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A new family of chromatically unique 6-bridge graph

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Abstract

For a graph G, let $P(G,\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. In this paper, the chromatic uniqueness of a new family of 6-bridge graph $\theta(a,a,b,b,b,c)$ where $2 \leq a \leq b \leq c$, is investigated.

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1. Introduction

All graphs considered here are simple graphs. For such a graph G, let $P(G,\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. The chromaticity of a graph G refers to questions about the chromatic equivalence class or chromatic uniqueness of G. For terminologies and notations which are not explained here, the reader is referred to [6, 21].

Let k be an integer with $k \geq 2$ and let $a_1, a_2, \ldots, a_k$ be natural numbers with $a_i + a_j \geq 3$ for all i, j and $1 \leq i \leq j \leq k$. Let $\theta(a_1, a_2, \ldots, a_k)$ denote the graph obtained by connecting two distinct vertices with k independent (internally disjoint) paths of length $a_1, a_2, \ldots, a_k$, respectively. The graph $\theta(a_1, a_2, \ldots, a_k)$ is called a multi-bridge (more spesifically k-bridge) graph.

Let $G = \theta(a_1, a_2, \cdots, a_k)$ and we then have (see [4])

$$\begin{aligned} P\Big(G,\lambda\Big) &= \frac{1}{\lambda^{k-1}(\lambda-1)^{k-1}}\prod_{i=1}^{k}\Big((\lambda-1)^{a_i+1}+(-1)^{a_i+1}(\lambda-1)\Big) \\ &\quad +\frac{1}{\lambda^{k-1}}\prod_{i=1}^{k}\Big((\lambda-1)^{a_i}+(-1)^{a_i}(\lambda-1)\Big) \end{aligned}$$

Let $\lambda = 1 - x$, then

$$\begin{aligned} P\Big(G,1-x\Big) &= \frac{(-1)^{a_1+a_2+\cdots+a_k+1}}{(1-x)^{k-1}}\Big(x\prod_{i=1}^{k}(x^{a_i}-1)-\prod_{i=1}^{k}(x^{a_i}-x)\Big) \\ &= \frac{(-1)^{e(G)+1}}{(1-x)^{e(G)-v(G)+1}}\Big(x\prod_{i=1}^{k}(x^{a_i}-1)-\prod_{i=1}^{k}(x^{a_i}-x)\Big) \end{aligned}$$

where $e(G) = \sum_{i=1}^{k} a_i$ and $v(G) = \sum_{i=1}^{k} a_i - k + 2$. Also define $Q(G,x)$ for any graph G and real number x as:

$$Q(G,x) = (-1)^{e(G)+1}(1-x)^{e(G)-v(G)+1}P(G,1-x).$$

Given positive integers $a_1, a_2, \ldots, a_k$, where $k \geq 2$, what is the necessary and sufficient condition on $a_1, a_2, \ldots, a_k$ for $\theta(a_1, a_2, \ldots, a_k)$ to be chromatically unique? Many papers [4, 5, 16, 17] have been published on this problem, but it is still far from being completely solved.

A 2-bridge graph is simply a cycle graph, which is $\chi-$unique. Chao and Whitehead Jr. [2] showed that every 3-bridge graph $\theta(1, a_2, a_3)$ (or a theta graph) is $\chi-$unique. Loerinc [20] extended the above result to all 3-bridge graphs by showing that all 3-bridge graphs (or generalized θ-graph) are χ-unique. For $k = 4$, Chen et al. [3] found that $\theta(a_1, a_2, a_3, a_4)$ may not be χ-unique.

Suppose $g_e\Big(G_1, G_2, \cdots, G_k\Big)$ be the collection of edge-gluing of all $G_1, G_2, \cdots, G_k$ where $k \geq 2$ and $e(G_i) \geq 1$ for all i.

Theorem 1.1. *(Chen et al. [3]) (a) Let a_1, a_2, a_3, a_4 be integers with $2 \leq a_1 \leq a_2 \leq a_3 \leq a_k$. Then $\theta(a_1, a_2, a_3, a_4)$ is χ-unique if and only if $(a_1, a_2, a_3, a_4) \neq (2, b, b+1, b+2)$ for any integer $b \geq 2$.*
(b) The χ-equivalence class of $\theta(2, b, b+1, b+2)$ is

$$\{\theta(2, b, b+1, b+2)\} \cup g_e\Big(\theta(3, b, b+1), C_{b+2}\Big).$$

Thus the problem of the chromaticity of $\theta(a_1, a_2, \cdots, a_k)$ has been completely settled for $k \leq 4$.

The results on the chromaticity of some families of 5-bridge graphs have been obtained by Bao and Chen [1], Li and Wei [19], Li [18], Khalaf [9], Khalaf and Peng [10], Khalaf et al. [15]. Ye [23, 24] proved that $\theta(2, 2, 2, 2, a, b)$ where $3 \leq a+1 \leq b$ and $\theta(2, 2, \ldots, 2, a, b)$ where $3 \leq a \leq b$ and $k \geq 5$ are χ-unique, respectively. Khalaf and Peng [11] also proved that $\theta(a, a, \ldots, a, b)$ for $a \leq b$ is χ-unique. The study on the chromaticity of 6-bridge graphs, $\theta(a_1, a_2, a_3, a_4, a_5, a_6)$ where $a_1, a_2, a_3, a_4, a_5, a_6$ assume exactly two distinct values and $\theta(3, 3, 3, 3, b, c)$ was done by Khalaf and Peng [12, 14]. Later on, Khalaf and Peng in [9, 13] solved the chromaticity of two types of 6-bridge graph $\theta(a_1, a_2, a_3, a_4, a_5, a_6)$ where $a_1, a_2, a_3, a_4, a_5, a_6$ assume exactly three distinct values, that is, the graphs $\theta(a, a, a, b, c, c)$ and $\theta(a, a, a, a, b, c)$, respectively. Recently, Karim et al. [7] investigated the chromaticity of another type of 6-bridge graphs, that is, $\theta(a, a, a, b, b, c)$. In this paper, we continue to investigate the chromaticity of new type of such graphs, that is, $\theta(a, a, b, b, b, c)$ (see Figure 1).

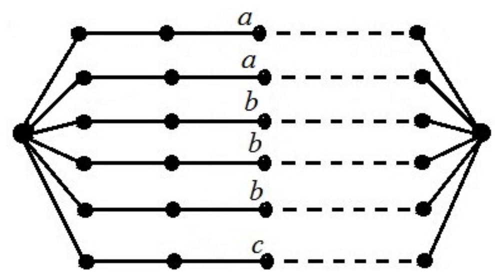

Figure 1 : $\theta(a, a, b, b, b, c)$

2. Preliminary Results and Notations

In this section, we cite some results to be used in sequel.

Lemma 2.1. *(Xu et al. [22]) For $k \geq 4$, $\theta(a_1, a_2, \ldots, a_k)$ is χ-unique if $k-1 \leq a_1 \leq a_2 \leq \ldots \leq a_k$.*

For each positive integer h, the graph $G(h)$ is obtained from G by replacing each edge of G by a path of length h, respectively and is called the h-uniform subdivision of G. Xu et al. [22] showed that any h-uniform subdivision of θ_k denoted as $\theta_k(h)$, is χ-unique, as stated in the following theorem.

Lemma 2.2. *(Xu et al. [22]) For $k \geq 2$, the graph $\theta_k(h)$ is χ-unique.*

Dong et al. [5] proved the following results.

Lemma 2.3. *(Dong et al. [5]) If $2 \leq a_1 \leq a_2 \leq \ldots \leq a_k < a_1 + a_2$ where $k \geq 3$, then the graph $\theta(a_1, a_2, \ldots, a_k)$ is χ-unique.*

Lemma 2.4. *(Dong et al. [5]) For any $k, a_1, a_2, \ldots, a_k \in N$,*

$$Q\Big(\theta(a_1, a_2, \ldots, a_k), x\Big) = x \prod_{i=1}^{k} \Big(x^{a_i} - 1\Big) - \prod_{i=1}^{k} \Big(x^{a_i} - x\Big).$$

Lemma 2.5. *(Dong et al. [5]) For any graphs G and H,*

1. *If $H \sim G$, then $Q(H,x) = Q(G,x)$,*
2. *If $Q(H,x) = Q(G,x)$ and $v(H) = v(G)$, then $H \sim G$.*

Lemma 2.6. *(Dong et al. [5]) Suppose that $\theta(a_1, a_2, \ldots, a_k) \sim \theta(b_1, b_2, \ldots, b_k)$ where $k \geq 3$, $2 \leq a_1 \leq a_2 \leq \ldots \leq a_k$ and $2 \leq b_1 \leq b_2 \leq \ldots \leq b_k$, then $a_i = b_i$ for all $i = 1, 2, \ldots, k$.*

Lemma 2.7. *(Dong et al. [5]) Let $H \sim \theta(a_1, a_2, \ldots, a_k)$ where $k \geq 3$ and $a_i \geq 2$ for all i, then one of the following is true:*

(i) $H \cong \theta(a_1, a_2, \ldots, a_k)$,

(ii) $H \in g_e\Big(\theta(b_1, b_2, \ldots, b_t), C_{b_{t+1}+1}, \ldots, C_{b_k+1}\Big)$, *where* $3 \leq t \leq k-1$ *and* $b_i \geq 2$ *for all* $i = 1, 2, \ldots, k$.

Lemma 2.8. *(Dong et al. [5]) Let $k, t, b_1, b_2, \ldots, b_k \in N$, where $3 \leq t \leq k-1$ and $b_i \geq 2$ for all $i = 1, 2, \ldots, k$. If $H \in g_e\Big(\theta(b_1, b_2, \ldots, b_t), C_{b_{t+1}+1}, \ldots, C_{b_k+1}\Big)$, then*

$$Q\Big(H, x\Big) = x \prod_{i=1}^{k} \Big(x^{b_i} - 1\Big) - \prod_{i=1}^{t} \Big(x^{b_i} - x\Big) \prod_{i=t+1}^{k} \Big(x^{b_i} - 1\Big).$$

Lemma 2.9. *(Koh & Teo [16]) If $G \sim H$, then*

(i) $v(G) = v(H)$,

(ii) $e(G) = e(H)$,

(iii) $g(G) = g(H)$,

(iv) *G and H have the same number of shortest cycle.*

where $v(G)$, $v(H)$, $e(G)$, $e(H)$, $g(G)$ and $g(H)$ denote the number of vertices, the number of edges and the girth of G and H, respectively.

