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On estimates for the generalized Fourier-Bessel transform

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Abstract

Two estimates useful in applications are proved for the generalized Fourier-Bessel transform in the space $L^2_{\alpha,n}$ as applied to some classes of functions characterized by a generalized modulus of continuity.

Keywords : *Generalized Fourier-Bessel transform; generalized translation operator; modulus of continuity.*

Mathematics Subject Classification :

1. Introduction and preliminaries

In [2], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator .

In this paper, we prove two useful estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier-Bessel transform in the space $L^2_{\alpha,n}$ analogs of the statements proved in [2, 4, 5]. For this purpose, we use a generalized translation operator.

Consider the second-order singular differential operator on the half line

$$Bf(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2}f(x),$$

where $\alpha > -\frac{1}{2}$ and $n = 0, 1, 2, ...$ For $n = 0$, we obtain the classical Bessel operator

$$B_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

For $\alpha > -\frac{1}{2}$ and $n = 0, 1, 2, ...$, let M be the map defined by

$$Mf(x) = x^{2n} f(x).$$

Let $L^2_{\alpha,n}$ be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{2,\alpha,n} = \|M^{-1}f\|_{2,\alpha+2n} < \infty,$$

where

$$\|f\|_{2,\alpha+2n} = \left(\int_0^{+\infty} |f(x)|^2 x^{2\alpha+4n+1} dx\right)^{1/2}.$$

For $\alpha > -\frac{1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(x)}{x^\alpha}, \tag{1.1}$$

where $J_\alpha(x)$ is a Bessel function of the first kind and $\Gamma(x)$ is the gamma-function. The function $y = j_\alpha(x)$ satisfies the differential equation

$$B_\alpha y + y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. The function $j_\alpha(x)$ is infinitely differentiable, even, and, moreover entire analytic.

Lemma 1.1. *The following inequalities are fulfilled:*

1. $1 - j_\alpha(x) = O(1), \ x \geq 1,$
2. $1 - j_\alpha(x) = O(x^2), \ 0 \leq x \leq 1,$
3. $\sqrt{hx} J_\alpha(hx) = O(1), \ hx \geq 0.$

Proof. (see [1])

For $\lambda \in \mathbf{C}$ and $x \in \mathbf{R}$, put

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x).$$

From [2] recall the following properties.

Proposition 1.2. *1. φ_λ satisfies the differential equation*

$$B\varphi_\lambda = -\lambda^2 \varphi_\lambda.$$

2. For all $\lambda \in \mathbf{C}$, and $x \in \mathbf{R}$

$$|\varphi_\lambda(x)| \leq x^{2n} e^{|Im\lambda||x|}$$

The generalized Fourier-Bessel transform we call the integral from [2]

$$\mathcal{F}_B(f)(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda(x) x^{2\alpha+1} dx, \lambda \geq 0, \ f \in L^1_{\alpha,n}$$

Let $f \in L^1_{\alpha,n}$, the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x) = \int_0^{+\infty} \mathcal{F}_B(f)(\lambda)\varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = \frac{1}{4^{\alpha+2n}\left(\Gamma(\alpha+2n+1)\right)^2} \lambda^{2\alpha+4n+1} d\lambda$$

From [2], we have

Theorem 1.3. *1. For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula*

$$\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1} dx = \int_0^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

2. *The generalized Fourier-Bessel transform $\mathcal{F}_B$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0,+\infty[, \mu_{\alpha+2n})$.*

Define the generalized translation operator T_h, $h > 0$ by the relation

$$T_h f(x) = (xh)^{2n} \tau^h_{\alpha+2n}(M^{-1}f)(x),\ x \geq 0,$$

where $\tau^h_{\alpha+2n}$ are the Bessel translation operators of order $\alpha+2n$ defined by

$$\tau^h_\alpha f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2+h^2-2xhcost}) sin^{2\alpha} t dt,$$

where

$$c_\alpha = \left(\int_0^\pi sin^{2\alpha} t dt\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}$$

