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Super vertex mean labeling of cycles through different ways

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Abstract

A super vertex mean labeling f of a (p, q) - graph $G = (V, E)$ is defined as an injection from E to the set $\{1, 2, 3, \cdots, p + q\}$ that induces for each vertex v the label defined by the rule $f^v(v) =$ Round $(\frac{\sum_{e\in E_v} f(e)}{d(v)})$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, p + q\}$. In this paper, we investigate the super vertex mean labeling behavior of cycles by giving various ways by which they can be labeled.

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1. Introduction

A graph is an ordered pair $(V(G), E(G))$, consisting of a finite non empty set $V(G)$ of objects called points or vertices and a set $E(G)$ of 2-element subsets of $V(G)$, known as edges. The sets $V(G)$ and $E(G)$ are the vertex set and edge set respectively. The cardinality of $V(G)$is the order of the graph G and is often denoted by $|V(G)|= p$, and that of $E(G)$ is the size of G and is denoted by $|E(G)| = q$. A graph of order p and size q is often called a (p, q) - graph.

A labeling of a graph G is an assignment of labels either to the vertices or edges. Harmonious labeling is one of the fundamental labeling introduced by Graham and Solane [4] in 1980 in connection with their study on error correction code. Mean labeling was introduced by Somasundaram and Ponraj [13]. Let $G = (V, E)$ be a simple graph with p vertices and q edges. A mean labeling f is an injection from V to the set $\{0, 1, 2, \cdots, q\}$ that induces for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the set of edge labels is $\{1, 2, \cdots, q\}$. A graph that accepts a mean labeling is known as mean graph.

A super mean labeling f is an injection from V to the set $\{1, 2, \cdots, p+q\}$ [10] that induces for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the set of all vertex labels and the induced edge labels is $\{1, 2, \cdots, p + q\}$. Some results on mean labeling and super mean labeling can be found in [6], [8], [9], [10] and [14].

Lourdusamy and Seenivasan [5] introduced vertex mean labeling as an edge analogue of mean labeling. A graph that has a vertex mean labeling is called a vertex mean graph or V − mean graph.

Lourdusamy et al. [7] brought in a new type of mean labeling, called Super vertex mean labeling of graphs.

2. Super Vertex Mean Labeling

Definition 2.1. *[7] A super vertex mean labeling f of a (p, q) - graph $G = (V, E)$ is defined as an injection from E to the set $\{1, 2, 3, \cdots, p + q\}$ that induces for each vertex v the label defined by the rule $f^v(v) = Round \left(\frac{\sum_{e \in E_v} f(e)}{d(v)} \right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, p + q\}$.*

A graph that accepts super vertex mean labeling is called a Super Vertex Mean, that is SVM - graph in short.

Observations 2.1: A graph having isolated vertices or leaves cannot be an SVM - graph. For, if $deg(v) = 0$ for any vertex v of G, the above definition is not defined and if $deg(v) = 1$ for any vertex v of G, the induced vertex label remains the same as the label of the edge that is incident on the vertex v. Therefore, for a graph to be SVM - graph it is necessary that $deg(v) \geq 2$ for all vertices v in $V(G)$. It is obvious that no tree is a SVM - graph.

3. Super Vertex Mean Labeling of Cycles

Theorem 3.1. *All the cycles except C_4 are SVM - graphs.*

Proof. Illustration: For C_4, we have $p = 4$ and $q = 4$.

$f(E) \cup f(V) = \{1, 2, 3, 4, 5, 6, 7, 8\} = \{1, 2, 3, \cdots, p + q\}$.

It is obvious that 1 and 8 cannot be induced vertex labels, so necessarily belong to $f(E)$. Since 2 cannot be an edge label, it belongs to $f(V)$ and for 2 to be a vertex label, it has to labeled on a vertex on which the edges that are labeled 1 and 3 lie. And so, 3 also belongs to $f(E)$.

