

## Edge fixed monophonic number of a graph

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### Abstract

*For an edge  $xy$  in a connected graph  $G$  of order  $p \geq 3$ , a set  $S \subseteq V(G)$  is an  $xy$ -monophonic set of  $G$  if each vertex  $v \in V(G)$  lies on an  $x - u$  monophonic path or a  $y - u$  monophonic path for some element  $u$  in  $S$ . The minimum cardinality of an  $xy$ -monophonic set of  $G$  is defined as the  $xy$ -monophonic number of  $G$ , denoted by  $m_{xy}(G)$ . An  $xy$ -monophonic set of cardinality  $m_{xy}(G)$  is called a  $m_{xy}$ -set of  $G$ . We determine bounds for it and find the same for special classes of graphs. It is shown that for any three positive integers  $r$ ,  $d$  and  $n \geq 2$  with  $2 \leq r \leq d$ , there exists a connected graph  $G$  with monophonic radius  $r$ , monophonic diameter  $d$  and  $m_{xy}(G) = n$  for some edge  $xy$  in  $G$ .*

**Key Words :** *Monophonic path, vertex monophonic number, edge fixed monophonic number.*

**Mathematics Subject Classification :** *05C12.*

## 1. Introduction

By a graph  $G = (V, E)$  we mean a finite undirected connected graph without loops or multiple edges. The order and size of  $G$  are denoted by  $p$  and  $q$  respectively. For basic graph theoretic terminology we refer to [1, 2]. For vertices  $x$  and  $y$  in a connected graph  $G$ , the *distance*  $d(x, y)$  is the length of a shortest  $x - y$  path in  $G$ . An  $x - y$  path of length  $d(x, y)$  is called an  $x - y$  *geodesic*. The *neighborhood* of a vertex  $v$  is the set  $N(v)$  consisting of all vertices  $u$  which are adjacent with  $v$ . A vertex  $v$  is a *simplicial vertex* if the subgraph induced by its neighbors is complete. A *non-separable* graph is connected, non-trivial, and has no cut-vertices. A *block* of a graph is a maximal non-separable subgraph. A *connected block graph* is a connected graph in which each of its blocks is complete. A *caterpillar* is a tree for which the removal of all the end vertices gives a path.

A *chord* of a path  $P$  is an edge joining two non-adjacent vertices of  $P$ . A path  $P$  is called *monophonic* if it is a chordless path. The *closed interval*  $I_m[x, y]$  consists of all vertices lying on some  $x - y$  monophonic of  $G$ . For any two vertices  $u$  and  $v$  in a connected graph  $G$ , the *monophonic distance*  $d_m(u, v)$  from  $u$  to  $v$  is defined as the length of a longest  $u - v$  monophonic path in  $G$ . The *monophonic eccentricity*  $e_m(v)$  of a vertex  $v$  in  $G$  is  $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$ . The *monophonic radius*,  $rad_m(G)$  of  $G$  is  $rad_m \{G\} = \min \{e_m(v) : v \in V(G)\}$  and the *monophonic diameter*,  $diam_m \{G\}$  of  $G$  is  $diam_m \{G\} = \max \{e_m(v) : v \in V(G)\}$ . The monophonic distance was introduced in [3] and further studied in [4]. The concept of vertex monophonic number was introduced by Santhakumaran and Titus [5]. A set  $S$  of vertices of  $G$  is an  *$x$ -monophonic set* if each vertex  $v$  of  $G$  lies on an  $x - y$  monophonic path in  $G$  for some element  $y$  in  $S$ . The minimum cardinality of an  $x$ -monophonic set of  $G$  is defined as the  *$x$ -monophonic number* of  $G$  and is denoted by  $m_x(G)$  or simply  $m_x$ . An  $x$ -monophonic set of cardinality  $m_x(G)$  is called a  *$m_x$ -set* of  $G$ . The following theorems will be used in the sequel.