Lemma 2.10. *(Khalaf & Peng [13]) A 6-bridge graph $\theta(a_1, a_2, \ldots, a_6)$ is χ-unique if the positive integers $a_1, a_2, \ldots, a_6$ assume exactly two distinct values.*

3. Main Results

In this section, we present our main result on the chromaticity of 6-bridge graph $\theta(a,a,b,b,b,c)$.

Theorem 3.1. *The graph 6-bridge $\theta(a,a,b,b,b,c)$, where $a \le b \le c$, is χ-unique.*

Proof. *Let G and H be two graphs such that G is a 6-bridge graph of the form $\theta(a,a,b,b,b,c)$ where $2 \le a \le b \le c$. By Lemmas 2.1 and 2.3, G is χ-unique if $c < 2a$ and $a \ge 5$, respectively. Suppose $H \sim G$, then $c \ge 2a$ and $a < 5$. We shall solve $Q(G) = Q(H)$ to get all solutions. Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively.*

*Considering Lemmas 2.6 and 2.7, we have three cases of H to consider, that are $H \in g_e\Big(\theta(b_1,b_2,b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1}\Big)$ where $2 \le b_1 \le b_2 \le b_3$ and $2 \le b_4, b_5, b_6$ (**Case A**) or $H \in g_e\Big(\theta(b_1,b_2,b_3,b_4), C_{b_5+1}, C_{b_6+1}\Big)$ where $2 \le b_1 \le b_2 \le b_3 \le b_4$ and $2 \le b_5, b_6$ (**Case B**) or $H \in g_e\Big(\theta(b_1,b_2,b_3,b_4,b_5), C_{b_6+1}\Big)$ where $2 \le b_1 \le b_2 \le b_3 \le b_4 \le b_5$ and $2 \le b_6$ (**Case C**).*

Since the method used is standard, long and rather repetitive, we shall not discuss all of the cases here. In the following we shall only show the detail proof for Cases A and B.

By Lemma 2.9, $g(G) = g(H) = 2a$ and H and G shall have the same number of cycles with length equal to their girth. We know that the size of G is equal to the size of H. Then, we have the following.

$$2a + 3b + c = b_1 + b_2 + b_3 + b_4 + b_5 + b_6 \tag{3.1}$$

<u>**Case A**</u> *$H \in g_e\Big(\theta(b_1,b_2,b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1}\Big)$ where $2 \le b_1 \le b_2 \le b_3$ and $2 \le b_4, b_5, b_6$. As $G = \theta(a,a,b,b,b,c)$ and $H \in g_e\Big(\theta(b_1,b_2,b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1}\Big)$, by Lemmas 2.4 and 2.8 we have the following.*

$$\begin{aligned} Q(G) &= x\Big(x^a-1\Big)^2\Big(x^b-1\Big)^3\Big(x^c-1\Big) - \Big(x^a-x\Big)^2\Big(x^b-x\Big)^3\Big(x^c-x\Big), \\ Q(H) &= x\Big(x^{b_1}-1\Big)\Big(x^{b_2}-1\Big)\Big(x^{b_3}-1\Big)\Big(x^{b_4}-1\Big)\Big(x^{b_5}-1\Big)\Big(x^{b_6}-1\Big) - \\ &\quad \Big(x^{b_1}-x\Big)\Big(x^{b_2}-x\Big)\Big(x^{b_3}-x\Big)\Big(x^{b_4}-1\Big)\Big(x^{b_5}-1\Big)\Big(x^{b_6}-1\Big). \end{aligned}$$

Considering Equation 3.1, $Q(G) = Q(H)$ yields,

$$
\begin{aligned}
Q_1(G) \;=\; & 3x^{2a+2b+1} + 3x^{2a+b+c+1} + 3x^{2a+b+3} + x^{2a+c+3} + x^{2a+1} + \\
& 2x^{a+3b+1} + 6x^{a+2b+c+1} + 6x^{a+2b+3} + 6x^{a+b+c+3} + 6x^{a+b+1} + \\
& 2x^{a+c+1} + 2x^{a+5} + x^{3b+c+1} + x^{3b+3} + 3x^{2b+c+3} + 3x^{2b+1} + \\
& 3x^{b+c+1} + 3x^{b+5} + x^{c+5} - \Big(3x^{2a+2b+2} + 3x^{2a+b+c+2} + \\
& 3x^{2a+b+1} + x^{2a+c+1} + x^{2a+4} + 2x^{a+3b+2} + 6x^{a+2b+c+2} + \\
& 6x^{a+2b+1} + 6x^{a+b+c+1} + 6x^{a+b+4} + 2x^{a+c+4} + 2x^{a+1} + \\
& x^{3b+c+2} + x^{3b+1} + 3x^{2b+c+1} + 3x^{2b+4} + 3x^{b+c+4} + 3x^{b+1} + \\
& x^{c+1} + x^6\Big), \\
Q_1(H) \;=\; & x^{b_1+b_2+b_3+b_4+b_5} + x^{b_1+b_2+b_3+b_4+b_6} + x^{b_1+b_2+b_3+b_4+1} + \\
& x^{b_1+b_2+b_3+b_5+b_6} + x^{b_1+b_2+b_3+b_5+1} + x^{b_1+b_2+b_3+b_6+1} + \\
& x^{b_1+b_2+b_3} + x^{b_1+b_4+b_5+b_6+1} + x^{b_1+b_4+b_5+2} + x^{b_1+b_4+b_6+2} + \\
& x^{b_1+b_4+1} + x^{b_1+b_5+b_6+2} + x^{b_1+b_5+1} + x^{b_1+b_6+1} + x^{b_1+2} + \\
& x^{b_2+b_4+b_5+b_6+1} + x^{b_2+b_4+b_5+2} + x^{b_2+b_4+b_6+2} + x^{b_2+b_4+1} + \\
& x^{b_2+b_5+b_6+2} + x^{b_2+b_5+1} + x^{b_2+b_6+1} + x^{b_2+2} + x^{b_3+b_4+b_5+b_6+1} + \\
& x^{b_3+b_4+b_5+2} + x^{b_3+b_4+b_6+2} + x^{b_3+b_4+1} + x^{b_3+b_5+b_6+2} + \\
& x^{b_3+b_5+1} + x^{b_3+b_6+1} + x^{b_3+2} + x^{b_4+b_5+b_6+3} + x^{b_4+b_5+1} + \\
& x^{b_4+b_6+1} + x^{b_4+3} + x^{b_5+b_6+1} + x^{b_5+3} + x^{b_6+3} - \\
& \Big(x^{b_1+b_2+b_3+b_4+b_5+1} + x^{b_1+b_2+b_3+b_4+b_6+1} + x^{b_1+b_2+b_3+b_4} + \\
& x^{b_1+b_2+b_3+b_5+b_6+1} + x^{b_1+b_2+b_3+b_5} + x^{b_1+b_2+b_3+b_6} + x^{b_1+b_2+b_3+1} + \\
& x^{b_1+b_4+b_5+b_6+2} + x^{b_1+b_4+b_5+1} + x^{b_1+b_4+b_6+1} + x^{b_1+b_4+2} + \\
& x^{b_1+b_5+b_6+1} + x^{b_1+b_5+2} + x^{b_1+b_6+2} + x^{b_1+1} + x^{b_2+b_4+b_5+b_6+2} + \\
& x^{b_2+b_4+b_5+1} + x^{b_2+b_4+b_6+1} + x^{b_2+b_4+2} + x^{b_2+b_5+b_6+1} + \\
& x^{b_2+b_5+2} + x^{b_2+b_6+2} + x^{b_2+1} + x^{b_3+b_4+b_5+b_6+2} + x^{b_3+b_4+b_5+1} + \\
& x^{b_3+b_4+b_6+1} + x^{b_3+b_4+2} + x^{b_3+b_5+b_6+1} + x^{b_3+b_5+2} + x^{b_3+b_6+2} + \\
& x^{b_3+1} + x^{b_4+b_5+b_6+1} + x^{b_4+b_5+3} + x^{b_4+b_6+3} + x^{b_4+1} + x^{b_5+b_6+3} + \\
& x^{b_5+1} + x^{b_6+1} + x^3\Big).
\end{aligned}
$$

Comparing the l.r.p. in $Q_1(G)$ and the l.r.p. in $Q_1(H)$, we have $a = 2$. Therefore, $g(G) = g(H) = 2a = 4$. Without loss of generality, we have four cases to consider, that are, $b_4 = b_5 = b_6 = 3$ or $b_4 = b_5 = 3, b_6 \neq 3$ or $b_4 = 3, b_5 \neq 3, b_6 \neq 3$ or $b_4 \neq 3, b_5 \neq 3, b_6 \neq 3$.

Case 1 $b_4 = b_5 = b_6 = 3$. *Now H has three cycles of length 4 while H shall has one cycles of length 4. Hence, a contradiction.*

Case 2 $b_4 = b_5 = 3, b_6 \neq 3$. *Now H has two cycles of length 4 while H shall has one cycle of length 4. Hence, a contradiction.*

Case 3 $b_4 = 3, b_5 \neq 3, b_6 \neq 3$. *Since $g(H) = 4$, then $b_5 \geq 4$ and $b_6 \geq 4$. From Equation 3.1, we have*

$$3b + c + 1 = b_1 + b_2 + b_3 + b_5 + b_6 \tag{3.2}$$

Cancelling the equal terms in $Q_1(G)$ and $Q_1(H)$, we obtain the following.