Therefore, 8 can be labeled on an edge that is adjacent to an edge labeled 3 or 1. The following cases arise:

Case 1: Let 8 be labeled on an edge adjacent to the edge labeled 3.

Now, 7 cannot be labeled on any edges. The remaining options are that, we label either 4 or 5 on the fourth edge.

Case 1(a): Let 4 be labeled on the fourth edge. This is not an SVM - labeling as the vertices that are incident on the edge labeled 8 get the same induced label 6.

Case 1(b): Let 5 be labeled on the fourth edge. This also is ruled out as one of the vertices incident on the edge labeled 5 gets the label 3, which is contrary to the assumption that 3 has to be an edge label.

Therefore, **Case 1** is not possible.

Case 2: Let 8 be labeled on an edge which is adjacent to the edge labeled 1.

In this case 7 cannot be an edge label and if 7 is an induced vertex label, then one of the induced vertex labels gets repeated. Therefore, **Case 2** also is impossible.

So, we conclude that the cycle C_4 is not a SVM - graph.

Now we prove that C_n, $n \geq 5$ is a SVM - graph. There can be two cases depending upon whether n is odd or even.

Case 3: $n \equiv 1(mod\ 2)$. Let C_n be an odd cycle with n vertices. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n, such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$.

Let $n = 2r + 1$. The edges of C_n are labeled as follows:

$$f(e_i) = \begin{cases} 2i-1 & \text{if } 1 \leq i \leq r+1 \\ 2i & \text{if } r+2 \leq i \leq n \end{cases}$$

It is easy to observe that f is injective. The induced vertex labels are given as follows:

$$f^v(v_i) = \begin{cases} n+1 & \text{if } i = 1 \\ 2i-2 & \text{if } 2 \leq i \leq r+1 \\ 2i-1 & \text{if } r+2 \leq i \leq n \end{cases}$$

It is clear that,

$$f(E) \cup f^v(V)$$
$$= \{1, 3, 5, \cdots, 2r+1, 2r+4, 2r+6, \cdots, 2n-2, 2n\} \cup$$

$$\{2r+2 = n+1, 2, 4, \cdots, 2r-2, 2r, 2r+3, 2r+5, \cdots, 2n-3, 2n-1\}$$

$$= \{1, 3, \cdots, 2r+1 = n, 2r+3, 2r+5, ..., 2n-1\} \cup$$

$$\{2, 4, \cdots, 2r = n-1, n+1 = 2r+2, 2r+4, 2r+6, \cdots, 2n-1, 2n\}$$

$$= \{2i-1 : 1 \leq i \leq n\} \cup \{2i : 1 \leq i \leq n\}$$

$$= \{1, 2, 3, \cdots, 2n\}$$

Case 4: $n \equiv 0(mod\ 2)$

Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$.

Let $n = 2r$.

The edges of C_n are labeled as follows:

$$f(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 3 & \text{if } i = 2 \\ 7 & \text{if } i = 3 \\ 4i - 4 & \text{if } 4 \leq i \leq r+1 \\ 4n - 4i + 5 & \text{if } r+2 \leq i \leq n-1 \\ 6 & \text{if } i = n \end{cases}.$$

It is easy to observe that f is injective. The induced vertex labels are given as follows:

$$f^v(v_i) = \begin{cases} 4 & \text{if } i = 1 \\ 2 & \text{if } i = 2 \\ 5 & \text{if } i = 3 \\ 4i - 6 & \text{if } 4 \leq i \leq r+1 \\ 4n - 4i + 7 & \text{if } r+2 \leq i \leq n-1 \\ 8 & \text{if } i = n \end{cases}$$

