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Bound on $H_3(1)$ Hankel determinant for pre-starlike functions of order α

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Abstract

The objective of this paper is to obtain best possible upper bound to the third Hankel determinant for the pre-starlike functions of order α ($0 \leq \alpha < 1$), using Toeplitz determinants.

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1. Introduction

Let A denote the class of functions f of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function its n^{th} Taylor coefficient is bounded by n (see [3]). The bounds for the coefficients give information about the geometric properties of these functions. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. Pommerenke [9] defined the following Hankel determinant of the function f for $q \geq 1$ and $n \geq 1$, which has been extensively studied

$$(1.2) \quad H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

It is easy to observe that $H_2(1)$ is the Fekete-Szegő functional. Fekete and Szegő then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali [1] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegő functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$, when $f \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α ($0 < \alpha \leq 1$). Further, sharp bounds for the second Hankel determinant functional

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2, \text{ when } q = 2 \text{ and } n = 2,$$

were obtained for various subclasses of univalent and multivalent analytic functions. In particular, Janteng et al. [6] obtained sharp bound to the functional $|H_2(2)|$ for the well known subclasses of S , namely, starlike and convex functions denoted by S^* and \mathcal{K} respectively and showed that $|a_2 a_4 - a_3^2| \leq 1$ and $|a_2 a_4 - a_3^2| \leq \frac{1}{8}$ respectively. For our discussion, in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$(1.3) \quad H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

On expanding the determinant by means of third column and then applying the triangle inequality, we obtain

$$(1.4) \quad |H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|.$$

Babalola [2] obtained sharp upper bounds to the functional $|a_2a_3 - a_4|$ and $|H_3(1)|$ for the familiar subclasses namely starlike and convex functions respectively denoted by S^* and \mathcal{K} of S . The sharp upper bound to the second Hankel functional $|H_2(2)| = |a_2a_4 - a_3^2|$ for these classes were obtained by Janteng et.al [6]. Motivated by the results obtained by different authors working in this direction, in this paper, using convolution technique, with the help of Toeplitz determinants, we seek an upper bound to the functional $|a_2a_3 - a_4|$ and hence for $|H_3(1)|$, when the function f given in (1.1) belongs to the class of pre-starlike functions of order α ($0 \leq \alpha < 1$), defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be starlike function of order α ($0 \leq \alpha < 1$), denoted by $f \in S^*(\alpha)$, if and only if

$$(1.5) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in E.$$

Definition 1.2. A function $f(z) \in A$ is said to be convex function of order α ($0 \leq \alpha < 1$), denoted by $f \in \mathcal{K}(\alpha)$, if and only if

$$(1.6) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad z \in E.$$

From the Definitions 1.1 and 1.2, we observe that $f \in \mathcal{K}(\alpha) \Leftrightarrow zf' \in S^*(\alpha)$.

Definition 1.3. A function $f \in A$ is said to be in the class of pre-starlike functions of order α ($0 \leq \alpha < 1$), denoted by R_α , if and only if

$$(1.7) \quad f(z) * s_\alpha(z) \in S^*(\alpha), \quad z \in E,$$

where $*$ denotes the convolution of two analytic functions and $s_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$ is the extremal function for the class $S^*(\alpha)$.

The class R_α was introduced and studied by Ruscheweyh [10].

$$(1.8) \quad \text{Take } c(\alpha, n) = \frac{\prod_{k=2}^n (k - 2\alpha)}{(n - 1)!}, \quad \text{for } n = 2, 3, \dots,$$

so that $s_\alpha(z)$ can be written in the form

$$(1.9) \quad s_\alpha(z) = z + \sum_{n=2}^{\infty} c(\alpha, n)z^n,$$

noting that $c(\alpha, n)$ is a decreasing function of α with

$$\lim_{n \rightarrow \infty} c(\alpha, n) = \begin{cases} \infty, & \text{if } \alpha < \frac{1}{2}, \\ 1, & \text{if } \alpha = \frac{1}{2}, \\ 0, & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

Ruscheweyh (see [11]) also showed that the necessary and sufficient condition for $f \in R_\alpha$ is that the function

$$G(\alpha, z) = \frac{f(z) * \frac{s_\alpha(z)}{(1-z)}}{f(z) * s_\alpha(z)},$$

satisfy $\operatorname{Re} G(\alpha, z) > \frac{1}{2}$, $z \in E$. Since $s_1(z) = z$, we say that f is pre-starlike function of order 1, if and only if

$$(1.10) \quad \operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}, \quad z \in E.$$

Note that $R_0 = \mathcal{K}$ and $R_{\frac{1}{2}} = S^*(\frac{1}{2})$.