$$\begin{aligned}
Q_2(G) &= 3x^{3b+3} + 9x^{2b+c+3} + 9x^{2b+5} + 3x^{2b+1} + 9x^{b+c+5} + 3x^{b+c+1} + \\
&\quad 3x^{b+7} + 6x^{b+3} + x^{c+7} + 2x^{c+3} + 2x^7 + x^5 - \Big(2x^{3b+4} + x^{3b+1} + \\
&\quad 6x^{2b+c+4} + 3x^{2b+c+1} + 3x^{2b+6} + 3x^{2b+4} + 6x^{2b+3} + 3x^{b+c+6} + \\
&\quad 3x^{b+c+4} + 6x^{b+c+3} + 6x^{b+6} + 3x^{b+1} + 2x^{c+6} + x^{c+1} + x^8 + \\
&\quad x^6 + x^3\Big), \\
Q_2(H) &= x^{b_1+b_2+b_3+b_5+3} + x^{b_1+b_2+b_3+b_5+1} + x^{b_1+b_2+b_3+b_6+3} + \\
&\quad x^{b_1+b_2+b_3+b_6+1} + x^{b_1+b_2+b_3+4} + x^{b_1+b_2+b_3} + x^{b_1+b_5+b_6+4} + \\
&\quad x^{b_1+b_5+b_6+2} + x^{b_1+b_5+5} + x^{b_1+b_5+1} + x^{b_1+b_6+5} + x^{b_1+b_6+1} + \\
&\quad x^{b_1+4} + x^{b_1+2} + x^{b_2+b_5+b_6+4} + x^{b_2+b_5+b_6+2} + x^{b_2+b_5+5} + \\
&\quad x^{b_2+b_5+1} + x^{b_2+b_6+5} + x^{b_2+b_6+1} + x^{b_2+4} + x^{b_2+2} + x^{b_3+b_5+b_6+4} + \\
&\quad x^{b_3+b_5+b_6+2} + x^{b_3+b_5+5} + x^{b_3+b_5+1} + x^{b_3+b_6+5} + x^{b_3+b_6+1} + \\
&\quad x^{b_3+4} + x^{b_3+2} + x^{b_5+b_6+6} + x^{b_5+b_6+1} + x^{b_5+4} + x^{b_5+3} + x^{b_6+4} + \\
&\quad x^{b_6+3} + x^6 - \Big(x^{b_1+b_2+b_3+b_5+4} + x^{b_1+b_2+b_3+b_5} + x^{b_1+b_2+b_3+b_6+4} + \\
&\quad x^{b_1+b_2+b_3+b_6} + x^{b_1+b_2+b_3+3} + x^{b_1+b_2+b_3+1} + x^{b_1+b_5+b_6+5} + \\
&\quad x^{b_1+b_5+b_6+1} + x^{b_1+b_5+4} + x^{b_1+b_5+2} + x^{b_1+b_6+4} + x^{b_1+b_6+2} + \\
&\quad x^{b_1+5} + x^{b_1+1} + x^{b_2+b_5+b_6+5} + x^{b_2+b_5+b_6+1} + x^{b_2+b_5+4} + x^{b_2+b_5+2} + \\
&\quad x^{b_2+b_6+4} + x^{b_2+b_6+2} + x^{b_2+5} + x^{b_2+1} + x^{b_3+b_5+b_6+5} + x^{b_3+b_5+b_6+1} + \\
&\quad x^{b_3+b_5+4} + x^{b_3+b_5+2} + x^{b_3+b_6+4} + x^{b_3+b_6+2} + x^{b_3+5} + x^{b_3+1} + \\
&\quad x^{b_5+b_6+4} + x^{b_5+b_6+3} + x^{b_5+6} + x^{b_5+1} + x^{b_6+6} + x^{b_6+1} + x^4\Big).
\end{aligned}$$

Comparing the l.r.p. in $Q_2(G)$ and $Q_2(H)$, we have $b_1 = 2$ or $b_2 = 2$ or $b_3 = 2$.

Case 3.1 $b_1 = 2$. *Then, we have to consider for* $b = 4$ *or* $c = 4$.

Case 3.1.1 $b = 4$. *Since the term* $-x^{b+1}$ *has coefficient 3, then there shall be another two terms in* $Q_2(H)$ *that are equal to* $-x^5$. *Hence, we have* $b_2 = b_3 = 4$ *or* $b_2 = b_5 = 4$ *or* $b_2 = b_6 = 4$ *or* $b_3 = b_5 = 4$ *or* $b_3 = b_6 = 4$ *or* $b_5 = b_6 = 4$.

Case 3.1.1.1 $b_2 = b_3 = 4$. *From Equation 3.2,* $c + 3 = b_5 + b_6$. *Then, we obtain the following after simplification.*

$$
\begin{aligned}
Q_3(G) &= 7x^{c+11} + 2x^{c+9} + 3x^{c+5} + 2x^{c+3} + 3x^{15} + 8x^{13} + 3x^9 + \\
&\quad 8x^7 - \Big(4x^{c+12} + 2x^{c+10} + x^{c+8} + 5x^{c+7} + x^{c+1} + 2x^{16} + \\
&\quad 3x^{14} + 3x^{12} + 2x^{11} + 6x^{10} + x^8 + x^6\Big), \\
Q_3(H) &= x^{b_5+b_6+1} + x^{b_5+13} + x^{b_5+11} + 2x^{b_5+9} + x^{b_5+7} + 2x^{b_5+5} + \\
&\quad 2x^{b_5+3} + x^{b_6+13} + x^{b_6+11} + 2x^{b_6+9} + x^{b_6+7} + 2x^{b_6+5} + \\
&\quad 2x^{b_6+3} + x^{14} + x^{10} + 2x^8 + 4x^6 - \Big(x^{b_5+14} + x^{b_5+10} + \\
&\quad 2x^{b_5+8} + 4x^{b_5+6} + x^{b_5+1} + x^{b_6+14} + x^{b_6+10} + 2x^{b_6+8} + \\
&\quad 4x^{b_6+6} + x^{b_6+1} + x^{13} + 2x^9 + x^7\Big).
\end{aligned}
$$

Comparing the l.r.p. in $Q_3(G)$ *and the l.r.p. in* $Q_3(H)$, *we have* $b_5 = b_6 = 5$. *Hence,* $c = 7$. *However,* $Q_3(G) \neq Q_3(H)$, *a contradiction.*

Case 3.1.1.2 $b_2 = b_5 = 4$. *From Equation 3.2,* $c + 3 = b_3 + b_6$. *Similar to Case 3.1.1.1, we obtain a contradiction.*

Case 3.1.1.3 $b_2 = b_6 = 4$. *From Equation 3.2,* $c + 3 = b_3 + b_5$. *Similar to Case 3.1.1.1, we obtain a contradiction.*

Case 3.1.1.4 $b_3 = b_5 = 4$. *From Equation 3.2,* $c + 3 = b_2 + b_6$. *Similar to Case 3.1.1.1, we obtain a contradiction.*

Case 3.1.1.5 $b_3 = b_6 = 4$. *From Equation 3.2,* $c + 3 = b_2 + b_5$. *Similar to Case 3.1.1.1, we obtain a contradiction.*

Case 3.1.1.6 $b_5 = b_6 = 4$. *From Equation 3.2,* $c + 3 = b_2 + b_3$. *Similar to Case 3.1.1.1, we obtain a contradiction.*

Case 3.1.2 $c = 4$. *Simplifying* $Q_2(G)$ *and* $Q_2(H)$, *we obtain the following.*

$$
Q_4(G) = 3x^{3b+3} + 9x^{2b+7} + 6x^{2b+5} + 3x^{2b+1} + 9x^{b+9} + 3x^{b+5} +
$$

$$6x^{b+3}+x^{11}+5x^{7}-\Big(2x^{3b+4}+x^{3b+1}+6x^{2b+8}+3x^{2b+6}+3x^{2b+4}+6x^{2b+3}+3x^{b+10}+3x^{b+8}+3x^{b+7}+6x^{b+6}+3x^{b+1}+2x^{10}+x^{8}\Big),$$

$$\begin{aligned}Q_4(H) &= x^{b_2+b_3+b_5+5}+x^{b_2+b_3+b_5+3}+x^{b_2+b_3+b_6+5}+x^{b_2+b_3+b_6+3}+\\ &\quad x^{b_2+b_3+6}+x^{b_2+b_3+2}+x^{b_2+b_5+b_6+4}+x^{b_2+b_5+b_6+2}+\\ &\quad x^{b_2+b_5+5}+x^{b_2+b_5+1}+x^{b_2+b_6+5}+x^{b_2+b_6+1}+x^{b_2+4}+\\ &\quad x^{b_2+2}+x^{b_3+b_5+b_6+4}+x^{b_3+b_5+b_6+2}+x^{b_3+b_5+5}+\\ &\quad x^{b_3+b_5+1}+x^{b_3+b_6+5}+x^{b_3+b_6+1}+x^{b_3+4}+x^{b_3+2}+\\ &\quad 2x^{b_5+b_6+6}+x^{b_5+b_6+1}+x^{b_5+7}+2x^{b_5+3}+x^{b_6+7}+\\ &\quad 2x^{b_6+3}+3x^{6}-\Big(x^{b_2+b_3+b_5+6}+x^{b_2+b_3+b_5+2}+x^{b_2+b_3+b_6+6}+\\ &\quad x^{b_2+b_3+b_6+2}+x^{b_2+b_3+5}+x^{b_2+b_3+3}+x^{b_2+b_5+b_6+5}+\\ &\quad x^{b_2+b_5+b_6+1}+x^{b_2+b_5+4}+x^{b_2+b_5+2}+x^{b_2+b_6+4}+\\ &\quad x^{b_2+b_6+2}+x^{b_2+5}+x^{b_2+1}+x^{b_3+b_5+b_6+5}+x^{b_3+b_5+b_6+1}+\\ &\quad x^{b_3+b_5+4}+x^{b_3+b_5+2}+x^{b_3+b_6+4}+x^{b_3+b_6+2}+x^{b_3+5}+\\ &\quad x^{b_3+1}+x^{b_5+b_6+7}+2x^{b_5+b_6+3}+2x^{b_5+6}+x^{b_5+1}+2x^{b_6+6}+\\ &\quad x^{b_6+1}\Big).\end{aligned}$$

Considering the l.r.p. in $Q_4(H)$, we have $b_2=b_3=b_5=5$ or $b_2=b_3=b_6=5$ or $b_2=b_5=b_6=5$ or $b_3=b_5=b_6=5$.

Case 3.1.2.1 $b_2=b_3=b_5=5$. *We obtain the following after simplification.*

$$\begin{aligned}Q_5(G) &= 3x^{3b+3}+9x^{2b+7}+6x^{2b+5}+3x^{2b+1}+9x^{b+9}+3x^{b+5}+6x^{b+3}+\\ &\quad x^{11}+3x^{7}-\Big(2x^{3b+4}+x^{3b+1}+6x^{2b+8}+3x^{2b+6}+3x^{2b+4}+\\ &\quad 6x^{2b+3}+3x^{b+10}+3x^{b+8}+3x^{b+7}+6x^{b+6}+3x^{b+1}+x^{8}\Big),\\ Q_5(H) &= 2x^{b_6+14}+x^{b_6+13}+2x^{b_6+10}+x^{b_6+6}+2x^{b_6+3}+x^{20}+x^{18}+\\ &\quad x^{16}+x^{15}+2x^{9}+2x^{8}-\Big(x^{b_6+16}+x^{b_6+15}+2x^{b_6+9}+2x^{b_6+8}+\\ &\quad x^{b_6+7}+x^{b_6+1}+x^{21}+x^{17}+2x^{14}+x^{13}\Big).\end{aligned}$$

Consider the l.r.p. in $Q_5(G)$ and the l.r.p. in $Q_5(H)$. Then, we have $b=6$ or $b_6=4$.