It is clear that,

$$\begin{aligned} & f(E) \cup f^v(V) \\ & = \{1, 3, 7, 12, 16, \cdots, 4r, 4r-3, 4r-7, \cdots, 13, 9, 6\} \cup \\ & \quad \{4, 2, 5, 10, 14, \cdots, 4r-6, 4r-2, 4r-1, 4r-5, \cdots, 15, 11, 8\} \\ & = \{1, 3, 6, 7, 12, 16, 20, \cdots, 4r, 9, 13, \cdots, 4r-7, 4r-3\} \cup \\ & \quad \{2, 4, 5, 8, 10, 14, 18, \cdots, 4r-6, 4r-2, 11, 15, 19, \cdots, 4r-5, 4r-1\} \\ & = \{1, 2, 3, 4, 5, 6, 7, 8\} \cup \{9, 13, 4r-3\} \cup \{10, 14, \cdots, 4r-2\} \\ & \quad \cup \{11, 15, \cdots, 4r-1\} \cup \{12, 16, \cdots, 4r\} \\ & = \{1, 2, 3, \cdots, 4r-3, 4r-2, 4r-1, 4r = 2n\} \\ & = \{1, 2, 3, \cdots, 2n\} \end{aligned}$$

Hence we have proved that all cycles C_n, except C_4, are Super Vertex Mean graphs. $\square$

Super vertex-mean labeling of C_9 and C_{10} is shown in Figure 3.1.

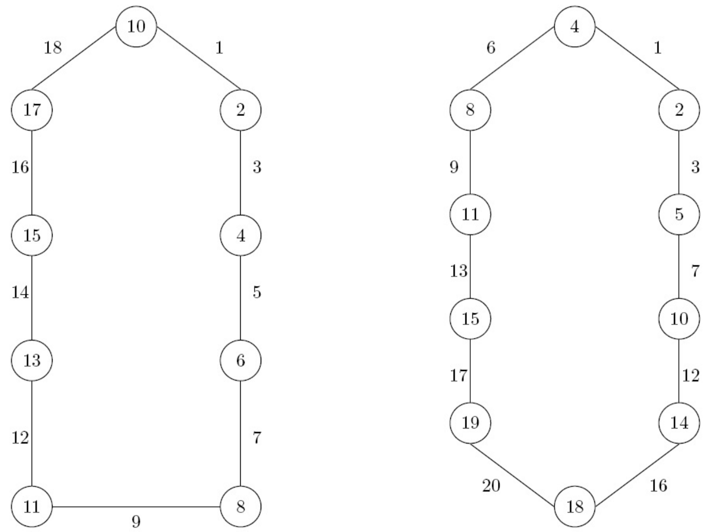

Figure 3.1

4. types of svm labeling of cycles

Any cycle C_n, $n \geq 3$ and $n \neq 4$ can be SVM labeled in a number of different ways. Therefore, the need arises to categorize various types of these labelings.

In Figure 4.1 we show that C_7 can be labeled in 3 different ways.