Proposition 1.4. *[2]*

1. *Let f be in $L^2_{\alpha,n}$. Then for all $h \geq 0$, the function $T_h f$ belongs to $L^2_{\alpha,n}$, and*

$$\|T_h f\|_{2,\alpha,n} \leq h^{2n} \|f\|_{2,\alpha,n}.$$

2. *For $f \in L^2_{\alpha,n}$, we have*

$$\mathcal{F}_B(T_h f)(\lambda) = \varphi_\lambda(h) \mathcal{F}_B(f)(\lambda),\ f \in L^2_{\alpha,n}$$

From [3], we have

$$\mathcal{F}_B(Bf)(\lambda) = -\lambda^2 \mathcal{F}_B(f)(\lambda),\ f \in L^2_{\alpha,n}$$

Then

$$\mathcal{F}_B(B^r f)(\lambda) = (-1)^r \lambda^{2r} \mathcal{F}_B(f)(\lambda) \tag{1.2}$$

where $r = 1, 2, ...$

The first and higher order finite differences of $f(x)$ are defined as follows

$$\Delta_h f(x) = T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x) = (T_h + T_{-h} - 2h^{2n} E) f(x),$$

where E is the identity operator in $L^2_{\alpha,n}$, and

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (T_h + T_h - 2h^{2n} E)^k f(x),$$

where $T_h^0 f(x) = f(x)$, $T_h^k f(x) = T_h(T_h^{k-1} f(x))$ for $k = 1, 2, ...$

The kth order generalized modulus of continuity of function $f \in L^2_{\alpha,n}$ is defined as

$$\Omega_k(f, \delta) = \sup_{0<h\leq\delta} \|\Delta_h^k f(x)\|_{2,\alpha,n}.$$

Let $W^{r,k}_{2,\psi}(B)$ denote the class of functions $f \in L^2_{\alpha,n}$ such that

$$\Omega_k(B^r f, \delta) = O(\psi(\delta^k)),$$

where $\psi(t)$ is any nonnegative function given on $[0, \infty)$ and $\psi(0) = 0$, for the generalized Bessel operator B, we have $B^0 f = f$, $B^r f = B(B^{r-1} f), r = 1, 2, ...$

Lemma 1.5. *For any function $f \in L^2_{\alpha,n}$ such that $B^r f \in L^2_{\alpha,n}$. Then*

$$\|\Delta_h^k B^r f(x)\|^2_{2,\alpha,n} = \int_0^{+\infty} 2^{2k} h^{4kn} \lambda^{4r} |1 - j_{\alpha+2n}(\lambda h)|^{2k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

Proof. From formulas (1.2) and (2) of Proposition 1.4, we have

$$\mathcal{F}_B(B^r f)(\lambda) = (-1)^r \lambda^{2r} \mathcal{F}_B(f)(\lambda)$$

and

$$\mathcal{F}_B(\Delta_h^k f)(\lambda) = 2^k h^{2kn} (j_{\alpha+2n}(\lambda h) - 1)^k \mathcal{F}_B(f)(\lambda)$$

Then

$$\mathcal{F}_B(\Delta_h^k B^r f)(\lambda) = (-1)^r 2^k h^{2kn} \lambda^{2r} (j_{\alpha+2n}(\lambda h) - 1)^k \mathcal{F}_B(f)(\lambda)$$

Plancherel's identity gives the result.

2. Main Result

In this section, we prove two estimates for the integral

$$\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Theorem 2.1. *For functions $f \in L^2_{\alpha,n}$ in the class $W^{r,k}_{2,\psi}(B)$*

$$\sup_{W^{r,k}_{2,\psi}(B)} \sqrt{\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)} = O\left(N^{2kn-2r}\psi\left(\frac{c}{N}\right)^k\right),$$

where $r = 0, 1, ...;$ $k = 1, 2....;$ $c > 0$ is a fixed constant, and $\psi(t)$ is any nonnegative function defined on the interval $[0, \infty)$.