If $b=6$, since $2\le b\le 4$, a contradiction.

If $b_6 = 4$*, from Equation 3.2 we obtain* $b = 5\frac{1}{3}$*. However,* $b \in N$*. Thus, a contradiction.*

Case 3.1.2.2 $b_2 = b_3 = b_6 = 5$*. Similar to Case 3.1.2.1, we obtain a contradiction.*

Case 3.1.2.3 $b_2 = b_5 = b_6 = 5$*. Similar to Case 3.1.2.1, we obtain a contradiction.*

Case 3.1.2.4 $b_3 = b_5 = b_6 = 5$*. Similar to Case 3.1.2.1, we obtain a contradiction.*

Case 3.2 $b_2 = 2$*. So,* $b_1 = 2$*. We obtain the following after simplification.*

$$\begin{aligned} Q_6(G) &= 3x^{3b+3} + 9x^{2b+c+3} + 9x^{2b+5} + 3x^{2b+1} + 9x^{b+c+5} + 3x^{b+c+1} + \\ &\quad 3x^{b+7} + 6x^{b+3} + x^{c+7} + 2x^{c+3} + 2x^7 + x^5 - \Big(2x^{3b+4} + x^{3b+1} + \\ &\quad 6x^{2b+c+4} + 3x^{2b+c+1} + 3x^{2b+6} + 3x^{2b+4} + 6x^{2b+3} + 3x^{b+c+6} + \\ &\quad 3x^{b+c+4} + 6x^{b+c+3} + 6x^{b+6} + 3x^{b+1} + 2x^{c+6} + x^{c+1} + x^8 + x^6\Big), \\ Q_6(H) &= x^{b_3+b_5+b_6+4} + x^{b_3+b_5+b_6+2} + x^{b_3+b_5+7} + 2x^{b_3+b_5+5} + x^{b_3+b_5+1} + \\ &\quad x^{b_3+b_6+7} + 2x^{b_3+b_6+5} + x^{b_3+b_6+1} + x^{b_3+8} + 2x^{b_3+4} + x^{b_3+2} + \\ &\quad 3x^{b_5+b_6+6} + x^{b_5+b_6+4} + x^{b_5+b_6+1} + 2x^{b_5+7} + 3x^{b_5+3} + 2x^{b_6+7} + \\ &\quad 3x^{b_6+3} + 3x^6 + x^4 - \Big(x^{b_3+b_5+b_6+5} + x^{b_3+b_5+b_6+1} + x^{b_3+b_5+8} + \\ &\quad 2x^{b_3+b_5+4} + x^{b_3+b_5+2} + x^{b_3+b_6+8} + 2x^{b_3+b_6+4} + x^{b_3+b_6+2} + \\ &\quad x^{b_3+7} + 2x^{b_3+5} + x^{b_3+1} + 2x^{b_5+b_6+7} + 3x^{b_5+b_6+3} + 3x^{b_5+6} + \\ &\quad x^{b_5+4} + x^{b_5+1} + 3x^{b_6+6} + x^{b_6+4} + x^{b_6+1} + 2x^7 + x^3\Big). \end{aligned}$$

Note that the l.r.p. in $Q_6(H)$ *is 3. Since* $b_3 \geq 2$ *and* $b_5, b_6 \geq 4$*, by comparing the l.r.p. in* $Q_6(G)$ *and the l.r.p. in* $Q_6(H)$*, we have* $b = 2$ *or* $c = 2$*.*

If $b = 2$*, then* $G = \theta(2, 2, 2, 2, 2, c)$*. By Lemma 2.10,* G *is* χ*-unique.*

If $c = 2$*, then* $b = 2$ as well*. Therefore,* $G = \theta(2, 2, 2, 2, 2, 2)$*. By Lemma 2.2,* G *is* χ*-unique.*

Case 3.3 $b_3 = 2$*. Therefore,* $b_1 = b_2 = 2$*. Similar to Case 3.2, we obtain that* G *is* χ*-unique.*

Case 4 $b_4 \neq 3, b_5 \neq 3, b_6 \neq 3$*. By Lemma 2.9,* $b_4 \geq 4$*,* $b_5 \geq 4$*,* $b_6 \geq 4$ *and* H *shall has one cycle of length 4. Therefore,* $b_1 + b_2 = 4$ *implying* $b_1 = b_2 = 2$*. From Equation 3.1, we obtain* $3b + c = b_3 + b_4 + b_5 + b_6$*. Similar to Case 3.2, we obtain that* G *is* χ*-unique.*

Case B $H \in g_e\Big(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1}\Big)$ where $2 \leq b_1 \leq b_2 \leq b_3 \leq b_4$ and $2 \leq b_5, b_6$. As $G = \theta(a, a, b, b, b, c)$ and $H \in g_e\Big(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1}\Big)$, *by Lemmas 2.4 and 2.8, we obtain the following.*

$$\begin{aligned} Q_7(G) &= x\Big(x^a-1\Big)^2\Big(x^b-1\Big)^3\Big(x^c-1\Big)-\Big(x^a-x\Big)^2\Big(x^b-x\Big)^3\Big(x^c-x\Big), \\ Q_7(H) &= x\Big(x^{b_1}-1\Big)\Big(x^{b_2}-1\Big)\Big(x^{b_3}-1\Big)\Big(x^{b_4}-1\Big)\Big(x^{b_5}-1\Big)\Big(x^{b_6}-1\Big)- \\ &\quad \Big(x^{b_1}-x\Big)\Big(x^{b_2}-x\Big)\Big(x^{b_3}-x\Big)\Big(x^{b_4}-x\Big)\Big(x^{b_5}-1\Big)\Big(x^{b_6}-1\Big). \end{aligned}$$

Similar to Case A, by considering Equation 3.1, let $Q_7(G) = Q_7(H)$, *we obtain that the l.r.p. is 4 in* $Q_7(H)$. *Since* $2 \leq a \leq b \leq c$, *by comparing the l.r.p. in* $Q_7(G)$ *and the l.r.p. in* $Q_7(H)$, *we have* $a = 2$ *or* $a = 3$.

Case 1 $a = 2$. *Therefore, we know that* $g(G) = g(H) = 2a = 4$. *Then, without loss of generality, we have three cases to consider, that are* $b_5 = b_6 = 3$ *or* $b_5 = 3, b_6 \neq 3$ *or* $b_5 \neq 3, b_6 \neq 3$.

Case 1.1 $b_5 = b_6 = 3$. *Since* H *shall has one cycle of length 4, then by Lemma 2.9, this is a contradiction.*

Case 1.2 $b_5 = 3, b_6 \neq 3$. *Since the girth of* H *is 4, then* $b_6 \geq 4$. *From Equation 3.1,* $3b + c + 1 = b_1 + b_2 + b_3 + b_4 + b_6$. *We obtain the following after simplification.*

$$\begin{aligned} Q_8(G) &= 3x^{3b+3}+9x^{2b+c+3}+9x^{2b+5}+3x^{2b+1}+9x^{b+c+5}+3x^{b+c+1}+ \\ &\quad 3x^{b+7}+6x^{b+3}+x^{c+7}+2x^{c+3}+x^7+x^5-\Big(2x^{3b+4}+x^{3b+1}+ \\ &\quad 6x^{2b+c+4}+3x^{2b+c+1}+3x^{2b+6}+3x^{2b+4}+6x^{2b+3}+3x^{b+c+6}+ \\ &\quad 3x^{b+c+4}+6x^{b+c+3}+6x^{b+6}+3x^{b+1}+2x^{c+6}+x^{c+1}+x^8+ \\ &\quad x^6+2x^3\Big), \\ Q_8(H) &= x^{b_1+b_2+b_3+b_4+3}+x^{b_1+b_2+b_3+b_4+1}+x^{b_1+b_2+b_6+4}+x^{b_1+b_2+b_6+2}+ \\ &\quad x^{b_1+b_2+5}+x^{b_1+b_2+1}+x^{b_1+b_3+b_6+4}+x^{b_1+b_3+b_6+2}+x^{b_1+b_3+5}+ \\ &\quad x^{b_1+b_3+1}+x^{b_1+b_4+b_6+4}+x^{b_1+b_4+b_6+2}+x^{b_1+b_4+5}+x^{b_1+b_4+1}+ \\ &\quad x^{b_1+b_6+6}+x^{b_1+b_6+1}+x^{b_1+4}+x^{b_1+3}+x^{b_2+b_3+b_6+4}+ \\ &\quad x^{b_2+b_3+b_6+2}+x^{b_2+b_3+5}+x^{b_2+b_3+1}+x^{b_2+b_4+b_6+4}+x^{b_2+b_4+b_6+2}+ \\ &\quad x^{b_2+b_4+5}+x^{b_2+b_4+1}+x^{b_2+b_6+6}+x^{b_2+b_6+1}+x^{b_2+4}+x^{b_2+3}+ \\ &\quad x^{b_3+b_4+b_6+4}+x^{b_3+b_4+b_6+2}+x^{b_3+b_4+5}+x^{b_3+b_4+1}+x^{b_3+b_6+6}+ \end{aligned}$$

$$x^{b_3+b_6+1} + x^{b_3+4} + x^{b_3+3} + x^{b_4+b_6+6} + x^{b_4+b_6+1} + x^{b_4+4} + x^{b_4+3} + 2x^{b_6+4} - \Big(x^{b_1+b_2+b_3+b_4+4} + x^{b_1+b_2+b_3+b_4} + x^{b_1+b_2+b_6+5} + x^{b_1+b_2+b_6+1} + x^{b_1+b_2+4} + x^{b_1+b_2+2} + x^{b_1+b_3+b_6+5} + x^{b_1+b_3+b_6+1} + x^{b_1+b_3+4} + x^{b_1+b_3+2} + x^{b_1+b_4+b_6+5} + x^{b_1+b_4+b_6+1} + x^{b_1+b_4+4} + x^{b_1+b_4+2} + x^{b_1+b_6+4} + x^{b_1+b_6+3} + x^{b_1+6} + x^{b_1+1} + x^{b_2+b_3+b_6+5} + x^{b_2+b_3+b_6+1} + x^{b_2+b_3+4} + x^{b_2+b_3+2} + x^{b_2+b_4+b_6+5} + x^{b_2+b_4+b_6+1} + x^{b_2+b_4+4} + x^{b_2+b_4+2} + x^{b_2+b_6+4} + x^{b_2+b_6+3} + x^{b_2+6} + x^{b_2+1} + x^{b_3+b_4+b_6+5} + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+4} + x^{b_3+b_4+2} + x^{b_3+b_6+4} + x^{b_3+b_6+3} + x^{b_3+6} + x^{b_3+1} + x^{b_4+b_6+4} + x^{b_4+b_6+3} + x^{b_4+6} + x^{b_4+1} + x^{b_6+7} + x^{b_6+1} + 2x^4\Big).$$

Since $2 \le b \le c$, by comparing the l.r.p. in $Q_8(G)$ and the l.r.p. in $Q_8(H)$, we have $b_1 = b_2 = 2$ or $b_1 = b_3 = 2$ or $b_1 = b_4 = 2$ or $b_2 = b_3 = 2$ or $b_2 = b_4 = 2$ or $b_3 = b_4 = 2$. However, we obtain that H has more than one cycle of length 4 for all cases. Then, by Lemma 2.9, this is a contradiction.