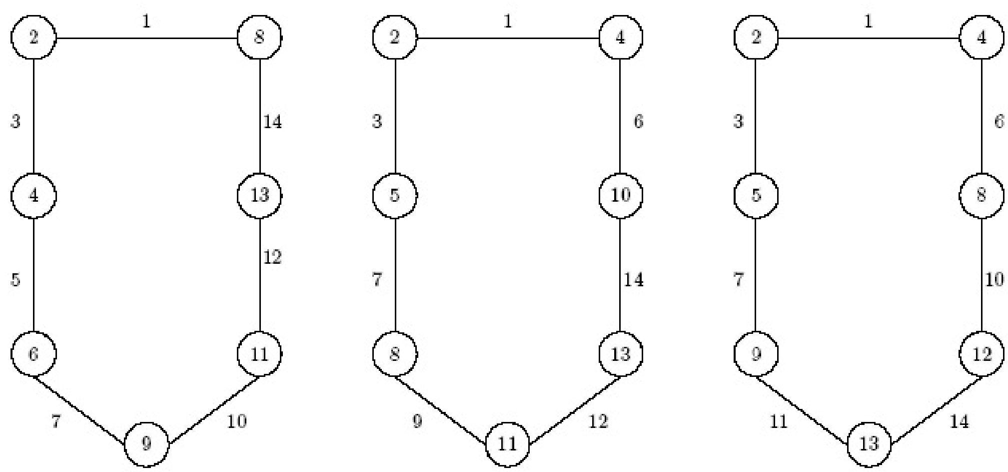

*Figure*4.1

Now we give $s-$ type labeling of Cycles, $C_m, m \geq 3$ and $m \neq 4$

Definition 4.1. *We denote a super vertex mean labeling $f : E \rightarrow \{1, 2, \cdots, 2m\}$ of a cycle C_m, $m \geq 3$ and $m \neq 4$, that places 1 and $2m$ on two edges such that the number of internal vertices along the shortest path connecting these two edges is s, as s-type labeling, where $1 \leq s \leq \lfloor \frac{m}{2} \rfloor$.*

5. $s-$ type labeling of all cycles

In order to define completely the various types of SVM labeling of $C_n, n \geq 3$ and $n \neq 4$, we have two cases, based on whether n is odd or even.

Case 1: $n \equiv 1(mod\ 2)$

Let $n = 2r + 1$, $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_i v_{i+1}$, $1 \leq i \leq n - 1$ and $e_n = v_n v_1$. 1 – type labeling of cycle $C_n, n \geq 3$, is given as follows;

$$f_1(e_i) = \begin{cases} 2i - 1 & \text{if } 1 \leq i \leq r + 1 \\ 2i & \text{if } r + 2 \leq i \leq n \end{cases}$$

or, when we reverse the order of naming the edges and vertices, we get equivalently

$$f_1(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4r - 2i + 6 & \text{if } 2 \leq i \leq r + 1 \\ 4r - 2i + 5 & \text{if } r + 2 \leq i \leq n. \end{cases}$$

2 – type labeling, then is defined as follows for $C_n, n \geq 5$;

$$f_2(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } i = 2 \\ 4r + 4 - 2i + 4 & \text{if } 3 \leq i \leq r + 1 \\ 4r + 4 - 2i + 3 & \text{if } r + 2 \leq i \leq 2r \\ 8r - 4i + 7 & \text{if } i = n \end{cases}$$

Similarly 3 – type labeling of $C_n, n \geq 7$, can be defined as,

$$f_3(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq 3 \\ 4r + 6 - 2i + 4 & \text{if } 4 \leq i \leq r + 1 \\ 4r + 6 - 2i + 3 & \text{if } r + 2 \leq i \leq 2r - 1 \\ 8r - 4i + 7 & \text{if } 2r \leq i \leq n. \end{cases}$$

And when $r = s$, r – type labeling of $C_n, n \geq 3$, is defined as,

$$f_r(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq r + 1 \\ 8r - 4i + 7 & \text{if } r + 2 \leq i \leq n. \end{cases}$$

or, equivalently

$$f_r(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq r = s \\ 4r + 2r - 2i + 4 & \text{if } i = r + 1 = s + 1 \\ 4r + 2r - 2i + 3 & \text{if } i = r + 2 = s + 2 \\ 8r - 4i + 7 & \text{if } r + 3 \leq i \leq n. \end{cases}$$

Therefore, when we consider odd cycles and all types of their SVM labeling in general, we have the following theorem.

Theorem 5.1. *Let $n = 2r+1$, $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n, such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$. Then the s - type $(1 \leq s \leq r)$ SVM labeling of cycle $C_n, n \equiv 1(mod\ 2), n \geq 3$, is given as follows:*

$$f_s(e_i) = \begin{cases} 1 & \textit{if } i = 1 \\ 4i-2 & \textit{if } 2 \le i \le s \\ 4r+2s-2i+4 & \textit{if } s+1 \le i \le r+1 \\ 4r+2s-2i+3 & \textit{if } r+2 \le i \le 2r-s+2 \\ 8r-4i+7 & \textit{if } 2r-s+3 \le i \le n. \end{cases}$$