It was shown that $R_\alpha \subset R_\beta$, for $0 \leq \alpha < \beta \leq 1$, which generalizes the well-known result that $\mathcal{K}(0) \subset S^*(\frac{1}{2})$.

In proving our results in section 3, we require the following Lemmas:

2. Preliminaries

Let P denote the class of functions consisting of p , such that

$$(2.1) \quad p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

which are analytic (regular) in the open unit disc E and satisfy $\operatorname{Re} p(z) > 0$ for any $z \in E$. Here $p(z)$ is called a Caratheodory function [4].

Lemma 2.1. ([8, 12]) *If $p \in P$, then $|c_k| \leq 2$, for each $k \geq 1$ and the extremal function in this case is $\frac{1+z}{1-z}$.*

Lemma 2.2. Let $p \in P$, then

$$(2.2) \quad 2c_2 = c_1^2 + y(4 - c_1^2)$$

and

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta\},$$

for some y, ζ such that $|y| \leq 1$ and $|\zeta| \leq 1$.

We apply the classical method initiated by Libera and Zlotkiewicz [7], which has been used widely, in obtaining our results.

3. Main Results

Theorem 3.1. If $f \in R_\alpha$ then

$$|a_2a_3 - a_4| \leq \frac{4}{3\sqrt{3}(3 - 2\alpha)}, \text{ for } (0 \leq \alpha < 1).$$

Proof. Let $f(z) = z + \sum_{n=2}^\infty a_n z^n \in R_\alpha$, by virtue of Definition 1.3, we have

$$(3.1) \quad f(z) * s_\alpha(z) \in S^*(\alpha), \quad \forall z \in E.$$

By the convolution, we obtain

$$g(z) = f(z) * s_\alpha(z) = \{z + \sum_{n=2}^\infty a_n z^n\} * \{z + \sum_{n=2}^\infty c(\alpha, n) z^n\} \\ = z + \sum_{n=2}^\infty c(\alpha, n) a_n z^n.$$

(3.2)

Since $g(z) \in S^*(\alpha)$, from Definition 1.1, there exists an analytic function $p \in P$ in the open unit disc E with $p(0) = 1$ and $\text{Re}p(z) > 0$ such that

$$(3.3) \quad \frac{zg'(z) - \alpha g(z)}{(1 - \alpha)g(z)} = p(z) \Leftrightarrow zg'(z) - \alpha g(z) = (1 - \alpha)g(z)p(z).$$

Using the series representations for $g(z)$ and $p(z)$ in (3.3), we have

$$z \{1 + \sum_{n=2}^\infty c(\alpha, n) n a_n z^{n-1}\} - \alpha \{z + \sum_{n=2}^\infty c(\alpha, n) a_n z^n\} \\ = (1 - \alpha) \{z + \sum_{n=2}^\infty c(\alpha, n) a_n z^n\} \{1 + \sum_{n=1}^\infty c_n z^n\}.$$

Upon simplification, we obtain

$$c(2, \alpha)a_2 + 2c(3, \alpha)a_3z + 3c(4, \alpha)a_4z^2 + 4c(5, \alpha)a_5z^3 + \dots =$$

$$(1 - \alpha) \times [c_1 + \{c_2 + c(2, \alpha)c_1a_2\}z + \{c_3 + c(2, \alpha)c_2a_2 + c(3, \alpha)c_1a_3\}z^2 +$$

$$\{c_4 + c(2, \alpha)c_3a_2 + c(3, \alpha)c_2a_3 + c(4, \alpha)c_1a_4\}z^3 + \dots].$$

(3.4)

Equating the coefficients of z^0 , z , z^2 and z^3 respectively in (3.4), after simplifying, we get

$$a_2 = \frac{c_1}{2}; \quad a_3 = \frac{1}{2(3-2\alpha)} \{c_2 + (1 - \alpha)c_1^2\};$$

$$a_4 = \frac{1}{4(2-\alpha)(3-2\alpha)} \{2c_3 + 3(1 - \alpha)c_1c_2 + (1 - \alpha)^2c_1^3\}.$$

$$a_5 = \frac{1}{2(5-2\alpha)(4-2\alpha)(3-2\alpha)} \times \\ \{6c_4 + 8(1 - \alpha)c_1c_3 + 3(1 - \alpha)c_2^2 + 6(1 - \alpha)^2c_1^2c_2 + (1 - \alpha)^3c_1^4\}.$$