Proof. Let $f \in W^{r,k}_{2,\psi}(B)$. Taking into account the Hölder inequality yields

$$\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) - \int_N^{+\infty} j_{\alpha+2n}(\lambda h)|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= \int_N^{+\infty} (1 - j_{\alpha+2n}(\lambda h))|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= \int_N^{+\infty} (1 - j_{\alpha+2n}(\lambda h))|\mathcal{F}_B(f)(\lambda)|^{2-\frac{1}{k}} |\mathcal{F}_B(f)(\lambda)|^{\frac{1}{k}} d\mu_{\alpha+2n}(\lambda)$$

$$\leq \left(\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{2k-1}{2k}}$$

$$\left(\int_N^{+\infty} |1 - j_{\alpha+2n}(\lambda h))|^{2k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{2k}}$$

$$\leq \left(\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{2k-1}{2k}}$$

$$\times \left(2^{-2k} h^{-4kn} \int_N^{+\infty} 2^{2k} h^{4kn} \lambda^{-4r} \lambda^{4r} |1 - j_{\alpha+2n}(\lambda h))|^{2k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{2k}}$$

$$\leq \left(\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{2k-1}{2k}} 2^{-1} h^{-2n} N^{-2r/k} \|\Delta_h^k B^r f(x)\|^{1/k}_{2,\alpha,n}$$

Therefore
$\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \leq \int_N^{+\infty} j_{\alpha+2n}(\lambda h)|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$
$+2^{-1}h^{-2n}N^{-2r/k}\left(\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{2k-1}{2k}}\|\Delta_h^k B^r f(x)\|_{2,\alpha,n}^{1/k}$

From formulas (1.1) and (3) of Lemma 1.1, we have

$$j_{\alpha+2n}(\lambda h) = O\left((\lambda h)^{-\alpha-2n-\frac{1}{2}}\right)$$

Then
$\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$

$= O(\int_N^{+\infty}(\lambda h)^{-\alpha-2n-\frac{1}{2}}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$

$+h^{-2n}N^{-2r/k}\left(\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{2k-1}{2k}}\|\Delta_h^k B^r f(x)\|_{2,\alpha,n}^{1/k})$

$= O((Nh)^{-\alpha-2n-\frac{1}{2}}\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$

$+h^{-2n}N^{-2r/k}\left(\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{2k-1}{2k}}\|\Delta_h^k B^r f(x)\|_{2,\alpha,n}^{1/k})$

Or

$$\left(1-O((Nh)^{-\alpha-2n-\frac{1}{2}}\right)\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O(h^{-2n}N^{-2r/k}\left(\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{2k-1}{2k}}\|\Delta_h^k B^r f(x)\|_{2,\alpha,n}^{1/k})$$

Setting $h = \frac{c}{N}$ in the last inequality and choosing $c > 0$ such that $1 - O(c^{-\alpha-2n-\frac{1}{2}}) \geq \frac{1}{2}$, we obtain

$$\left(\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{2k}} = O(N^{2n-2r/k})\|\Delta_h^k B^r f(x)\|_{2,\alpha,n}^{1/k}$$

Then

$$\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(N^{4kn-4r}\psi^2\left(\frac{c}{N}\right)^k\right).$$

which proves theorem 2.1.

Theorem 2.2. *Let $\psi(t) = t^{\nu}$. Then the following are equivalents*

1. $\left(\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{1/2} = O\left(N^{2kn-2r-k\nu}\right)$,

2. $f \in W^{r,k}_{2,\psi}(B)$,

where $r = 0, 1, 2.....;\ k = 1, 2, ...;\ 0 < \nu < 2$.

1) $\Longrightarrow$ 2) Assume that

$$\left(\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{1/2} = O\left(N^{2kn-2r-k\nu}\right).$$

Then

$$\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(N^{4kn-4r-2k\nu}\right)$$

From Lemma 1.5, we have

$$\|\Delta_h^k B^r f(x)\|^2_{2,\alpha,n} = \int_0^{+\infty} 2^{2k} h^{4kn} \lambda^{4r} |1 - j_{\alpha+2n}(\lambda h)|^{2k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

This integral is divided into two

$\int_0^{+\infty} \lambda^{4r} |1 - j_{\alpha+2n}(\lambda h)|^{2k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$

$= \int_0^{N} \lambda^{4r} |1 - j_{\alpha+2n}(\lambda h)|^{2k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$

$+ \int_N^{+\infty} \lambda^{4r} |1 - j_{\alpha+2n}(\lambda h)|^{2k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$

$= I_1 + I_2$,
where $N = [h^{-1}]$ let us estimate them separately.