Case 1.3 *$b_5 \ne 3, b_6 \ne 3$. Since the girth of H is 4 and H shall has one cycle of length equal to its girth, then we know that $b_5, b_6 \ge 4$ and $b_1 + b_2 = 4$. Therefore, $b_1 = b_2 = 2$. From Equation 3.1, we have*

$$3b + c = b_3 + b_4 + b_5 + b_6 \tag{3.3}$$

Since the l.r.p. in $Q_7(H)$ is 4, by considering $2 \le b_3 \le b_4$ and $b_5, b_6 \ge 4$, we have $b = 3$ or $c = 3$.

Case 1.3.1 *$b = 3$. Since the coefficient of $-x^{b+1}$ in $Q_7(G)$ is 3, then there are another two terms in $Q_7(H)$ that equal to $-x^4$. Hence, we have $b_3 = b_4 = 3$. From Equation 3.3, $c + 3 = b_5 + b_6$. Then, we obtain the following after simplification.*

$$Q_9(G) = x^{c+9} + 6x^{c+8} + 2x^{c+4} + 2x^{c+3} + 8x^{11} + 5x^7 - \Big(6x^{c+10} + 2x^{c+7} + 6x^{c+6} + x^{c+1} + 2x^{13} + 12x^9\Big),$$

$$Q_9(H) = x^{b_5+10} + x^{b_5+8} + 3x^{b_5+7} + 3x^{b_5+4} + 2x^{b_5+3} + x^{b_6+10} + x^{b_6+8} + 3x^{b_6+7} + 3x^{b_6+4} + 2x^{b_6+3} + 2x^5 - \Big(4x^{b_5+b_6+7} + x^{b_5+11} + 5x^{b_5+6} + 3x^{b_5+5} + x^{b_5+1} + x^{b_6+11} + 5x^{b_6+6} + 3x^{b_6+5} + x^{b_6+1} + 3x^7\Big).$$

Considering the l.r.p. in $Q_9(H)$, we have $b_5 = b_6 = 4$. Therefore, $c = 5$. However, we obtain $Q_9(G) \neq Q_9(H)$, a contradiction.

Case 1.3.2 *$c = 3$. Therefore, $2 \leq b \leq 3$. If $b = 2$, then $G = \theta(2,2,2,2,2,3)$. By Lemma 2.10, G is χ-unique. If $b = 3$, then $G = \theta(2,2,3,3,3,3)$. By Lemma 2.10, G is χ-unique.*

Case 2 *$a = 3$. We know that $g(G) = g(H) = 2a = 6$. Without loss of generality, we have three cases to consider, that are $b_5 = b_6 = 5$ or $b_5 = 5, b_6 \neq 5$ or $b_5 \neq 5, b_6 \neq 5$.*

Case 2.1 *$b_5 = b_6 = 5$. Since H shall has only one cycle of length 6, then by Lemma 2.9, this is a contradiction.*

Case 2.2 *$b_5 = 5, b_6 \neq 5$. Since the girth of H is 6, then $b_6 \geq 6$. From Equation 3.1, we have*

$$3b + c + 1 = b_1 + b_2 + b_3 + b_4 + b_6 \tag{3.4}$$

Note that the term with the l.r.p. is $-x^4$ in $Q_7(G)$. Since $3 \leq b \leq c$, by comparing the l.r.p. in $Q_7(G)$ and the l.r.p. in $Q_7(H)$, we have $b_1 = 3$ or $b_2 = 3$ or $b_3 = 3$ or $b_4 = 3$.

Case 2.2.1 *$b_1 = 3$. We obtain the following after simplification.*

$$\begin{aligned}
Q_{10}(G) \;=\;& 2x^{3b+4} + x^{3b+3} + 6x^{2b+c+4} + 3x^{2b+c+3} + 3x^{2b+7} + 6x^{2b+6} + \\
& 3x^{2b+1} + 3x^{b+c+7} + 6x^{b+c+6} + 3x^{b+c+1} + 3x^{b+9} + 3x^{b+5} + \\
& 6x^{b+4} + x^{c+9} + x^{c+5} + 2x^{c+4} + 2x^{8} + x^{7} - \Big(2x^{3b+5} + \\
& x^{3b+1} + 6x^{2b+c+5} + 3x^{2b+c+1} + 3x^{2b+8} + 9x^{2b+4} + 3x^{b+c+8} + \\
& 9x^{b+c+4} + 9x^{b+7} + 3x^{b+1} + 3x^{c+7} + x^{c+1} + x^{10}\Big), \\
Q_{10}(H) \;=\;& x^{b_2+b_3+b_4+8} + x^{b_2+b_3+b_4+4} + x^{b_2+b_3+b_6+6} + x^{b_2+b_3+b_6+2} + \\
& x^{b_2+b_3+7} + x^{b_2+b_3+1} + x^{b_2+b_4+b_6+6} + x^{b_2+b_4+b_6+2} + x^{b_2+b_4+7} + \\
& x^{b_2+b_4+1} + x^{b_2+b_6+9} + x^{b_2+b_6+8} + x^{b_2+b_6+5} + x^{b_2+b_6+1} + \\
& x^{b_2+10} + x^{b_2+6} + x^{b_2+4} + x^{b_2+3} + x^{b_3+b_4+b_6+6} + x^{b_3+b_4+b_6+2} + \\
& x^{b_3+b_4+7} + x^{b_3+b_4+1} + x^{b_3+b_6+9} + x^{b_3+b_6+8} + x^{b_3+b_6+5} + \\
& x^{b_3+b_6+1} + x^{b_3+10} + x^{b_3+6} + x^{b_3+4} + x^{b_3+3} + x^{b_4+b_6+9} + \\
& x^{b_4+b_6+8} + x^{b_4+b_6+5} + x^{b_4+b_6+1} + x^{b_4+10} + x^{b_4+6} + x^{b_4+4} + \\
& x^{b_4+3} + x^{b_6+11} + 2x^{b_6+4} + 2x^{9} + x^{6} - \Big(x^{b_2+b_3+b_4+9} + \\
& x^{b_2+b_3+b_4+3} + x^{b_2+b_3+b_6+7} + x^{b_2+b_3+b_6+1} + x^{b_2+b_3+6} +
\end{aligned}$$

$$x^{b_2+b_3+2} + x^{b_2+b_4+b_6+7} + x^{b_2+b_4+b_6+1} + x^{b_2+b_4+6} + x^{b_2+b_4+2} + x^{b_2+b_6+10} + x^{b_2+b_6+6} + x^{b_2+b_6+4} + x^{b_2+b_6+3} + x^{b_2+9} + x^{b_2+8} + x^{b_2+5} + x^{b_2+1} + x^{b_3+b_4+b_6+7} + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+6} + x^{b_3+b_4+2} + x^{b_3+b_6+10} + x^{b_3+b_6+6} + x^{b_3+b_6+4} + x^{b_3+b_6+3} + x^{b_3+9} + x^{b_3+8} + x^{b_3+5} + x^{b_3+1} + x^{b_4+b_6+10} + x^{b_4+b_6+6} + x^{b_4+b_6+4} + x^{b_4+b_6+3} + x^{b_4+9} + x^{b_4+8} + x^{b_4+5} + x^{b_4+1} + 2x^{b_6+9} + x^{b_6+1} + x^{11}\Big).$$

Considering the l.r.p. in $Q_{10}(H)$, we have $b_2 = 5$ or $b_3 = 5$ or $b_4 = 5$.

Case 2.2.1.1 $b_2 = 5$. *Simplifying the equal terms in $Q_{10}(G)$ and $Q_{10}(H)$, we have $b = 6$ or $c = 6$.*

Case 2.2.1.1(a) $b = 6$. *Since the coefficient of $-x^{b+1}$ in $Q_{10}(G)$ is 3, therefore, there are another two terms in $Q_{10}(H)$ that equal to $-x^7$. Hence, we have $b_3 = b_4 = 6$ or $b_3 = b_6 = 6$ or $b_4 = b_6 = 6$.*

Case 2.2.1.1(a-i) $b_3 = b_4 = 6$. *From Equation 3.4, $c = b_6 + 1$. Then, we obtain $c = 7$ and $b_6 = 6$. However, $Q_{10}(G) \neq Q_{10}(H)$, a contradiction.*

Case 2.2.1.1(a-ii) $b_3 = b_6 = 6$. *From Equation 3.4, $c = b_4 + 1$. Then, we obtain $c = 7$ and $b_4 = 6$. However, $Q_{10}(G) \neq Q_{10}(H)$, a contradiction.*

Case 2.2.1.1(a-iii) $b_4 = b_6 = 6$. *From Equation 3.4, $c = b_3 + 1$. Then, we obtain $c = 7$ and $b_3 = 6$. However, $Q_{10}(G) \neq Q_{10}(H)$, a contradiction.*

Case 2.2.1.1(b) $c = 6$. *Then, we have to consider $b_3 = 5$ or $b_4 = 5$ since $3 \leq b \leq 6$.*

Case 2.2.1.1(b-i) $b_3 = 5$. *From Equation 3.4, $3b = b_4 + b_6 + 6$. We obtain the following after simplification.*

$$\begin{aligned} Q_{11}(G) &= 2x^{3b+4} + 6x^{2b+10} + 3x^{2b+9} + 6x^{2b+6} + 3x^{2b+1} + 3x^{b+13} + 6x^{b+12} + 3x^{b+9} + 3x^{b+5} + 6x^{b+4} + 2x^{10} - \Big(2x^{3b+5} + x^{3b+1} + 6x^{2b+11} + 3x^{2b+8} + 9x^{2b+4} + 3x^{b+14} + 9x^{b+10} + 6x^{b+7} + 3x^{b+1} + x^{13}\Big), \\ Q_{11}(H) &= 2x^{b_4+b_6+11} + x^{b_4+b_6+8} + 2x^{b_4+b_6+7} + x^{b_4+b_6+5} + x^{b_4+b_6+1} + x^{b_4+18} + x^{b_4+14} + 2x^{b_4+12} + x^{b_4+10} + 3x^{b_4+6} + x^{b_4+4} + x^{b_4+3} + x^{b_6+16} + 2x^{b_6+14} + 2x^{b_6+13} + x^{b_6+12} + \end{aligned}$$

$$2x^{b_6+10} + 2x^{b_6+6} + 2x^{b_6+4} + x^{17} + x^{15} + x^{11} + 4x^9 - \Big(2x^{b_4+b_6+12} + x^{b_4+b_6+10} + 3x^{b_4+b_6+6} + x^{b_4+b_6+4} + x^{b_4+b_6+3} + x^{b_4+19} + x^{b_4+13} + 2x^{b_4+11} + x^{b_4+9} + x^{b_4+8} + 2x^{b_4+7} + x^{b_4+5} + x^{b_4+1} + x^{b_6+17} + 2x^{b_6+15} + 2x^{b_6+11} + 4x^{b_6+9} + 2x^{b_6+8} + x^{b_6+1} + x^{16} + 2x^{14} + x^{12} + x^{10} + x^6\Big).$$

Since $b_4 \geq 5$ and $b_6 \geq 6$, by considering the l.r.p. in $Q_{11}(G)$ and the l.r.p. in $Q_{11}(H)$, we have $b = 5$ and $b_4 = 5$. However, $Q_{11}(G) \neq Q_{11}(H)$, a contradiction.

