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An Algorithmic Approach to Equitable Total Chromatic Number of Graphs

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Abstract

The equitable total coloring of a graph G is a combination of vertex and edge coloring whose color classes differs by atmost one. In this paper, we find the equitable total chromatic number for S_n, W_n, H_n and G_n.

Keywords: *Equitable total coloring, Wheel, Helm, Gear, Sunlet*

1. Introduction

Graphs in this paper are finite, simple and undirected graphs without loops. The total coloring was introduced by Behzad and Vizing in 1964. A total coloring of a graph G is a coloring of all elements (i.e,vertices and edges) of G, such that no two adjacent or incident elements receive the same color. The minimum number of colors is called the total chromatic number of G and is denoted by $\chi''(G)$. In 1973, Meyer[7] presented the concept of equitable coloring and conjectured that the equitable chromatic number of a connected graph G, is atmost $\Delta(G)$. In 1994, Hung-lin Fu first introduced the concepts of equitable total coloring and equitable total chromatic number of a graph. Furthermore Fu presented a conjecture concerning the equitable total chromatic number, $\chi''_{=}(G) \leq \Delta + 2$.
Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Clearly $\chi''_{=}(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. In 1989, Sanchez Arroyo[8] proved that the problem of determining the total chromatic number of an arbitrary graph is NP-hard. It is also NP - Hard to decide $\chi''_{=}(G) \leq \Delta(G) + 1$ or $\chi''_{=}(G) \leq \Delta(G) + 2$. Graphs with $\chi''_{=}(G) \leq \Delta(G) + 1$ are said to be of Type 1, and graphs with $\chi''_{=}(G) \leq \Delta(G) + 2$ are said to be of Type 2. The problem of deciding whether a graph is Type 1 has been shown NP-Complete in this paper for S_n, W_n, H_n and G_n.

2. Preliminaries

Definition 2.1. *For any integer $n \geq 4$, the wheel graph W_n is the n-vertex graph obtained by joining a vertex v_0 to each of the $n-1$ vertices $\{v_1, v_2, \ldots, v_n\}$ of the cycle graph C_{n-1}.*

Definition 2.2. *The Helm graph H_n is the graph obtained from a Wheel graph W_n by adjoining a pendant edge to each vertex of the $n-1$ cycle in W_n.*

Definition 2.3. *The Gear graph G_n is the graph obtained from a Wheel graph W_n by adding a vertex to each edge of the $n-1$ cycle in W_n.*

Definition 2.4. *The n- sunlet graph on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.*

Definition 2.5. *[6] For a simple graph $G(V, E)$, let f be a proper $k-$total coloring of G*

$$||T_i| - |T_j|| \leq 1, \; i, j = 1, 2, \ldots, k.$$

The partition $\{T_i\} = \{V_i \cup E_i : 1 \leq i \leq k\}$ is called a $k-$equitable total coloring ($k-$ETC of G in brief), and

$$\chi''_{=}(G) = \min\{k : \text{there exists a } k - ETC \text{ of } G\}$$

is called the equitable total chromatic number of G, where $\forall x \in T_i = V_i \cup E_i$, $f(x) = i,\ i = 1, 2, \ldots, k$.

Following [4], let us denote the Total Coloring Conjecture by TCC.

Conjecture 2.6. *[TCC] For any graph G, $\Delta(G)+1 \leq \chi''(G) \leq \Delta(G)+2$.*

Conjecture 2.7. *[4][10] For every graph G, G has an equitable total $k-$coloring for each $k \geq max\{\chi''(G), \Delta(G)+2\}$.*

Conjecture 2.8. *[4] [ETCC] For every graph G, $\chi''_{=}(G) \leq \Delta(G)+2$.*

Lemma 2.9. *[6] For complete graph K_p with order p,*

$$\chi''_{=}(K_p) = \begin{cases} p, & p \equiv 1 \bmod 2 \\ p+1, & p \equiv 0 \bmod 2. \end{cases}$$

Lemma 2.10. *[10] Let G be a graph consisting of two components G_1 and G_2. If G_1 and G_2 are equitably total $k-$colorable, then so is G.*

Proof. Let $\left(\widetilde{T_1}, \widetilde{T_2}, \ldots, \widetilde{T_k}\right)$ and $\left(\overline{T_1}, \overline{T_2}, \ldots, \overline{T_k}\right)$ be equitable total $k-$colorings of G_1 and G_2 repectively, satisfying $\left|\widetilde{T_1}\right| \leq \left|\widetilde{T_2}\right| \leq \ldots \leq \left|\widetilde{T_k}\right|$ and $\left|\overline{T_1}\right| \leq \left|\overline{T_2}\right| \leq \ldots \leq \left|\overline{T_k}\right|$. Then we put

$$T_i = \widetilde{T}_i \cup \overline{T}_{k-i+1}, \quad i = 1, 2, \ldots, k.$$

It is easy to see that $(T_1, T_2, \ldots, T_k)$ is an equitable total $k-$coloring of G. □

In the following section, we determine the equitable total chromatic number of S_n, W_n, H_n and G_n.