From formula (1) of Lemma 1.1, we have

$$I_2 = \int_N^{+\infty} \lambda^{4r} |1 - j_{\alpha+2n}(\lambda h)|^{2k} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O\left(\int_N^{+\infty} \lambda^{4r} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)$$

$$= O\left(\sum_{n=N}^{\infty} \int_n^{n+1} \lambda^{4r} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)$$

$$= \left(\sum_{n=N}^{\infty} n^{4r} \int_n^{n+1} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)$$

$$= O\left(\sum_{n=N}^{\infty} n^{4r} \left[\int_n^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right.\right.$$

$$- \int_{n+1}^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O\left(\sum_{n=N}^{\infty} n^{4r} \int_n^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right.$$

$$- \sum_{n=N}^{\infty} n^{4r} \int_{n+1}^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O(N^{4r} \int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$+ \sum_{n=N+1}^{\infty} n^{4r} \int_n^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$- \sum_{n=N}^{\infty} n^{4r} \int_{n+1}^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda))$$

$$= O(N^{4r} \int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$+ \sum_{n=N}^{\infty} (n+1)^{4r} \int_{n+1}^{+\infty} |\mathcal{F}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$- \sum_{n=N}^{\infty} n^{4r} \int_{n+1}^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda))$$

$$= O\Big(N^{4r}\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$+\sum_{n=N}^{\infty}((n+1)^{4r}-n^{4r})\int_{n+1}^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O\Big(N^{4r}\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$+\sum_{n=N}^{\infty}((n+1)^{4r}-n^{4r})\int_{n}^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O\Big(N^{4r}\int_N^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$+\sum_{n=N}^{\infty}n^{4r-1}\int_{n}^{+\infty}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O(N^{4r}N^{4kn-4r-2k\nu}) + O(\sum_{n=N}^{\infty}n^{4r-1}n^{4kn-4r-2k\nu})$$

$$= O(N^{4kn-2k\nu}) + O(N^{4kn-2k\nu})$$

$$= O(h^{-4kn+2k\nu}).$$

Then

$$2^{2k}h^{4kn}I_2 = O(h^{2k\nu}). \tag{2.1}$$

Now we estimate I_1. By virtue of formula (2) of Lemma 1.1.

$$\mathbf{I}_1 = \int_0^N \lambda^{4r}|1-j_{\alpha+2n}(\lambda h)|^{2k}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O(h^{4k}) \int_0^N \lambda^{4r+4k}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O(h^{4k}) \sum_{n=0}^{N} \int_n^{n+1} \lambda^{4r+4k}|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O(h^{4k}) \sum_{n=0}^{N} (n+1)^{4r+4k} \int_n^{n+1} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O(h^{4k}) \sum_{n=0}^{N} (n+1)^{4r+4k} \Big[\int_n^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$- \int_n^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$= O(h^{4k}) \left[1 + \sum_{n=0}^{N} ((n+1)^{4r+4k} - n^{4r+4k}) \int_n^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right]$$

$$= O(h^{4k}) \left[1 + \sum_{n=0}^{N} n^{4r+4k-1} n^{4kn-4r-2k\nu}\right]$$

$$= O(h^{4k}) \left[1 + \sum_{n=0}^{N} n^{4k+4kn-2k\nu-1}\right]$$

$$= O(h^{4k}) O(N^{4k+4kn-2k\nu})$$

$$= O(h^{-4kn+2k\nu})$$

Then

$$2^{2k} h^{4kn} I_1 = O(h^{2k\nu}) \tag{2.2}$$

Combining the estimates for formulas (3) and (4) gives

$$\|\Delta_h^k B^r f(x)\|_{2,\alpha,n} = O(h^{k\nu})$$

which means that $f \in W^{r,k}_{2,\psi}(B)$.

2) $\Longrightarrow$ 1) Suppose that $f \in W^{r,k}_{2,\psi}(B)$ and $\psi(t) = t^\nu$. By theorem 2.1, we have

$$\left(\int_N^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{1/2} = O\left(N^{2kn-2r-k\nu}\right)$$

Thus, the proof is finiched.

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