Proof. Let $n = 2r+1$ and $n \equiv 1(mod\,2)$. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_iv_{i+1}, 1 \le i \le n-1$ and $e_n = v_nv_1$. The edges of C_n can be s - type labeled, $1 \le s \le r$, as given in the Theorem 5.1. Clearly f_s is an injective function with range from $\{1, 2, \cdots, 2n\}$.
The induced vertex labeling is given as follows:

When $s = 1$,

$$f_1^v(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ n+1 & \text{if } i = 2 \\ 4r-2i+7 & \text{if } 3 \le i \le r+2 \\ 4r-2i+6 & \text{if } r+3 \le i \le n. \end{cases}$$

It is evident that,
$f_1(E) \cup f_1^v(V)$
$= \{1, 2n, 2n-2, 2n-5, \cdots, n+5, n+3, n, n-2, n-4, \cdots, 5, 3\} \cup$

$\{2, n+1, 2n-1, 2n-3, \cdots, n+4, n+2, n-1, n-3, \cdots, 6, 4\}$

$= \{1, 3, \cdots, n, n+3, n+5, \cdots, 2n\} \cup$

$\{2, 4, \cdots, n-1, n+1, n+2, n+4, \cdots, 2n-1\}$

$= \{1, 2, 3, \cdots, 2n\}$.
When $s = r$,

$$f_r^v(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ 4i-4 & \text{if } 2 \le i \le r+1 \\ 4r+1 = 2n-1 & \text{if } i = r+2 \\ 8r-4i+9 & \text{if } r+3 \le i \le n. \end{cases}$$

It is clear that,

$f_r(E) \cup f_r^v(V)$
$= \{1, 6, 10, \cdots, 2n, 2n-3, 2n-7, \cdots, 3\} \cup$

$\{2, 4, 8, \cdots, 2n-2, 2n-1, 2n-5, \cdots, 9, 5\}$

$= \{1, 3, 7, \cdots, 2n-3, 6, 10, \cdots, 2n\} \cup$

$\{2, 4, 8, \cdots, 2n-2, 2n-1, 5, 9, \cdots, 2n-5\}$

$= \{1, 2, 3, 4, 5, \cdots, 2n-3, 2n-2, 2n-1, 2n\}.$

Therefore, in the more general case, the induced vertex labels are given as follows:

$$f_s^v(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ 4i-4 & \text{if } 2 \leq i \leq s \\ 2r+2s & \text{if } i = s+1 \\ 4r+2s-2i+5 & \text{if } s+2 \leq i \leq r+2 \\ 4r+2s-2i+4 & \text{if } r+3 \leq i \leq 2r-s+2 \\ 8r-4i+9 & \text{if } 2r-s+3 \leq i \leq n. \end{cases}$$

Clearly it is injective and

$$f_s(E) \cup f_s^v(V) = \{1, 2, 3, 4, 5, \cdots, 2n-3, 2n-2, 2n-1, 2n\}.$$

Therefore,

$$f_s(E) \cup f_s^v(V) = \{1, 2, 3, 4, 5, \cdots, 2n-3, 2n-2, 2n-1, 2n\}.$$

Hence we have proved that all odd cycles C_n, can be s - type labeled, where $1 \leq s \leq r$ and $n = 2r+1$. Hence the proof. □

Case 2: $n \equiv 0 (mod\ 2)$
Let C_n be an even cycle and $n = 2r$ where, $n \geq 6$.

Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$.

By checking various possibilities we find that 1 – type labeling is not possible for even cycles. So we assume that $2 \leq s \leq r$.