3. Main Results

Theorem 3.1. *For Sunlet graph S_n with $n \geq 3$, $\chi''_{=}(S_n) = 4$.*

Proof. Let S_n be the sunlet graph on $2n$ vertices and $2n$ edges.

$$\text{Let } V\left(S_n\right) = \{v_1, v_2, v_3, \ldots, v_n\} \bigcup \{u_1, u_2, u_3, \ldots, u_n\} \text{ and}$$

$$E\left(S_n\right) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e_n\} \bigcup \{e_i' : 1 \leq i \leq n\}$$

where e_i is the edge $v_i v_{i+1}\left(1 \leq i \leq n-1\right)$, e_n is the edge $v_n v_1$ and e_i' is the edge $v_i u_i\left(1 \leq i \leq n\right)$.

We define an equitable total coloring f, such that $f : S \to C$ where $S = V\left(S_n\right) \cup E\left(S_n\right)$ and $C = \{1, 2, 3, 4\}$. The order of coloring is followed by coloring the pendant vertices first followed by pendant edges, rim vertices and rim edges respectively. In this total coloration, $C(u_i)$ means the color of the i^{th} pendant vertex u_i, $C(e_i)$ means the color of the i^{th} rim edge e_i and $C(e_i')$ means the color of the i^{th} pendant edge e_i'. While coloring, when the value mod 4 is equal to 0 it should be replaced by 4.

Case 1: $n \equiv 0 (\text{mod } 4)$

$$f\left(u_i\right) = \begin{cases} 1, \text{ if } i \equiv 1 (\text{mod } 4) \\ 2, \text{ if } i \equiv 2 (\text{mod } 4) \\ 3, \text{ if } i \equiv 3 (\text{mod } 4) \\ 4, \text{ if } i \equiv 0 (\text{mod } 4) \end{cases} \text{ for } 1 \leq i \leq n$$

$$f\left(e_i'\right) = \{C(u_i) + 1\} (\text{mod } 4), \quad \text{for } 1 \leq i \leq n$$

$$f\left(v_i\right) = \{C(e_i') + 1\} (\text{mod } 4), \quad \text{for } 1 \leq i \leq n$$

$$f\left(e_i\right) = C(u_i), \quad \text{for } 1 \leq i \leq n$$

Case 2: $n \equiv 1 (\text{mod } 4)$

$$f\left(u_i\right) = \begin{cases} 1, \text{ if } i \equiv 1 (\text{mod } 4) \\ 2, \text{ if } i \equiv 2 (\text{mod } 4) \\ 3, \text{ if } i \equiv 3 (\text{mod } 4) \\ 4, \text{ if } i \equiv 0 (\text{mod } 4) \end{cases} \text{ for } 1 \leq i \leq n-2$$

$$f\left(u_{n-1}\right) = 1$$

$$f\left(u_n\right) = 4$$

$$f\,(e'_i) = \{C(u_i) + 1\}(\text{mod } 4), \quad \text{for } 1 \leq i \leq n-2$$
$$f\,(e'_{n-1}) = 2$$
$$f\,(e'_n) = 3$$

$$f\,(v_i) = \{C(e'_i) + 1\}(\text{mod } 4), \quad \text{for } 1 \leq i \leq n-2$$
$$f\,(v_{n-1}) = 4$$
$$f\,(v_n) = 2$$

$$f\,(e_i) = C(u_i), \quad \text{for } 1 \leq i \leq n$$

Case 3: $n \equiv 2(\text{mod } 4)$

$$f\,(u_i) = \begin{cases} 1, \text{ if } i \equiv 1(\text{mod } 4) \\ 2, \text{ if } i \equiv 2(\text{mod } 4) & \text{for } 1 \leq i \leq n-1 \\ 3, \text{ if } i \equiv 3(\text{mod } 4) \\ 4, \text{ if } i \equiv 0(\text{mod } 4) \end{cases}$$
$$f\,(u_n) = 4$$

$$f\,(e'_i) = \begin{cases} \{C(u_i) + 1\}(\text{mod } 4), & \text{for } 1 \leq i \leq n-1 \\ 3, \text{ for } i = n \end{cases}$$

$$f\,(v_i) = \begin{cases} \{C(e'_i) + 1\}(\text{mod } 4), & \text{for } 1 \leq i \leq n-1 \\ 2, \text{ for } i = n \end{cases}$$