Case 2.2.1.1(b-ii) *$b_4 = 5$. Then, we have $b_3 = 5$ as well. From Equation 3.4, $3b = b_6 + 11$. Similar to Case 2.2.1.1(b-i), we obtain $Q_{11}(G) \neq Q_{11}(H)$, a contradiction.*

Case 2.2.1.2 *$b_3 = 5$. Simplifying the equal terms in $Q_{10}(G)$ and $Q_{10}(H)$, we have $b = 6$ or $c = 6$.*

Case 2.2.1.2(a) *$b = 6$. Since the coefficient of $-x^{b+1}$ in $Q_{10}(G)$ is 3, then there are another two terms in $Q_{10}(H)$ that equal to $-x^7$. Considering $3 \leq b_2 \leq 5 \leq b_4$, we have $b_4 = b_6 = 6$. From Equation 3.4, $c = b_2 + 1$. We obtain the following after simplification.*

$$\begin{aligned} Q_{12}(G) &= 5x^{c+16} + 3x^{c+15} + 5x^{c+12} + x^{c+9} + 2x^{c+4} + 2x^{22} + x^{21} + \\ &\quad 6x^{18} + 3x^{15} + 3x^{11} + 3x^{10} + x^8 - \Big(6x^{c+17} + 3x^{c+14} + \\ &\quad 9x^{c+10} + x^{c+1} + x^{23} + 3x^{20} + 9x^{16} + 5x^{13}\Big), \\ Q_{12}(H) &= 2x^{b_2+15} + x^{b_2+14} + x^{b_2+7} + x^{b_2+6} + x^{b_2+4} + x^{b_2+3} + \\ &\quad x^{23} + 2x^{20} + x^{16} + 3x^{12} + 4x^9 - \Big(x^{b_2+20} + x^{b_2+16} + \\ &\quad 2x^{b_2+12} + 2x^{b_2+9} + 2x^{b_2+8} + x^{b_2+5} + x^{b_2+1} + x^{24} + x^{22} + \\ &\quad x^{18} + 4x^{15} + 3x^{14} + x^{11}\Big). \end{aligned}$$

Considering the l.r.p. in $Q_{12}(G)$, we have $c = 7$ and $b_2 = 6$. However, $Q_{12}(G) \neq Q_{12}(H)$, a contradiction.

Case 2.2.1.2(b) *$c = 6$. Considering the term x^8 in $Q_{10}(G)$ and $3 \leq b \leq 6$, we have $b_2 = 4$ or $b_2 = 5$ or $b_4 = 5$.*

Case 2.2.1.2(b-i) *$b_2 = 4$. From Equation 3.4, $3b = b_4 + b_6 + 5$. Then, we obtain $b = 4$. After simplification, we obtain $-2x^5$ in $Q_{10}(G)$ but not in $Q_{10}(H)$, a contradiction.*

Case 2.2.1.2(b-ii) $b_2 = 5$. *From Equation 3.4,* $3b = b_4 + b_6 + 6$. *Then, we obtain* $b = 5$. *After simplification, we obtain* $-2x^6$ *in* $Q_{10}(G)$ *but not in* $Q_{10}(H)$, *a contradiction.*

Case 2.2.1.2(b-iii) $b_4 = 5$. *From Equation 3.4,* $3b = b_2 + b_6 + 6$. *Then, we obtain* $b_2 = 3$. *After simplification, we obtain* $-2x^4$ *in* $Q_{10}(G)$ *but not in* $Q_{10}(H)$, *a contradiction.*

Case 2.2.1.3 $b_4 = 5$. *Simplifying the equal terms in* $Q_{10}(G)$ *and* $Q_{10}(H)$, *we obtain* $b = 6$ *or* $c = 6$.

Case 2.2.1.3(a) $b = 6$. *Since the coefficient of* $-x^{b+1}$ *in* $Q_{10}(G)$ *is 3, then there are another two terms in* $Q_{10}(H)$ *that equal to* $-x^7$. *However,* $3 \le b_2 \le b_3 \le 5$. *Therefore, there exist* $-2x^7$ *in* $Q_{10}(G)$ *but not in* $Q_{10}(H)$, *a contradiction.*

Case 2.2.1.3(b) $c = 6$. *Considering the term* x^8 *in* $Q_{10}(G)$ *and* $3 \le b \le 6$, *we have* $b_2 = 4$ *or* $b_2 = 5$ *or* $b_3 = 4$ *or* $b_3 = 5$.

Case 2.2.1.3(b-i) $b_2 = 4$. *From Equation 3.4,* $3b = b_3 + b_6 + 5$. *Then, we obtain the following after simplification.*

$$
\begin{aligned}
Q_{13}(G) \;=\;& x^{3b+4} + 6x^{2b+10} + 3x^{2b+9} + 6x^{2b+6} + 3x^{2b+1} + 3x^{b+13} + \\
& 6x^{b+12} + 3x^{b+9} + 3x^{b+5} + 6x^{b+4} + x^{11} - \Big(2x^{3b+5} + \\
& 6x^{2b+11} + 3x^{2b+8} + 9x^{2b+4} + 3x^{b+14} + 9x^{b+10} + 6x^{b+7} + \\
& 3x^{b+1} + x^{13}\Big), \\
Q_{13}(H) \;=\;& x^{b_3+b_6+7} + x^{b_3+b_6+1} + x^{b_3+17} + x^{b_3+13} + x^{b_3+6} + x^{b_3+4} + \\
& x^{b_3+3} + 2x^{b_6+13} + x^{b_6+12} + x^{b_6+11} + x^{b_6+6} + x^{b_6+5} + 2x^{b_6+4} + \\
& x^{16} + 2x^9 + x^7 - \Big(x^{b_3+b_6+12} + x^{b_3+b_6+4} + x^{b_3+b_6+3} + x^{b_3+18} + \\
& x^{b_3+9} + x^{b_3+8} + x^{b_3+7} + x^{b_3+1} + x^{b_6+16} + x^{b_6+10} + 2x^{b_6+9} + \\
& 2x^{b_6+8} + x^{b_6+7} + x^{b_6+1} + x^{15} + x^{12} + x^{11} + x^5\Big).
\end{aligned}
$$

Comparing the l.r.p. in $Q_{13}(G)$ *and the l.r.p. in* $Q_{13}(H)$, *we have* $b = 4$. *Since the coefficient of* $-x^{b+1}$ *is 3, then* $b_3 = b_6 = 4$. *However,* $3b = 12 \ne b_3 + b_6 + 5 = 13$, *a contradiction.*

Case 2.2.1.3(b-ii) $b_2 = 5$. *From Equation 3.4,* $3b = b_3 + b_6 + 6$. *Similar to Case 2.2.1.3(b-i), we obtain a contradiction.*

Case 2.2.1.3(b-iii) $b_3 = 4$. *From Equation 3.4,* $3b = b_2 + b_6 + 5$. *Similar to Case 2.2.1.3(b-i), we obtain a contradiction.*

Case 2.2.1.3(b-iv) $b_3 = 5$. *From Equation 3.4,* $3b = b_2 + b_6 + 6$. *Similar to Case 2.2.1.3(b-i), we obtain a contradiction.*

