}{n+1}\eta(b,a,m)\Big)\Big|^q dt\Bigg]^{\frac{1}{q}} \\
&\le \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Bigg[\tfrac{q-1}{(\alpha+3-\mu)q-1}\Bigg]^{1-\frac{1}{q}} \\
&\quad\times\Bigg\{\int_0^1(1-t)^{\mu q}\Bigg[mh_1\Big(\tfrac{1-t}{n+1}\Big)\Big|f'''(a)\Big|^q+h_2\Big(\tfrac{1-t}{n+1}\Big)\Big|f'''(b)\Big|^q\Bigg]dt\Bigg\}^{\frac{1}{q}} \\
&\quad+\tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\Bigg[\tfrac{q-1}{(\alpha+3-\lambda)q-1}\Bigg]^{1-\frac{1}{q}} \\
&\quad\times\Bigg\{\int_0^1(1-t)^{\lambda q}\Bigg[mh_1\Big(\tfrac{n+t}{n+1}\Big)\Big|f'''(a)\Big|^q+h_2\Big(\tfrac{n+t}{n+1}\Big)\Big|f'''(b)\Big|^q\Bigg]dt\Bigg\}^{\frac{1}{q}},
\end{aligned}
$$

which completes the proof.

We point out, now, some special cases of Theorem 3.4.

Corollary 3.20. *In Theorem 3.4, if we take $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$ and $\mu = \lambda$, then we have the following inequality for generalized (m,s)-Breckner preinvex functions*

$$\begin{aligned}
&\left|R(\alpha; n, m, a, b)(f)\right| \\
&\leq \frac{\eta^3(b,a,m)}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+3-\mu)q-1}\right]^{1-\frac{1}{q}} \\
&\times\left\{\left[m\int_0^1(1-t)^{\mu q}(n+t)^s dt\left|f'''(a)\right|^q + \frac{1}{\mu q+s+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{m}{\mu q+s+1}\left|f'''(a)\right|^q + \int_0^1(1-t)^{\mu q}(n+t)^s dt\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\} \\
&= \frac{\eta^3(b,a,m)}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+3-\mu)q-1}\right]^{1-\frac{1}{q}} \\
&\times\left\{\left[\frac{mn^s{}_2F_1\left[-s,1;\mu q+2;-\frac{1}{n}\right]}{\mu q+1}\left|f'''(a)\right|^q + \frac{1}{\mu q+s+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{m}{\mu q+s+1}\left|f'''(a)\right|^q + \frac{n^s{}_2F_1\left[-s,1;\mu q+2;-\frac{1}{n}\right]}{\mu q+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 3.21. *In Theorem 3.4, taking $h_1(t) = h_2(t) = 1$, we have the following inequality for generalized (m,P)-preinvex functions*

$$
\begin{aligned}
&\left|R(\alpha;n,m,a,b)(f)\right| \\
&\le \tfrac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)} \\
&\times\left\{\left[\tfrac{q-1}{(\alpha+3-\mu)q-1}\right]^{1-\frac{1}{q}}\left(\tfrac{1}{\mu q+1}\right)^{\frac{1}{q}}+\left[\tfrac{q-1}{(\alpha+3-\lambda)q-1}\right]^{1-\frac{1}{q}}\left(\tfrac{1}{\lambda q+1}\right)^{\frac{1}{q}}\right\} \\
&\times\left(m\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right)^{\frac{1}{q}},
\end{aligned}
\tag{3.13}
$$

specially, if we put $\eta(b,a,m) = b - ma$ with $m = 1$ in (3.13), and take $n = \alpha = 1$, then we have Theorem 3.4 in [39].

Remark 3.1. *(i) In Corollary 3.21, putting $\mu = \lambda = 1$, we obtain*

$$
\begin{aligned}
&\left|R(\alpha;n,m,a,b)(f)\right| \\
&\le \tfrac{2\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\tfrac{q-1}{(\alpha+2)q-1}\right)^{1-\frac{1}{q}}\left(\tfrac{1}{q+1}\right)^{\frac{1}{q}}\left[m\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