**Theorem 1.1.** [2] *Let  $v$  be a vertex of a connected graph  $G$ . The following statements are equivalent:*

- i)  $v$  is a cut-vertex of  $G$ .
- ii) There exist vertices  $u$  and  $w$  distinct from  $v$  such that  $v$  is on every  $u - w$  path.
- iii) There exists a partition of the set of vertices  $V - \{v\}$  into subsets  $U$  and  $W$  such that for any vertices  $u \in U$  and  $w \in W$ , the vertex  $v$  is on every  $u - w$  path.

**Theorem 1.2.** [2] *Every non-trivial connected graph has at least two vertices which are not cut-vertices.*

**Theorem 1.3.** [2] *Let  $G$  be a connected graph with at least three vertices. The following statements are equivalent:*

- i)  $G$  is a block.
- ii) Every two vertices of  $G$  lie on a common cycle.

Throughout this paper  $G$  denotes a connected graph with at least three vertices.

## 2. Edge fixed monophonic number

**Definition 2.1.** Let  $e = xy$  be any edge of a connected graph  $G$  of order at least three. A set  $S$  of vertices of  $G$  is an  $xy$ -monophonic set if every vertex of  $G$  lies on either an  $x - u$  monophonic path or a  $y - u$  monophonic path in  $G$  for some element  $u$  in  $S$ . The minimum cardinality of an  $xy$ -monophonic set of  $G$  is defined as the  $xy$ -monophonic number of  $G$  and is denoted by  $m_{xy}(G)$  or  $m_e(G)$ . An  $xy$ -monophonic set of cardinality  $m_{xy}(G)$  is called a  $m_{xy}$ -set or  $m_e$ -set of  $G$ .

**Example 2.2.** For the graph  $G$  given in Figure 2.1, the minimum edge fixed monophonic sets and the edge fixed monophonic numbers are given in Table 2.1.

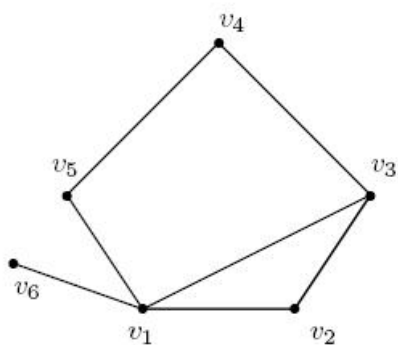


Figure 2.1 :  $G$

**Theorem 2.3.** For any edge  $xy$  in a connected graph  $G$  of order at least three, the vertices  $x$  and  $y$  do not belong to any minimum  $xy$ -monophonic set of  $G$ .

**Proof.** Suppose that  $x$  belongs to a minimum  $xy$ -monophonic set, say  $S$ , of  $G$ . Since  $G$  is a connected graph with at least three vertices and  $xy$  in an edge, it follows from the definition of an  $xy$ -monophonic set that  $S$  contains a vertex  $v$  different from  $x$  and  $y$ . Since the vertex  $x$  lies on every  $x - v$  monophonic path in  $G$ , it follows that  $T = S - \{x\}$  is an  $xy$ -monophonic set of  $G$ , which is a contradiction to  $S$  a minimum  $xy$ -monophonic set of  $G$ . Similarly,  $y$  does not belong to any minimum  $xy$ -monophonic set of  $G$ .  $\square$

Table 2.1: The Edge Fixed Monophonic Number of a Graph

Edge $e$	minimum $e$ -monophonic sets	$e$ -monophonic number
$v_1v_2$	$\{v_4, v_6\}, \{v_5, v_6\}$	2
$v_2v_3$	$\{v_4, v_6\}, \{v_5, v_6\}$	2
$v_3v_4$	$\{v_2, v_6\}$	2
$v_4v_5$	$\{v_2, v_6\}$	2
$v_5v_1$	$\{v_2, v_6\}$	2
$v_1v_6$	$\{v_2, v_4\}$	2
$v_1v_3$	$\{v_2, v_6, v_4\}, \{v_2, v_6, v_5\}$	3