Case 2.2.2 $b_2 = 3$. *We obtain the following after simplification.*

$$\begin{aligned}
Q_{14}(G) &= 2x^{3b+4} + x^{3b+3} + 6x^{2b+c+4} + 3x^{2b+c+3} + 3x^{2b+7} + 6x^{2b+6} + \\
&\quad 3x^{2b+1} + 3x^{b+c+7} + 6x^{b+c+6} + 3x^{b+c+1} + 3x^{b+9} + 3x^{b+5} + \\
&\quad 6x^{b+4} + x^{c+9} + x^{c+5} + 2x^{c+4} + 2x^8 + x^7 - \Big(2x^{3b+5} + x^{3b+1} + \\
&\quad 6x^{2b+c+5} + 3x^{2b+c+1} + 3x^{2b+8} + 9x^{2b+4} + 3x^{b+c+8} + 9x^{b+c+4} + \\
&\quad 9x^{b+7} + 3x^{b+1} + 3x^{c+7} + x^{c+1} + x^{10}\Big), \\
Q_{14}(H) &= x^{b_1+b_3+b_4+8} + x^{b_1+b_3+b_4+4} + x^{b_1+b_3+b_6+6} + x^{b_1+b_3+b_6+2} + \\
&\quad x^{b_1+b_3+7} + x^{b_1+b_3+1} + x^{b_1+b_4+b_6+6} + x^{b_1+b_4+b_6+2} + x^{b_1+b_4+7} + \\
&\quad x^{b_1+b_4+1} + x^{b_1+b_6+9} + x^{b_1+b_6+8} + x^{b_1+b_6+5} + x^{b_1+b_6+1} + \\
&\quad x^{b_1+10} + x^{b_1+6} + x^{b_1+4} + x^{b_1+3} + x^{b_3+b_4+b_6+6} + x^{b_3+b_4+b_6+2} + \\
&\quad x^{b_3+b_4+7} + x^{b_3+b_4+1} + x^{b_3+b_6+9} + x^{b_3+b_6+8} + x^{b_3+b_6+5} + \\
&\quad x^{b_3+b_6+1} + x^{b_3+10} + x^{b_3+6} + x^{b_3+4} + x^{b_3+3} + x^{b_4+b_6+9} + \\
&\quad x^{b_4+b_6+8} + x^{b_4+b_6+5} + x^{b_4+b_6+1} + x^{b_4+10} + x^{b_4+6} + x^{b_4+4} + \\
&\quad x^{b_4+3} + x^{b_6+11} + 2x^{b_6+4} + 2x^9 + x^6 - \Big(x^{b_1+b_3+b_4+9} + \\
&\quad x^{b_1+b_3+b_4+3} + x^{b_1+b_3+b_6+7} + x^{b_1+b_3+b_6+1} + x^{b_1+b_3+6} + \\
&\quad x^{b_1+b_3+2} + x^{b_1+b_4+b_6+7} + x^{b_1+b_4+b_6+1} + x^{b_1+b_4+6} + x^{b_1+b_4+2} + \\
&\quad x^{b_1+b_6+10} + x^{b_1+b_6+6} + x^{b_1+b_6+4} + x^{b_1+b_6+3} + x^{b_1+9} + x^{b_1+8} + \\
&\quad x^{b_1+5} + x^{b_1+1} + x^{b_3+b_4+b_6+7} + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+6} + \\
&\quad x^{b_3+b_4+2} + x^{b_3+b_6+10} + x^{b_3+b_6+6} + x^{b_3+b_6+4} + x^{b_3+b_6+3} + \\
&\quad x^{b_3+9} + x^{b_3+8} + x^{b_3+5} + x^{b_3+1} + x^{b_4+b_6+10} + x^{b_4+b_6+6} + \\
&\quad x^{b_4+b_6+4} + x^{b_4+b_6+3} + x^{b_4+9} + x^{b_4+8} + x^{b_4+5} + x^{b_4+1} + \\
&\quad 2x^{b_6+9} + x^{b_6+1} + x^{11}\Big).
\end{aligned}$$

Considering the term x^6 *in* $Q_{14}(H)$ *and* $2 \le b_1 \le 3 \le b_3 \le b_4$, *we have* $b_3 = 5$ *or* $b_4 = 5$.

Case 2.2.2.1 $b_3 = 5$. *Comparing the l.r.p. in* $Q_{14}(G)$ *and the l.r.p. in* $Q_{14}(H)$, *we have* $b = 6$ *or* $c = 6$.

Case 2.2.2.1(a) $b = 6$. *Since the coefficient of* $-x^{b+1}$ *is 3, then* $b_4 = b_6 = 6$. *From Equation 3.4,* $c = b_1 + 1$. *After simplification, we obtain* $c = 7$ *and* $b_1 = 6$. *However,* $2 \le b_1 \le 3$, *a contradiction.*

Case 2.2.2.1(b) $c = 6$. *Simplifying* $Q_{14}(G)$ *and* $Q_{14}(H)$, *we have* $b_1 = 2$ *or* $b_4 = 5$.

Case 2.2.2.1(b-i) $b_1 = 2$. *From Equation 3.4,* $3b = b_4 + b_6 + 3$. *Then, we obtain* $b = 2$. *However,* $3 \le b \le 6$, *a contradiction.*

Case 2.2.2.1(b-ii) $b_4 = 5$. *From Equation 3.4,* $3b = b_1 + b_6 + 6$. *Since* $2 \le b_1 \le 3$, *we have* $b_1 = 2$ *or* $b_1 = 3$.

If $b_1 = 2$, *similar to Case 2.2.2.1(b-i), we obtain a contradiction.*

If $b_1 = 3$, *similar to Case 2.2.2.1(b-i), we obtain a contradiction.*

Case 2.2.2.2 $b_4 = 5$. *Comparing the l.r.p. in* $Q_{14}(G)$ *and the l.r.p. in* $Q_{14}(H)$, *we have* $b = 6$ *or* $c = 6$.

Case 2.2.2.2(a) $b = 6$. *Considering the coefficient of* $-x^{b+1}$ *is 3, we have* $b_1 = 2$ *and* $b_6 = 6$. *From Equation 3.4,* $c + 3 = b_3$. *Then, we obtain the following after simplification.*

$$
\begin{aligned}
Q_{15}(G) \;=\;& 5x^{c+16} + 3x^{c+15} + 6x^{c+12} + x^{c+5} + 2x^{c+4} + 2x^{22} + x^{21} + \\
& 6x^{18} + 3x^{15} + 3x^{11} + 5x^{10} - \Big(6x^{c+17} + 3x^{c+14} + 9x^{c+10} + \\
& x^{c+1} + 2x^{23} + 3x^{20} + 9x^{16} + 5x^{13}\Big), \\
Q_{15}(H) \;=\;& x^{b_3+17} + x^{b_3+15} + 2x^{b_3+14} + x^{b_3+11} + x^{b_3+6} + 2x^{b_3+3} + x^{17} + \\
& 2x^{16} + x^{12} + 3x^{9} + x^{8} + x^{6} + x^{5} - \Big(x^{b_3+18} + 2x^{b_3+16} + x^{b_3+12} + \\
& 2x^{b_3+9} + 2x^{b_3+8} + x^{b_3+5} + x^{b_3+1} + x^{21} + x^{18} + x^{15} + 3x^{14} + \\
& 2x^{11} + x^{3}\Big).
\end{aligned}
$$

Since $c \ge 6$, $3 \le b_3 \le 5$ *and* $b_6 \ge 6$, *we know that* $-x^3$ *is in* $Q_{15}(H)$ *but not in* $Q_{15}(G)$, *a contradiction.*

Case 2.2.2.2(b) $c = 6$. *Simplifying* $Q_{14}(G)$ *and* $Q_{14}(H)$, *we have* $b_1 = 2$ *or* $b_3 = 4$ *or* $b_3 = 5$.

Case 2.2.2.2(b-i) $b_1 = 2$. *From Equation 3.4,* $3b = b_3 + b_6 + 3$. *Similar to Case 2.2.2.2(a), we obtain a contradiction.*

Case 2.2.2.2(b-ii) $b_3 = 4$. *From Equation 3.4,* $3b = b_1 + b_6 + 5$. *Similar to Case 2.2.2.2(a), we obtain a contradiction.*

Case 2.2.2.2(b-iii) $b_3 = 5$. *From Equation 3.4,* $3b = b_1 + b_6 + 6$. *Simplifying* $Q_{14}(G)$ *and* $Q_{14}(H)$, *we obtain* $b_1 = 2$ *or* $b_1 = 3$.

If $b_1 = 2$, similar to Case 2.2.2.2(a), we obtain a contradiction.

If $b_1 = 3$, similar to Case 2.2.2.2(a), we obtain a contradiction.

Case 2.2.3 $b_3 = 3$. *Then, we have $b_4 = 5$. Cancelling the equal terms, we have $b = 6$ or $c = 6$.*

Case 2.2.3.1 $b = 6$. *Since $-x^{b+1}$ has coefficient 3, by considering $2 \leq b_1 \leq b_2 \leq 3$, we have $b_1 = b_2 = 2$ or $b_1 = 2, b_6 = 6$ or $b_2 = 2, b_6 = 6$.*

Case 2.2.3.1(a) $b_1 = b_2 = 2$. *Then, we obtain the following after simplification.*

$$
\begin{aligned}
Q_{16}(G) &= 6x^{c+16} + 3x^{c+15} + 6x^{c+12} + x^{c+9} + x^{c+5} + 2x^{c+4} + 2x^{22} + \\
&\quad x^{21} + 2x^{19} + 6x^{18} + 2x^{15} + 3x^{11} + 6x^{10} - \Big(6x^{c+17} + 3x^{c+14} + \\
&\quad 9x^{c+10} + x^{c+1} + 2x^{23} + 3x^{20} + 9x^{16} + 4x^{13}\Big), \\
Q_{16}(H) &= 3x^{b_6+13} + x^{b_6+11} + 4x^{b_6+10} + 2x^{b_6+7} + 2x^{b_6+4} + 2x^{b_6+3} + \\
&\quad x^{17} + x^{14} + x^{12} + x^{9} + 3x^{8} + x^{6} + 3x^{5} - \Big(x^{b_6+15} + x^{b_6+14} + \\
&\quad 2x^{b_6+12} + x^{b_6+9} + 5x^{b_6+8} + 3x^{b_6+5} + x^{b_6+1} + x^{18} + x^{11} + \\
&\quad 3x^{10} + 2x^{3}\Big).
\end{aligned}
$$

Since $c \geq 6$ and $b_6 \geq 6$, we know that $-2x^3$ is in $Q_{16}(H)$ but not in $Q_{16}(G)$. Then, $Q_{16}(G) \neq Q_{16}(H)$, a contradiction.

Case 2.2.3.1(b) $b_1 = 2, b_6 = 6$. *Similar to Case 2.2.3.1(a), we obtain a contradiction.*

Case 2.2.3.1(c) $b_2 = 2, b_6 = 6$. *Then, $b_1 = 2$ as well. Similar to Case 2.2.3.1(a), we obtain a contradiction.*

Case 2.2.3.2 $c = 6$. *Cancelling the equal terms, we have $b_1 = 2$ or $b_2 = 2$.*

Case 2.2.3.2(a) $b_1 = 2$. *From Equation 3.4, $3b = b_2 + b_6 + 3$. Similar to Case 2.2.3.1(a), we obtain a contradiction.*

Case 2.2.3.2(b) $b_2 = 2$. *Then $b_1 = 2$ as well. From Equation 3.4, $3b = b_6 + 5$. Similar to Case 2.2.3.1(a), we obtain a contradiction.*