(3.5)

Substituting the values of a_2 , a_3 and a_4 from (3.5) in the functional $|a_2a_3 - a_4|$, which simplifies to give

$$|a_2a_3 - a_4| = \frac{1}{4(2-\alpha)(3-2\alpha)} |-2c_3 + (-1 + 2\alpha)c_1c_2 + (1 - \alpha)c_1^3|.$$

The above expression is equivalent to

$$(3.6) \quad |a_2a_3 - a_4| = \frac{1}{4(2-\alpha)(3-2\alpha)} |d_1c_3 + d_2c_1c_2 + d_3c_1^3|,$$

$$(3.7) \quad \text{where } d_1 = -2; \quad d_2 = -1 + 2\alpha; \quad d_3 = 1 - \alpha.$$

Using the values of c_2 and c_3 from (2.2) and (2.3) respectively from Lemma 2.2 on the right-hand side of (3.6), we have

$$|d_1c_3 + d_2c_1c_2 + d_3c_1^3| = \left| d_1 \times \frac{1}{4} \{c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 \right. \\ \left. + 2(4 - c_1^2)(1 - |y|^2)\zeta\} + d_2c_1 \times \frac{1}{2} \{c_1^2 + y(4 - c_1^2)\} + d_3c_1^3 \right|.$$

(3.8)

Applying the triangle inequality and the fact $|\zeta| \leq 1$ in the expression (3.8), which simplifies to

$$4|d_1c_3 + d_2c_1c_2 + d_3c_1^3| \leq |(d_1 + 2d_2 + 4d_3)c_1^3 + 2d_1(4 - c_1^2) + 2(d_1 + d_2)c_1(4 - c_1^2)|y| - d_1(c_1 + 2)(4 - c_1^2)|y|^2|. \tag{3.9}$$

From (3.7), we can now write

$$d_1 + 2d_2 + 4d_3 = 0; \quad d_1 + d_2 = -3 + 2\alpha. \tag{3.10}$$

From the expressions (3.9) and (3.10), we have

$$4|d_1c_3 + d_2c_1c_2 + d_3c_1^3| \leq |4(4 - c_1^2) + 2(-3 + 2\alpha)c_1(4 - c_1^2)|y| + 2(c_1 + 2)(4 - c_1^2)|y|^2|. \tag{3.11}$$

Since $c_1 = c \in [0, 2]$, noting that $c_1 + a \geq c_1 - a$, where $a \geq 0$, applying the triangle inequality and replacing $|y|$ by μ on the right-hand side of the above inequality, we get

$$2|d_1c_3 + d_2c_1c_2 + d_3c_1^3| \leq [2 + (3 - 2\alpha)c\mu + (c - 2)\mu^2](4 - c^2) = F(c, \mu), \quad \text{for } 0 \leq \mu = |y| \leq 1, \tag{3.12}$$

where

$$F(c, \mu) = [2 + (3 - 2\alpha)c\mu + (c - 2)\mu^2](4 - c^2). \tag{3.13}$$

Further, we maximize the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ partially with respect to μ and c respectively, we obtain

$$\frac{\partial F}{\partial \mu} = [(3 - 2\alpha)c + (c - 2)\mu](4 - c^2) \tag{3.14}$$

$$\text{and } \frac{\partial F}{\partial c} = -4c + (3 - 2\alpha)(4 - 3c^2)\mu + (4 + 4c - 3c^2)\mu^2. \tag{3.15}$$

For the extreme values of $F(c, \mu)$, consider

$$\frac{\partial F}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial F}{\partial c} = 0. \tag{3.16}$$

In view of (3.16), on solving the equations in (3.14) and (3.15), we obtain the critical point for the function $F(c, \mu)$ which lies in the closed region $[0, 2] \times [0, 1]$ is $(0, 0)$ only. At the critical point $(0, 0)$, we observe that

$$\frac{\partial^2 F}{\partial \mu^2} = -4 < 0; \quad \frac{\partial^2 F}{\partial c^2} = -16 < 0; \quad \frac{\partial^2 F}{\partial c \partial \mu} = 4(3 - 2\alpha)$$

and

$$\left[\left(\frac{\partial^2 F}{\partial \mu^2} \right) \left(\frac{\partial^2 F}{\partial c^2} \right) - \left(\frac{\partial^2 F}{\partial c \partial \mu} \right)^2 \right] = 16(-4\alpha^2 + 12\alpha - 5) < 0, \quad \text{for } 0 \leq \alpha < 1.$$