**Theorem 2.4.** *Let  $xy$  be any edge of a connected graph  $G$  of order at least three. Then*

- i) *every simplicial vertex of  $G$  other than the vertices  $x$  and  $y$  (whether  $x$  or  $y$  is simplicial or not) belongs to every  $m_{xy}$ -set.*
- ii) *no cut-vertex of  $G$  belongs to any  $m_{xy}$ -set.*

**Proof.** (i) By Theorem 2.3, the vertices  $x$  and  $y$  do not belong to any  $m_{xy}$ -set. So, let  $u \neq x, y$  be a simplicial vertex of  $G$ . Let  $S$  be a  $m_{xy}$ -set of  $G$  such that  $u \notin S$ . Then  $u$  is an internal vertex of either an  $x - v$  monophonic path or a  $y - v$  monophonic path for some element  $v$  in  $S$ . Without loss of generality, let  $P$  be an  $x - v$  monophonic path with  $u$  is an internal vertex. Then both the neighbors of  $u$  on  $P$  are not adjacent and hence  $u$  is not a simplicial vertex, which is a contradiction.

(ii) Let  $v$  be a cut-vertex of  $G$ . Then by Theorem 1.1, there exists a partition of the set of vertices  $V - \{v\}$  into subsets  $U$  and  $W$  such that for any vertex  $u \in U$  and  $w \in W$ , the vertex  $v$  lies on every  $u - w$  path. Let  $S$  be a  $m_{xy}$ -set of  $G$ . We consider three cases.

**Case (i):** Both  $x$  and  $y$  belong to  $U$ . Suppose that  $S \cap W = \emptyset$ . Let  $w_1 \in W$ . Since  $S$  is an  $xy$ -monophonic set, there exists an element  $z$  in  $S$  such that  $w_1$  lies on either an  $x - z$  monophonic path or a  $y - z$  monophonic path in  $G$ . Suppose that  $w_1$  lies on an  $x - z$  monophonic path  $P : x = z_0, z_1, \dots, w_1, \dots, z_n = z$  in  $G$ . Then the  $x - w_1$  subpath of  $P$  and  $w_1 - z$  subpath of  $P$  both contain  $v$  so that  $P$  is not a path in  $G$ , which is a contradiction. Hence  $S \cap W \neq \emptyset$ . Let  $w_2 \in S \cap W$ . Then  $v$  is an internal vertex of any  $x - w_2$  monophonic path and  $v$  is also an internal vertex of any  $y - w_2$  monophonic path. If  $v \in S$ , then let  $S' = S - \{v\}$ . It is clear that every vertex that lies on an  $x - v$  monophonic path also lies on an  $x - w_2$  monophonic path. Hence it follows that  $S'$  is an  $xy$ -monophonic set of  $G$ , which is a contradiction to  $S$  a minimum  $xy$ -monophonic set of  $G$ . Thus  $v$  does not belong to any minimum  $xy$ -monophonic set of  $G$ .

**Case (ii):** Both  $x$  and  $y$  belong to  $W$ . It is similar to Case (i).

**Case (iii):** Either  $x = v$  or  $y = v$ . By Theorem 2.3,  $v$  does not belong to any  $m_{xy}$ -set.  $\square$

**Corollary 2.5.** *Let  $T$  be a tree with  $k$  end vertices. Then  $m_{xy}(T) = k - 1$  or  $k$  according as  $xy$  is an end edge or cut-edge.*

**Proof.** This follows from Theorem 2.4.  $\square$

**Corollary 2.6.** Let  $K_{1,n}$  ( $n \geq 2$ ) be a star. Then  $m_{xy}(K_{1,n}) = n - 1$  for any edge  $xy$  in  $K_{1,n}$ .