2 – type labeling of $C_n, n \geq 6$, is given as follows:

$$f_2(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 4r - 2i + 6 & \text{if } 3 \leq i \leq r \\ 4r - 2i + 5 & \text{if } r + 1 \leq i \leq 2r - 2 \\ 6 & \text{if } i = 2r - 1 \\ 3 & \text{if } i = 2r. \end{cases}$$

Similarly, 3 – type labeling of C_n, $n \geq 8$, is as follows;

$$f_3(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 12 & \text{if } i = 3 \\ 4r - 2i + 8 & \text{if } 4 \leq i \leq r \\ 4r - 2i + 7 & \text{if } r + 1 \leq i \leq 2r - 3 \\ 9 & \text{if } i = 2r - 2 \\ 6 & \text{if } i = 2r - 1 \\ 3 & \text{if } i = 2r \end{cases}$$

And 4 – type labeling of C_n, $n \geq 10$, is given below;

$$f_4(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 4i & \text{if } 3 \leq i \leq 4 \\ 4r - 2i + 10 & \text{if } 5 \leq i \leq r \\ 4r - 2i + 9 & \text{if } r + 1 \leq i \leq 2r - 4 \\ 8r - 4i + 1 & \text{if } 2r - 3 \leq i \leq 2r - 2 \\ 6 & \text{if } i = 2r - 1 \\ 3 & \text{if } i = 2r \end{cases}$$

and, when $s = r$, the r – type labeling is given by,

$$f_r(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq r - 1 \\ r + 3i - 3 & \text{if } r \leq i \leq r + 1 \\ 8r - 4i + 3 & \text{if } r + 2 \leq i \leq n. \end{cases}$$

As in the previous case, for all even cycles, s – type labeling is defined in the following theorem:

Theorem 5.2. *Let $n = 2r$. Let C_n be an even cycle with n vertices and $\{e_1, e_2, \cdots, e_n\}$ be the edge set of C_n such that $e_i = v_iv_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_nv_1$. The s - type $(2 \leq s \leq r-1)$ SVM labeling of $C_n, n \geq 6$, is given as follows:*

$$f_s(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 4i & \text{if } 3 \leq i \leq s \\ 4r - 2i + 2s + 2 & \text{if } s+1 \leq i \leq r \\ 4r - 2i + 2s + 1 & \text{if } r+1 \leq i \leq 2r - s \\ 8r - 4i + 1 & \text{if } 2r - s + 1 \leq i \leq 2r - 2 \\ 6r - 3i + 3 & \text{if } 2r - 1 \leq i \leq n \end{cases}$$

and, when $s = r$, the r – type labeling is given by,

$$f_r(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq r-1 \\ r + 3i - 3 & \text{if } r \leq i \leq r+1 \\ 8r - 4i + 3 & \text{if } r+2 \leq i \leq n. \end{cases}$$

Proof. Let $n \equiv 0(mod\ 2)$, and $n = 2r$ and $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_iv_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_nv_1$.

Case 1: When $2 \leq s \leq r-1$, the edges of $C_n, n \geq 6$, can be s – type labeled as given below:

$$f_s(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 4i & \text{if } 3 \leq i \leq s \\ 4r - 2i + 2s + 2 & \text{if } s+1 \leq i \leq r \\ 4r - 2i + 2s + 1 & \text{if } r+1 \leq i \leq 2r - s \\ 8r - 4i + 1 & \text{if } 2r - s + 1 \leq i \leq 2r - 2 \\ 6r - 3i + 3 & \text{if } 2r - 1 \leq i \leq n \end{cases}$$

Clearly f_s is an injective function with range from $\{1, 2, ..., 2n\}$. The induced vertex labeling is given as follows:

When $s = 2$

$$f_s^v(v_i) = \begin{cases} 2i & \text{if } 1 \le i \le 2 \\ 2r+4 & \text{if } i = 3 \\ 4r-2i+2s+3 & \text{if } 4 \le i \le r+1 \\ 4r-2i+2s+2 & \text{if } r+2 \le i \le 2r-2 \\ 6r-3i+5 & \text{if } 2r-1 \le i \le n \end{cases}$$

and when $s \ge 3$, we have

$$f_s^v(v_i) = \begin{cases} 2i & \text{if } 1 \le i \le 2 \\ 2i+4 & \text{if } i = 3 \\ 4i-2 & \text{if } 4 \le i \le s \\ 2r+2s & \text{if } i = s+1 \\ 4r-2i+2s+3 & \text{if } s+2 \le i \le r+1 \\ 4r-2i+2s+2 & \text{if } r+2 \le i \le 2r-s \\ 4s-1 & \text{if } i = 2r-s+1 \\ 8r-4i+3 & \text{if } 2r-s+2 \le i \le 2r-2 \\ 6r-3i+5 & \text{if } 2r-1 \le i \le n \end{cases}$$