Therefore, at the point $(0, 0)$, the function $F(c, \mu)$ has local minimum. Now, we search for the critical points on the boundary of the region $[0, 2] \times [0, 1]$. (i) At $L_1 = \{(2, \mu)/0 \leq \mu \leq 1\}$, we have $F(2, \mu) = 0$, which is a constant.

(ii) At $L_2 = \{(0, \mu)/0 \leq \mu \leq 1\}$, we obtain $F(2, \mu) = 8 - 8\mu^2$, which gives us the same critical point $(0, 0)$. (iii) At $L_3 = \{(c, 0)/0 \leq c \leq 2\}$, we get $F(c, 0) = 8 - 2c^2$, which also gives us the same critical point $(0, 0)$. (iv) At $L_4 = \{(c, 1)/0 \leq c \leq 2\}$, we get $F(c, 1) = 8(2 - \alpha) - 2(2 - \alpha)c^3$.

$$(3.17) \quad \text{Let } G(c) = F(c, 1) = 8(2 - \alpha) - 2(2 - \alpha)c^3,$$

$$(3.18) \quad G'(c) = 8(2 - \alpha) - 6(2 - \alpha)c^2,$$

$$(3.19) \quad G''(c) = -12(2 - \alpha)c.$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From (3.18), we get $c^2 = \frac{4}{3}$. Since $c \in [0, 2]$, consider $c = \frac{2}{\sqrt{3}}$ only. Substituting the value of $c = \frac{2}{\sqrt{3}}$ in (3.19), we observe that $G''(c) < 0$, for $0 \leq \alpha < 1$. Therefore, by the second derivative test, $G(c)$ has maximum value at $c = \frac{2}{\sqrt{3}}$, from (3.17), it is obtained to be

$$(3.20) \quad G_{max} = \frac{32(2 - \alpha)}{3\sqrt{3}}.$$

Simplifying the expressions (3.6), (3.12) and (3.20), we get

$$(3.21) \quad |a_2a_3 - a_4| \leq \frac{4}{3\sqrt{3}(3 - 2\alpha)}.$$

This completes the proof of Theorem 3.1. \square

Remark 3.2. Choosing $\alpha = 0$ in (3.21), we obtain $|a_2a_3 - a_4| \leq \frac{4}{9\sqrt{3}}$. This inequality is sharp and coincides with that of Babalola [2].

The following Theorem is a straightforward verification on applying Theorem 3.1.

Theorem 3.3. If $f \in R_\alpha$ then $|a_3 - a_2^2| \leq \frac{1}{(3-2\alpha)}$, for $(0 \leq \alpha < 1)$.

Remark 3.4. If $\alpha = 0$ in the above Theorem, we get $|a_3 - a_2^2| \leq \frac{1}{3}$, this result is same as the result obtained by Babalola [2].

Theorem 3.5. If $f(z) \in R_\alpha$ then we have $|a_k| \leq 1$, for $k \geq 2$.

Proof. Using the fact that $|c_n| \leq 2$, for $n \in N = \{1, 2, 3 \dots\}$, with the help of c_2 and c_3 values given in (2.2) and (2.3) respectively together with the values obtained in (3.5), we obtain $|a_k| \leq 1$, for $k \geq 2$. This completes the proof of our Theorem 3.5. \square

Substituting the results of Theorems 3.1, 3.3, 3.5 and the known result $|a_2a_4 - a_3^2| \leq \frac{1}{8(1-\alpha)}$, for $(0 \leq \alpha < \frac{1}{2})$ (see [14]) in the inequality given in (1.4), which simplifies to obtain the following Corollary.

Corollary 3.6. If $f(z) \in R_\alpha$, then

$$|H_3(1)| \leq \left[\frac{32(1 - \alpha) + \sqrt{3}(33 - 30\alpha)}{24\sqrt{3}(1 - \alpha)(3 - 2\alpha)} \right].$$

Remark 3.7. Choosing $\alpha = 0$, the above corollary coincides with the result obtained by Babalola [2].

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