**Corollary 2.7.** Let  $G$  be a complete graph  $K_p$  ( $p \geq 3$ ). Then  $m_{xy}(G) = p - 2$  for any edge  $xy$  in  $G$ .

**Theorem 2.8.** For any edge  $xy$  in the cube  $Q_n$  ( $n \geq 3$ ),  $m_{xy}(Q_n) = 1$ .

**Proof.** Let  $e = xy$  be an edge in  $Q_n$  and let  $x = (a_1, a_2, \dots, a_n)$ , where  $a_i \in \{0, 1\}$ . Let  $x' = (a'_1, a'_2, \dots, a'_n)$  be another vertex of  $Q_n$  such that  $a'_i$  is the compliment of  $a_i$ . Let  $u$  be any vertex in  $Q_n$ . For convenience, let  $u = (a_1, a'_2, a_3, \dots, a_n)$ . Then  $u$  lies on an  $x - x'$  monophonic path  $P : x = (a_1, a_2, \dots, a_n), (a_1, a'_2, a_3, \dots, a_n), \dots, (a'_1, a'_2, \dots, a'_{n-1}, a_n), (a'_1, a'_2, \dots, a'_n) = x'$ . Hence  $\{x'\}$  is an  $xy$ -monophonic set of  $Q_n$  and so  $m_{xy}(Q_n) = 1$ .  $\square$

**Theorem 2.9.** i) For any edge  $xy$  in the wheel  $W_n = K_1 + C_{n-1}$  ( $n \geq 5$ ),

$$m_{xy}(W_n) = 1.$$

ii) For any edge  $xy$  in the complete bipartite graph  $K_{m,n}$  ( $1 \leq m \leq n$ ),

$$m_{xy}(K_{m,n}) = \begin{cases} n - 1 & \text{if } m = 1 \\ 1 & \text{if } m = 2 \\ 2 & \text{if } m \geq 3. \end{cases}$$

**Proof.** (i) Let  $xy$  be an edge in  $W_n$ . Then either  $x$  or  $y$  is a vertex of  $C_{n-1}$ . Let  $x \in V(C_{n-1})$  and let  $z$  be a non-adjacent vertex of  $x$  in  $C_{n-1}$ . It is clear that every vertex of  $W_n$  lies on an  $x - z$  monophonic path. Hence  $\{z\}$  is a  $m_{xy}$ -set of  $W_n$  and so  $m_{xy}(W_n) = 1$ .

(ii) Let  $U = \{u_1, u_2, \dots, u_m\}$  and  $W = \{w_1, w_2, \dots, w_n\}$  be the vertex subsets of the bipartition of the vertices of  $K_{m,n}$ . If  $m = 1$ , then by Corollary 2.6,  $m_{xy}(K_{1,n}) = n - 1$  for any edge  $xy$  in  $K_{1,n}$ . If  $m = 2$ , let  $e$  be an edge in  $K_{m,n}$ , say  $e = u_1w_1$ . It is clear that every vertex of  $K_{m,n}$  lies on an  $u_1 - u_2$  monophonic path. Hence  $\{u_2\}$  is an  $e$ -monophonic set of  $K_{m,n}$  and so  $m_e(K_{m,n}) = 1$ . If  $m \geq 3$ , then it is clear that no singleton subset of  $V$  is an  $e$ -monophonic set of  $K_{m,n}$  and so  $m_e(K_{m,n}) \geq 2$ . Without loss of generality, take  $e = u_1w_1$ . Let  $S = \{u_2, w_2\}$ . Then every vertex of  $U$  lies on a  $w_1 - w_2$  monophonic path and every vertex of  $W$  lies on a  $u_1 - u_2$  monophonic path. Hence  $S$  is an  $e$ -monophonic set of  $K_{m,n}$  and so  $m_e(K_{m,n}) = 2$ .  $\square$