Case 2.2.4 $b_4 = 3$. *We obtain the following after simplification.*

$$\begin{aligned}
Q_{17}(G) &= 2x^{3b+4} + x^{3b+3} + 6x^{2b+c+4} + 3x^{2b+c+3} + 3x^{2b+7} + 6x^{2b+6} + \\
&\quad 3x^{2b+1} + 3x^{b+c+7} + 6x^{b+c+6} + 3x^{b+c+1} + 3x^{b+9} + 3x^{b+5} + \\
&\quad 6x^{b+4} + x^{c+9} + x^{c+5} + 2x^{c+4} + 2x^{8} + x^{7} - \Big(2x^{3b+5} + x^{3b+1} + \\
&\quad 6x^{2b+c+5} + 3x^{2b+c+1} + 3x^{2b+8} + 9x^{2b+4} + 3x^{b+c+8} + 9x^{b+c+4} + \\
&\quad 9x^{b+7} + 3x^{b+1} + 3x^{c+7} + x^{c+1} + x^{10}\Big), \\
Q_{17}(H) &= x^{b_1+b_2+b_3+8} + x^{b_1+b_2+b_3+4} + x^{b_1+b_2+b_6+6} + x^{b_1+b_2+b_6+2} + \\
&\quad x^{b_1+b_2+7} + x^{b_1+b_2+1} + x^{b_1+b_3+b_6+6} + x^{b_1+b_3+b_6+2} + x^{b_1+b_3+7} + \\
&\quad x^{b_1+b_3+1} + x^{b_1+b_6+9} + x^{b_1+b_6+8} + x^{b_1+b_6+5} + x^{b_1+b_6+1} + \\
&\quad x^{b_1+10} + x^{b_1+6} + x^{b_1+4} + x^{b_1+3} + x^{b_2+b_3+b_6+6} + x^{b_2+b_3+b_6+2} + \\
&\quad x^{b_2+b_3+7} + x^{b_2+b_3+1} + x^{b_2+b_6+9} + x^{b_2+b_6+8} + x^{b_2+b_6+5} + \\
&\quad x^{b_2+b_6+1} + x^{b_2+10} + x^{b_2+6} + x^{b_2+4} + x^{b_2+3} + x^{b_3+b_6+9} + \\
&\quad x^{b_3+b_6+8} + x^{b_3+b_6+5} + x^{b_3+b_6+1} + x^{b_3+10} + x^{b_3+6} + x^{b_3+4} + \\
&\quad x^{b_3+3} + x^{b_6+11} + 2x^{b_6+4} + 2x^{9} + x^{6} - \Big(x^{b_1+b_2+b_3+9} + \\
&\quad x^{b_1+b_2+b_3+3} + x^{b_1+b_2+b_6+7} + x^{b_1+b_2+b_6+1} + x^{b_1+b_2+6} + \\
&\quad x^{b_1+b_2+2} + x^{b_1+b_3+b_6+7} + x^{b_1+b_3+b_6+1} + x^{b_1+b_3+6} + x^{b_1+b_3+2} + \\
&\quad x^{b_1+b_6+10} + x^{b_1+b_6+6} + x^{b_1+b_6+4} + x^{b_1+b_6+3} + x^{b_1+9} + x^{b_1+8} + \\
&\quad x^{b_1+5} + x^{b_1+1} + x^{b_2+b_3+b_6+7} + x^{b_2+b_3+b_6+1} + x^{b_2+b_3+6} + \\
&\quad x^{b_2+b_3+2} + x^{b_2+b_6+10} + x^{b_2+b_6+6} + x^{b_2+b_6+4} + x^{b_2+b_6+3} + \\
&\quad x^{b_2+9} + x^{b_2+8} + x^{b_2+5} + x^{b_2+1} + x^{b_3+b_6+10} + x^{b_3+b_6+6} + \\
&\quad x^{b_3+b_6+4} + x^{b_3+b_6+3} + x^{b_3+9} + x^{b_3+8} + x^{b_3+5} + x^{b_3+1} + \\
&\quad 2x^{b_6+9} + x^{b_6+1} + x^{11}\Big).
\end{aligned}$$

Considering the term with the l.r.p. in $Q_{17}(H)$, that is x^6, since $2 \le b_1 \le b_2 \le b_3 \le 3$, $b_6 \ge 6$ and $3 \le b \le c$, we know that x^6 is in $Q_{17}(H)$ but not in $Q_{17}(H)$, a contradiction.

<u>Case 2.3</u> *$b_5 \ne 5, b_6 \ne 5$. Since $g(H) = 6$, we have $b_5, b_6 \ge 6$ and $b_1 + b_2 = 6$. Then, $b_1 = 2, b_2 = 4$ or $b_1 = b_2 = 3$.*

<u>Case 2.3.1</u> *$b_1 = 2, b_2 = 4$. From Equation 3.1, $3b + c = b_3 + b_4 + b_5 + b_6$. Then, since $4 \le b_3 \le 4$ and $3 \le b \le c$, after simplification, we obtain $-x^3$ in $Q_7(H)$ but not in $Q_7(H)$, a contradiction.*

<u>Case 2.3.2</u> *$b_1 = b_2 = 3$. From Equation 3.1, $3b + c = b_3 + b_4 + b_5 + b_6$. Then, we obtain the term with the l.r.p. is $-x^4$ in $Q_7(H)$. Since $3 \le b_3 \le$*

b_4, *by comparing the l.r.p. in* $Q_7(G)$ *and the l.r.p. in* $Q_7(H)$, *we have* $b = 3$ *or* $c = 3$.

If $b = 3$, *then* $G = \theta(3,3,3,3,3,c)$. *By Lemma 2.10, we know that* G *is* χ*-unique.*

If $c = 3$, *then* $G = \theta(3,3,3,3,3,3)$. *By Lemma 2.2, we know that* G *is* χ*-unique.*

Case C $H \in g_e\Big(\theta(b_1,b_2,b_3,b_4,b_5), C_{b_6+1}\Big)$ *where* $2 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5$ *and* $b_6 \geq 2$.

The proof for Case C is similar to that of Cases A and B. The detail proof can be obtained by e-mail from the first author [8].

Thus, this completes the proof. ∎

4. Conclusion

Reviewing the literature results on the chromaticity of k*-bridge graphs, for example* $k = 6$, *we understand that the only technique to sovle this problem is by comparing the chromatic polynomial of graphs under investigation. The technique leads, of course, to quite a number of cases and subcases and very routine. Therefore, it is challenging problem for next researchers to apply new and effective technique to solve the chromaticity of 6-bridge graphs.*

It is also natural to ask the following question: for which choices $(a_1, a_2, \cdots, a_6)$ *where* $a_1 \leq a_2 \leq \cdots \leq a_6$, *the graph* $\theta(a_1, a_2, \cdots, a_6)$ *is* χ*-unique? In general, this problem still remains open. Also the chromaticity of other types of the graph* $\theta(a_1, a_2, \cdots, a_6)$ *where* $a_1, a_2, \cdots, a_6$ *assume exactly three distinct values is still worth for further investigation.*

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References

[1] X. W. Bao and X. E. Chen, *Chromaticity of the graph* $\theta(a,b,c,d,e)$ *(in Chinese), J. Xinjiang Univ. Natur. Sci.* **11**, *pp. 19-22, (1994).*

[2] C. Y. Chao and E. G. Whitehead Jr., On chromatic equivalence of graphs, *Theory and Applications of Graphs, Lecture Notes in Math.* **642**, Springer, Berlin, pp. 121-131, (1978).

[3] X. E. Chen, X. W. Bao and K. Z. Ouyang, Chromaticity of the graph $\theta(a, b, c, d)$, *J. Shaanxi Normal Univ.* **20**, pp. 75-79, (1992).

[4] F. M. Dong, K. M. Koh and K. L. Teo, Chromatic polynomials and chromaticity of graphs, *World Scientific Publishing Ptd. Ltd.*, Singapore, pp. 105-123, (2005).

[5] F. M. Dong, K. L. Teo, C. H. C. Little, M. Hendy and K. M. Koh, Chromatically unique multibridge graphs, *The Electric Journal of Combinatorics* **11** (2004) #R12.

[6] F. Harary, Graph theory, Addison-Wesley Publishing Company Inc., United States of America, (1972).

[7] N. S. A. Karim, R. Hasni and G.C. Lau, Chromatic uniqueness of 6-bridge graph $\theta(a, a, a, b, b, c)$, *Electronic Journal of Graph Theory and Applications* **4(1)**, pp. 60–78, (2016).

[8] N. S. A. Karim, Chromaticity of K_4-homeomorphs with girth 9 and 6-bridge graph, *Doctoral Dissertation*, Universiti Malaysia Terengganu, Terengganu, Malaysia, (2015).

[9] A. M. Khalaf, Chromaticity of a certain k-bridge graphs, *Doctoral Dissertation*, Universiti Putra Malaysia, Serdang, Malaysia, (2010).

[10] A. M. Khalaf and Y. H. Peng, Another proof of a family of chromatically unique 5-bridge graphs, *International Journal of Applied Mathematics* **22(6)**, pp. 1009-1020, (2009).

[11] A. M. Khalaf and Y. H. Peng, A note on chromaticity of k-bridge graphs, *Far East J. Math. Sci.* **34(3)**, pp. 353-359, (2009).

[12] A. M. Khalaf and Y. H. Peng, The chromatic uniqueness of a family of 6-bridge graphs, *AKCE J. Graphs Combin.* **6(3)**, pp. 393-400, (2009).

[13] A. M. Khalaf and Y. H. Peng, Chromaticity of the 6-bridge graph $\theta(a, a, a, b, c, c)$, *International Journal of Applied Mathematics* **22(7)**, pp. 1087-1111, (2009).

[14] A. M. Khalaf and Y. H. Peng, A family of chromatically unique 6-bridge graphs, *Ars Combin.* **94**, pp. 211-220, (2010).

[15] A. M. Khalaf, Y. H. Peng and K. A. Atan, Chromaticity of 5-bridge graphs (II), *International Journal of Applied Mathematics* **23**, pp. 791-808, (2010).

[16] K. M. Koh and K. L. Teo, The search for chromatically unique graphs, *Graphs Combin.* **6**, pp. 259-285, (1990).

[17] K. M. Koh and K. L. Teo, The search for chromatically unique graphs-II, *Discrete Math.* **172**, pp. 59-78, (1997).

[18] X. F. Li, A family of chromatically unique 5-bridge graphs, *Ars Combin.* **88**, pp. 415-428, (2008).

[19] X. F. Li and X. S. Wei, The chromatic uniqueness of a family of 5-bridge graphs (in Chinese), *J. Qinghai Normal Univ.* **2**, pp. 12-17, (2001).

[20] B. Loerinc, Note: Chromatic uniqueness of the generalized θ-graph, *Discrete Math.* **23**, pp. 313-316, (1978).

[21] D. B. West, Introduction to graph theory 2nd ed., *Prentice Hall, Inc.*, United States of America, (2001).

[22] S. Xu, J. Liu and Y. H. Peng, The chromaticity of s-bridge graphs and related graphs, *Discrete Math.* **135**, pp 349-358, (1994).

[23] C. F. Ye, The chromatic uniqueness of 6-bridge graphs (in Chinese), *J. Math. Study* **34(4)**, pp. 399-421, (2001).

[24] C. F. Ye, The chromatic uniqueness of s-bridge graphs (in Chinese), *J. Xinjiang Univ. Natur. Sci.* **19(3)**, pp. 246-265, (2002).

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