Clearly it is an injective function and, it is also evident that, when $s = 2$,

$$f_2(E) \cup f_2^v(V)$$
$$= \{1, 7, 4r, 4r-2, \cdots, 2r+6, 2r+3, 2r+1, \cdots, 9, 6, 3\} \cup$$
$$\{2, 4, 2r+4, 4r-5, 4r-3, \cdots, 2r+5, 2r+2, \cdots, 10, 8, 5\}$$
$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots, 4r-3, 4r-2, 4r-1, 4r\}.$$

And for $3 \le s \le r-1$,

$f_s(E) \cup f_s^v(V)$
$= \{1, 7, 12, \cdots, 4s, 4r, 4r-2, \cdots, 2r+2s+2, 2r+2s-1,$

$2r+2s-3, \cdots, 4s+3, 4s+1, 4s-3, 4s-7, \cdots, 9, 6, 3\} \cup$

$\{2, 4, 10, 14, 18, \cdots, 4s-2, 2r+2s, 4r-1, 4r-3, \cdots, 2r+2s+1,$

$2r+2s-2, 2r+2s-4, \cdots, 4s+2, 4s-1, 4s-5, 4s-9, \cdots, 11, 8, 5\}$

$= \{1, 3, 6, 7, 9, 13, \cdots, 4s-7, 4s-3, 12, 16, \cdots, 4s, 4r, 4r-2, \cdots,$

$2r+2s+2, 2r+2s-1, 2r+2s-3, \cdots, 4s+3, 4s+1\} \cup$

$\{2, 4, 5, 8, 10, 14, \cdots, 4s-2, 11, 15, \cdots, 4s-9, 4s-5, 4s-1, 4s+2,$

$4s, \cdots, 2r+2s-4, 2r+2s, 4r-1, 4r-3, \cdots, 2r+2s+1\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \cdots, 4r-2, 4r-1, 4r\}.$

Case 2: When $s = r$, the edges of C_n, $n \geq 6$, can be $r-$type labeled as given below:

$$f_r(e_i) = \begin{cases} 1 \\ \text{if} \quad \text{i=1} \\ \text{4i-2} \\ \text{if } 2 \leq i \leq r-1 \\ \text{r+3i-3} \\ \text{if r} \leq i \leq r+1 \\ \text{8r-4i+3} \\ \text{if r+2} \leq i \leq n. \end{cases}$$

Clearly, f_r is an injective function with range from $\{1, 2, 3, \cdots, 2n\}$. The induced vertex labeling is given as follows:

$$f_r^v(v_i) = \begin{cases} 2i & \text{if } 1 \leq i \leq 2 \\ 4i-4 & \text{if } 3 \leq i \leq r \\ 4r-1 & \text{if } i = r+1 \\ 4r-2 & \text{if } i = r+2 \\ 8r-4i+5 & \text{if } r+3 \leq i \leq n. \end{cases}$$

It is clear now that,

$$f_r(E) \cup f_r^v(V) = \{1, 2, 3, \cdots, 2n-2, 2n-1, 2n\}.$$

Hence we have proved that all even cycles C_n, can be s - type, ($1 \le s \le r$, $n \ge 6$ and $n = 2r$), SVM – labeled. Thus the proof. □

6. Conclusion

While analyzing the Super Vertex Mean labeling for cycles, C_n, we observe that the ideal situation would have been that the sum of all the edge labels to be equal to the sum of all vertex labels, as the induced vertex labels are the averages of the two edge labels of the edges that are incident on the vertex and each edge is considered twice to obtain the induced vertex labels.