**Theorem 2.10.** For any edge  $xy$  in a connected graph  $G$  of order  $p \geq 3$ ,  $1 \leq m_{xy}(G) \leq p - 2$ .

**Proof.** It is clear from the definition of  $m_{xy}$ -set that  $m_{xy}(G) \geq 1$ . Also, since the vertices  $x$  and  $y$  do not belong to any  $m_{xy}$ -set, it follows that  $m_{xy}(G) \leq p - 2$ .  $\square$

**Remark 2.11.** The bounds for  $m_{xy}(G)$  in Theorem 2.10 are sharp. If  $C$  is any cycle, then  $m_{xy}(C) = 1$  for any edge  $xy$  in  $C$ . For any edge  $xy$  in a complete graph  $K_p$  ( $p \geq 3$ ),  $m_{xy}(K_p) = p - 2$ .

Now we proceed to characterize graphs for which the upper bound in Theorem 2.10 is attained.

**Theorem 2.12.** *Let  $G$  be a connected graph of order at least 3. Then  $G$  is either  $K_p$  or  $K_{1,p-1}$  if and only if  $m_{xy}(G) = p - 2$  for every edge  $xy$  in  $G$ .*

**Proof.** If  $G = K_p$ , then by Corollary 2.7,  $m_{xy}(G) = p - 2$  for every edge  $xy$  in  $G$ . If  $G = K_{1,p-1}$ , then by Corollary 2.6,  $m_{xy}(G) = p - 2$  for any edge  $xy$  in  $G$ . Conversely, suppose that  $m_{xy}(G) = p - 2$  for every edge  $xy$  in  $G$ . By Theorem 1.2,  $G$  has at least two vertices which are not cut-vertices. Let  $xy$  be an edge of  $G$  with  $x$  is not a cut-vertex. If  $G$  has two or more cut-vertices, then by Theorem 2.4(ii),  $m_{xy}(G) \leq p - 3$ , which is a contradiction. Thus the number of cut-vertices  $k$  of  $G$  is at most one.

**Case (i)**  $k = 0$ . Then the graph  $G$  is a block. Now we claim that  $G$  is complete. If  $G$  is not complete, then there exist two vertices  $x$  and  $y$  in  $G$  such that  $d(x, y) \geq 2$ . By Theorem 1.3,  $x$  and  $y$  lie on a common cycle and hence  $x$  and  $y$  lie on a smallest cycle  $C : x, x_1, x_2, \dots, y, \dots, x_n, x$  of length at least 4. Then  $(V(G) - V(C)) \cup \{y\}$  is an  $xx_1$ -monophonic set of  $G$  and so  $m_{xx_1}(G) \leq p - 3$ , which is a contradiction. Hence  $G$  is the complete graph.

**Case (ii)**  $k = 1$ . Let  $x$  be the cut-vertex of  $G$ . If  $p = 3$ , then  $G = P_3$ , a star with three vertices. If  $p \geq 4$ , we claim that  $G = K_{1,p-1}$ . It is enough to prove that degree of every vertex other than  $x$  is one. Suppose that there exists a vertex, say  $y$ , with  $\deg y \geq 2$ . Let  $z \neq x$  be an adjacent vertex of  $y$  in  $G$ . Let  $e = yz$ . Since the vertices  $y$  and  $z$  do not lie on any minimum  $yz$ -monophonic set of  $G$  and by Theorem 2.4(ii), we have  $m_{yz}(G) \leq p - 3$ , which is a contradiction. Thus every vertex of  $G$  other than  $x$  is of degree one. Hence  $G$  is a star.  $\square$

**Theorem 2.13.** *For any edge  $xy$  in a connected graph  $G$ , every  $x$ -monophonic set of  $G$  is an  $xy$ -monophonic set of  $G$ .*

**Proof.** Let  $S$  be an  $x$ -monophonic set of  $G$ . Then every vertex of  $G$  lies on an  $x - z$  monophonic path for some  $z$  in  $S$ . It follows that  $S$  is an  $xy$ -monophonic set of  $G$ .  $\square$