(ii) In Corollary 3.21, putting $\mu = \lambda = 2$, we obtain

$$
\begin{aligned}
&\left|R(\alpha;n,m,a,b)(f)\right| \\
&\le \tfrac{2\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\tfrac{q-1}{(\alpha+1)q-1}\right)^{1-\frac{1}{q}}\left(\tfrac{1}{2q+1}\right)^{\frac{1}{q}}\left[m\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.22. *In Theorem 3.4, if we take $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$ and $\mu = \lambda$, then we have the following inequality for generalized (m,s)-Godunova-Levin-Dragomir preinvex functions*

$$\begin{aligned}
&\left|R(\alpha;n,m,a,b)(f)\right| \\
&\leq \frac{\eta^3(b,a,m)}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+3-\mu)q-1}\right]^{1-\frac{1}{q}} \\
&\times\left\{\left[m\int_0^1(1-t)^{\mu q}(n+t)^{-s}dt\left|f'''(a)\right|^q+\frac{1}{\mu q-s+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{m}{\mu q-s+1}\left|f'''(a)\right|^q+\int_0^1(1-t)^{\mu q}(n+t)^{-s}dt\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\} \\
&= \frac{\eta^3(b,a,m)}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+3-\mu)q-1}\right]^{1-\frac{1}{q}} \\
&\times\left\{\left[\frac{mn^{-s}{}_2F_1\left[s,1;\mu q+2;-\frac{1}{n}\right]}{\mu q+1}\left|f'''(a)\right|^q+\frac{1}{\mu q-s+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{m}{\mu q-s+1}\left|f'''(a)\right|^q+\frac{n^{-s}{}_2F_1\left[s,1;\mu q+2;-\frac{1}{n}\right]}{\mu q+1}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 3.23. *In Theorem 3.4, if we take $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$ with $\mu = \lambda$, then we have the following inequality for generalized (m, h)-preinvex functions*

$$\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+3-\mu)q-1}\right]^{1-\frac{1}{q}} \\
&\quad\times\Bigg\{\left[\int_0^1(1-t)^{\mu q}\left(mh\left(\frac{n+t}{n+1}\right)\Big|f'''(a)\Big|^q+h\left(\frac{1-t}{n+1}\right)\Big|f'''(b)\Big|^q\right)dt\right]^{\frac{1}{q}} \\
&\quad+\left[\int_0^1(1-t)^{\mu q}\left(mh\left(\frac{1-t}{n+1}\right)\Big|f'''(a)\Big|^q+h\left(\frac{n+t}{n+1}\right)\Big|f'''(b)\Big|^q\right)dt\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Corollary 3.24. *In Theorem 3.4, if we take $h_1(t) = h_2(t) = t(1-t)$ with $\mu = \lambda$, then we have the following inequality for generalized (m, tgs)-preinvex functions*

$$\begin{aligned}
&\Big|R(\alpha;n,m,a,b)(f)\Big| \\
&\leq \frac{2\eta^3(b,a,m)}{(n+1)^{4+\frac{2}{q}}(\alpha+1)(\alpha+2)}\left[\frac{q-1}{(\alpha+3-\mu)q-1}\right]^{1-\frac{1}{q}}\left[\frac{n(\mu q+3)+1}{(\mu q+2)(\mu q+3)}\right]^{\frac{1}{q}} \\
&\quad\times\left(m|f'''(a)|^q+|f'''(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3.25. *In Theorem 3.4, if we take $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ with $\mu = \lambda$, then we have the following inequality for generalized m-MT-preinvex functions*

$$\begin{aligned}&\left|R(\alpha;n,m,a,b)(f)\right|\\&\leq \frac{\eta^3(b,a,m)}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{q-1}{(\alpha+3-\mu)q-1}\right)^{1-\frac{1}{q}}\\&\quad\times\left\{\left[\left(\frac{mn^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2},1;\mu q+\frac{3}{2};-\frac{1}{n}\right]}{2\mu q+1}\right)\left|f'''(a)\right|^q\right.\right.\\&\quad\left.+\frac{n^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2},1;\mu q+\frac{5}{2};-\frac{1}{n}\right]}{2\mu q+3}\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\\&\quad+\left[\frac{mn^{-\frac{1}{2}}{}_2F_1\left[\frac{1}{2},1;\mu q+\frac{5}{2};-\frac{1}{n}\right]}{2\mu q+3}\left|f'''(a)\right|^q\right.\\&\quad\left.\left.+\left(\frac{n^{\frac{1}{2}}{}_2F_1\left[-\frac{1}{2},1;\mu q+\frac{3}{2};-\frac{1}{n}\right]}{2\mu q+1}\right)\left|f'''(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

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