But we notice that in the case of odd cycles, C_n, $n \equiv 1(mod\ 2)$, be it any type of SVM - labeling, there are exactly two vertices which have such edges incident on it, that are labeled with two integers one of which is odd and the other is even. Therefore, the induced vertex labels of these two vertices are 0.5 each more than the actual average of the labels of the incident edges on it, as per the definition of the SVM labeling. When we sum up all the induced vertex labels, we get an integer which is exactly one more than the sum of all the edge labels. Or in other words, this sum of all induced vertex labels is 0.5 more than the half of the sum of first $2n$ positive integers. Similarly the sum of all edge labels is 0.5 less than the half of the sum of the first $2n$ positive integers.

We also know that the half of the sum the of first $2n$ positive integers is

$$\frac{(2n)(2n+1)}{4}$$

For example, 2 – type labeling of C_5, where, $2n = 10$, and

Half of the sum of first 10 positive integers =

$$\frac{10 \times 11}{4} = 27.5$$

The sum of the vertex labels is $2 + 4 + 8 + 9 + 5 = 28$, and
The sum of the edge labels is $1 + 6 + 10 + 7 + 3 = 27$.
Therefore, the sum of the vertex labels for $C_n, n \equiv 1(mod\ 2)$, is given by

the following equation:

$$\sum_{i=1}^{n} f^{v}(v_i) = (\frac{(2n)(2n+1)}{4} + 0.5)$$

and,
the sum of the edge labels for $C_n, n \equiv 1(mod\ 2)$, is

$$\sum_{i=1}^{n} f(e_i) = (\frac{(2n)(2n+1)}{4} - 0.5).$$

Similarly, for even cycles, $C_n, n \equiv 0(mod\ 2)$, there are exactly 4 vertices which have edges incident on them in such a manner that they are labeled with integers of which one is odd and the other is even, resulting in an increase of 2 in the sum of the vertex labels to that of the edge labels.

Therefore, sum of the edge labels = sum of the vertex labels -2

Also, sum of the first $2n$ positive integers = $\frac{(2n)(2n+1)}{2}$

So, sum of the vertex labels = $\frac{(2n)(2n+1)}{2}$ - sum of the edge labels

i.e., = $\frac{(2n)(2n+1)}{2}$ - sum of the vertex labels $+2$

i.e., $2\times$ sum of the vertex labels = $\frac{(2n)(2n+1)}{2} + 2$

Therefore, the sum of the vertex labels of $C_n, n \equiv 0(mod\ 2)$, is given by the following equation,

$$\sum_{i=1}^{n} f^{v}(v_i) = (\frac{(2n)(2n+1)}{4} + 1).$$

and sum of the edge labels for $C_n, n \equiv 0(mod\ 2)$ is

$$\sum_{i=1}^{n} f(e_i) = (\frac{(2n)(2n+1)}{4} - 1).$$

We conclude by stating that the above equations are not sufficient but necessary conditions for a set of integers from the set of first $2n$ positive integers to be the edge label set, $f(E)$ or the induced vertex label set, $f^{v}(V)$

of a Super Vertex Mean labeling of any type for any cycle C_n. It is given as follows:

$$\sum_{i=1}^{n} f(e_i) = \begin{cases} (\frac{(2n)(2n+1)}{4} - 0.5) & \text{if } n \equiv 1(mod\ 2) \\ (\frac{(2n)(2n+1)}{4} - 1) & \text{if } \ n \equiv 0(mod\ 2). \end{cases}$$

$$\sum_{i=1}^{n} f^v(v_i) = \begin{cases} (\frac{(2n)(2n+1)}{4} + 0.5) & \text{if } n \equiv 1(mod\ 2) \\ (\frac{(2n)(2n+1)}{4} + 1) & \text{if } \ n \equiv 0(mod\ 2). \end{cases}$$

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