**Corollary 2.14.** *For any edge  $xy$  in a connected graph  $G$ ,  $m_{xy}(G) \leq \min\{m_x(G), m_y(G)\}$ .*

**Theorem 2.15.** *For every pair  $a, b$  of integers with  $1 \leq a \leq b$ , there is a connected graph  $G$  with  $m_{xy}(G) = a$  and  $m_x(G) = b$  for some edge  $xy$  in  $G$ .*

**Proof.** Let  $C_4 : x, y, z, u, x$  be a cycle of order 4. Add  $b-1$  new vertices  $v_1, v_2, \dots, v_{a-1}, w_1, w_2, \dots, w_{b-a}$  and joining each  $v_i (1 \leq i \leq a-1)$  to  $x$  and joining each  $w_j (1 \leq j \leq b-a)$  to the vertices  $y$  and  $u$ , thereby producing the graph  $G$  given in Figure 2.2. Let  $S = \{v_1, v_2, \dots, v_{a-1}\}$  be the set of all simplicial vertices of  $G$ . Since  $S$  is not an  $xy$ -monophonic set, it follows from Theorem 2.4(i) that  $m_{xy}(G) \geq a$ . On the other hand,  $S_1 = S \cup \{u\}$

is an  $xy$ -monophonic set of  $G$  and so  $m_{xy}(G) = |S_1| = a$ . Clearly,  $S_2 = \{v_1, v_2, \dots, v_{a-1}, z, w_1, w_2, \dots, w_{b-a}\}$  is the unique  $x$ -monophonic set of  $G$  and so  $m_x(G) = |S_2| = b$ .

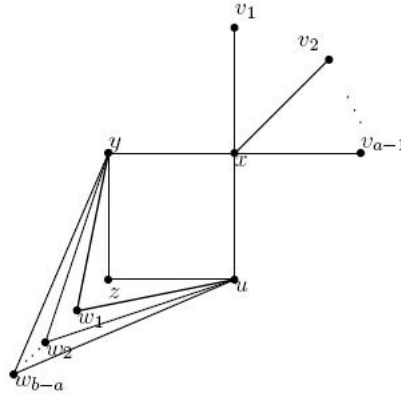


Figure 2.2 :  $G$

We have seen that if  $G$  is a connected graph of order  $p \geq 3$ , then  $1 \leq m_{xy}(G) \leq p - 2$  for any edge  $xy$  in  $G$ . In the following theorem we give an improved upper bound for the edge fixed monophonic number of a tree in terms of its order and monophonic diameter.

**Theorem 2.16.** *If  $T$  is a tree of order  $p$  and monophonic diameter  $d_m$ , then  $m_{xy}(T) \leq p - d_m + 1$  for any edge  $xy$  in  $T$ .*

**Proof.** Let  $P : v_0, v_1, v_2, \dots, v_{d_m}$  be a monophonic path of length  $d_m$ . Now, let  $S = V(G) - \{v_1, v_2, \dots, v_{d_m-1}\}$ . If  $e$  is an internal edge of  $P$ , then clearly  $S$  is an  $e$ -monophonic set of  $T$  so that  $m_e(T) \leq |S| = p - d_m + 1$ . If  $e$  is an end edge of  $P$ , say  $e = v_0v_1$ , then  $S_1 = S - \{v_0\}$  is an  $e$ -monophonic set of  $T$  so that  $m_e(T) \leq |S_1| = p - d_m$ . If  $e = xy$  is an edge lies outside  $P$ , then  $S_2 = S - \{x, y\}$  is an  $e$ -monophonic set of  $T$  so that  $m_e(T) \leq |S_2| = p - d_m$ . Hence for any edge  $xy$  in  $T$ ,  $m_{xy}(T) \leq p - d_m + 1$ .  $\square$

**Remark 2.17.** *The bound in Theorem 2.16 is not true for any graph. For example, consider the graph  $G$  given in Figure 2.3. Here  $p = 7$ ,  $d_m(G) = 4$ ,  $m_e(G) = 5$  and  $p - d_m + 1 = 4$ . Hence  $m_e(G) > p - d_m + 1$ .*

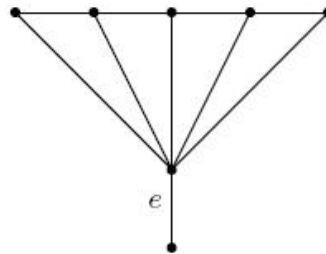


Figure 2.3 :  $G$

**Theorem 2.18.** *For any edge  $xy$  in a non-trivial tree  $T$  of order  $p$  and monophonic diameter  $d_m$ ,  $m_{xy}(T) = p - d_m$  or  $p - d_m + 1$  if and only if  $T$  is a caterpillar.*

**Proof.** Let  $T$  be any non-trivial tree. Let  $P : v_0, v_1, \dots, v_d$  be a monophonic path of length  $d_m$ . Let  $k$  be the number of end vertices of  $T$  and let  $l$  be the number of internal vertices of  $T$  other than  $v_1, v_2, \dots, v_{d-1}$ . Then  $d_m - 1 + l + k = p$ . By Corollary 2.5,  $m_{xy}(T) = k$  or  $k - 1$  for any edge  $xy$  in  $T$  and so  $m_{xy}(T) = p - d_m - l + 1$  or  $p - d_m - l$  for any edge  $xy$  in  $T$ . Hence  $m_{xy}(T) = p - d_m + 1$  or  $p - d_m$  for any edge  $xy$  in  $T$  if and only if  $l = 0$ , if and only if all the internal vertices of  $T$  lie on the monophonic path  $P$ , if and only if  $T$  is a caterpillar.

For any connected graph  $G$ ,  $rad_m(G) \leq diam_m(G)$ . It is shown in [3] that every two positive integers  $a$  and  $b$  with  $a \leq b$  are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This result can be extended so that the edge fixed monophonic number can be prescribed.  $\square$

**Theorem 2.19.** *For positive integers  $r, d$  and  $n \geq 2$  with  $2 \leq r \leq d$ , there exists a connected graph  $G$  with  $rad_m(G) = r$ ,  $diam_m(G) = d$  and  $m_{xy}(G) = n$  for some edge  $xy$  in  $G$ .*

**Proof.** **Case (i)**  $2 \leq r = d$ . Let  $C_{r+2} : v_1, v_2, \dots, v_{r+2}, v_1$  be the cycle of order  $r+2$ . Let  $G$  be the graph obtained from  $C_{r+2}$  by adding  $n$  vertices  $u_1, u_2, \dots, u_n$  and joining each vertex  $u_i$  ( $1 \leq i \leq n$ ) to both  $v_2$  and  $v_{r+2}$ , and also adding the edge  $v_1u_1$ . The graph  $G$  is shown in Figure 2.4. It is easily verified that the monophonic eccentricity of each vertex of  $G$  is  $r$  and so  $rad_m(G) = diam_m(G) = r$ . Also, for the edge  $v_1u_1$ , it is clear that  $S = \{v_{r+1}, u_2, \dots, u_n\}$  is a minimum  $xy$ -monophonic set of  $G$  and so  $m_{xy}(G) = n$ .

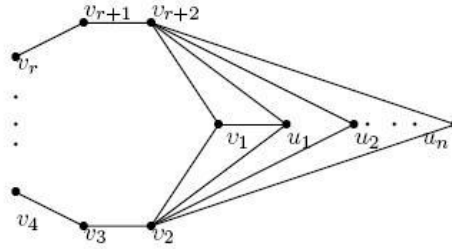


Figure 2.4 :  $G$

**Case (ii)**  $2 \leq r < d \leq 2r$ . Let  $C_{r+2} = v_1, v_2, \dots, v_{r+2}, v_1$  be the cycle of order  $r + 2$  and let  $P_{d-r+1} : u_0, u_1, \dots, u_{d-r}$  be a path of order  $d - r + 1$ . Let  $H$  be the graph obtained from  $C_{r+2}$  and  $P_{d-r+1}$  by identifying  $v_1$  in  $C_{r+2}$  and  $u_0$  in  $P_{d-r+1}$ . Let  $G$  be the graph obtained from  $H$  by adding  $n - 1$  new vertices  $w_1, w_2, \dots, w_{n-1}$  and joining each  $w_i$  ( $1 \leq i \leq n - 1$ ) with  $u_{d-r-1}$ . The graph  $G$  is shown in Figure 2.5. It is easily verified that  $r \leq e_m(x) \leq d$  for any vertex  $x$  in  $G$ ,  $e_m(v_1) = r$  and  $e_m(v_3) = d$ . Thus  $rad_m(G) = r$  and  $diam_m(G) = d$ . For the edge  $e = u_{d-r-1}u_{d-r}$ ,  $S = \{w_1, w_2, \dots, w_{n-1}, v_3\}$  is a minimum  $e$ -monophonic set of  $G$  and so  $m_e(G) = n$ .



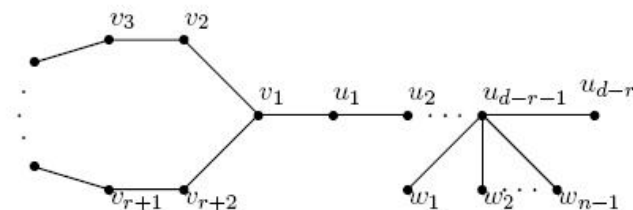


Figure 2.5 :  $G$

**Case (iii)**  $d > 2r$ . Let  $P_{2r-1} : v_1, v_2, \dots, v_{2r-1}$  be a path of order  $2r - 1$ . Let  $G$  be the graph obtained from the wheel  $W = K_1 + C_{d+2}$  and the complete graph  $K_n$  by identifying the vertex  $v_1$  of  $P_{2r-1}$  with the central vertex of  $W$ , and identifying the vertex  $v_{2r-1}$  of  $P_{2r-1}$  with a vertex of  $K_n$ . The graph  $G$  is shown in Figure 2.6. Since  $d > 2r$ , we have  $e_m(x) = d$  for any vertex  $x \in V(C_{d+2})$ . Also,  $e_m(x) = 2r$  for any vertex  $x \in V(K_n) - v_{2r-1}$ ;  $r \leq e_m(x) \leq 2r - 1$  for any vertex  $x \in V(P_{2r-1})$ ; and  $e_m(x) = r$  for the central vertex  $x$  of  $P_{2r-1}$ . Thus  $rad_m(G) = r$  and  $diam_m(G) = d$ .

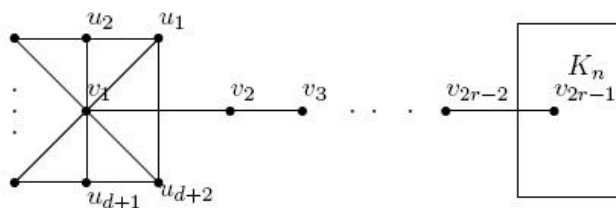


Figure 2.6 :  $G$

Let  $S = V(K_n) - \{v_{2r-1}\}$  be the set of all simplicial vertices of  $G$ . Then by Theorem 2.4(i), every  $m_e$ -set contains  $S$  for the edge  $e = u_1u_2$ . It is clear that  $S$  is not an  $e$ -monophonic set of  $G$  and so  $m_e(G) > |S| = n - 1$ . Since  $S' = S \cup \{u_{d+1}\}$  is an  $e$ -monophonic set of  $G$ , we have  $m_e(G) = n$ .  $\square$

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