$$f\,(e_i) = C(u_i), \quad \text{for } 1 \leq i \leq n$$

Case 4: $n \equiv 3(\text{mod } 4)$

$$f\,(u_i) = \begin{cases} 1, \text{ if } i \equiv 1(\text{mod } 4) \\ 2, \text{ if } i \equiv 2(\text{mod } 4) & \text{for } 1 \leq i \leq n-1 \\ 3, \text{ if } i \equiv 3(\text{mod } 4) \\ 4, \text{ if } i \equiv 0(\text{mod } 4) \end{cases}$$

$$f\left(u_n\right)=4$$

$$f\left(e_i'\right)=\begin{cases}\{C(u_i)+1\}(\bmod\ 4), & \text{for } 1\le i\le n-1\\ 3,\ \text{for } i=n\end{cases}$$

$$f\left(v_i\right)=\begin{cases}\{C(e_i')+1\}(\bmod\ 4), & \text{for } 1\le i\le n-1\\ 1,\ \text{for } i=n\end{cases}$$

$$f\left(e_i\right)=C(u_i), \quad \text{for } 1\le i\le n$$

Based on the above mehod of coloring, we observe that S_n is equitably total colorable with 4 colors, such that its color classes are $T\left(S_n\right)=\{T_1,T_2,T_3,T_4\}$. Clearly these color classes T_1,T_2,T_3,T_4 are independent sets of S_n with no vertices and edges in common and satisfies $||T_i|-|T_j||\le 1$, for $i\ne j$. For example consider the case $n\equiv 0(\bmod\ 4)$ (See Figure 1), in this $|T_1|=|T_2|=|T_3|=|T_4|=n$ which implies $||T_i|-|T_j||\le 1$, for $i\ne j$ and so it is equitably total colorable with 4 colors. Hence $\chi''_{=}\left(S_n\right)\le 4$. Since $\Delta=3$, we have $\chi''_{=}\left(S_n\right)\ge\chi''\left(S_n\right)\ge\Delta+1\ (=4)$. Therefore $\chi'_{=}\left(S_n\right)=4$. Similarly this is true for all other cases. Hence f is an equitable total 4-coloring of S_n. □

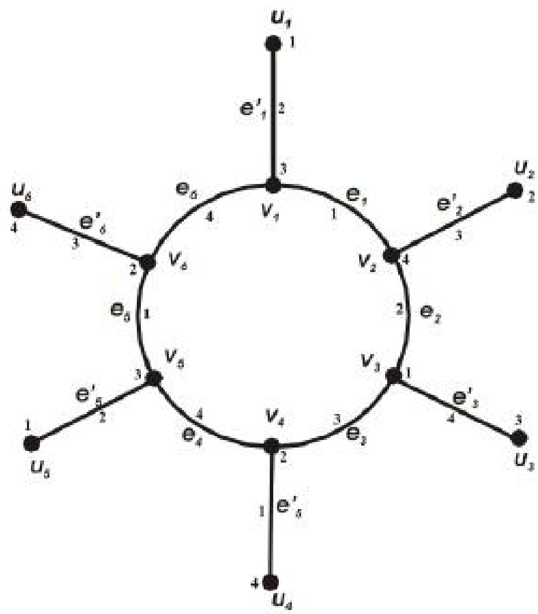

Figure 1: Sunlet S_6.

Algorithm : Equitable total coloring of Sunlet graph
Input: n, the number of vertices of S_n
Output: Equitably total colored S_n

Initialize S_n with $2n$ vertices, the rim vertices by $v_1, v_2, v_3, \ldots, v_n$ and pendant vertices by $u_1, u_2, u_3, \ldots, u_n$.

Initialize the adjacent edges on the rim by $e_1, e_2, e_3, \ldots, e_n$ and pendant edges by $e'_1, e'_2, e'_3, \ldots, e'_n$.

Let f be the coloring of vertices and edges in S_n such that $f : S \rightarrow \{1, 2, 3, 4\}$ where $S = V(S_n) \cup E(S_n)$.

Apply the coloring rules of Theorem 3.1 for each of the following cases if ($n \equiv 0 \bmod 4$)

for $i = 1$ to n
{
$e'_i = \{C(u_i) + 1\} (\bmod\ 4);$
$v_i = \{C(e'_i) + 1\} (\bmod\ 4);$
$e_i = C(u_i);$
}
end for
if ($n \equiv 1 \bmod\ 4$)
for $i = 1$ to $n - 2$
{
if ($i = n - 1$)
$u_i = 1;$
if ($i = n$)
$u_i = 4;$
$e'_i = \{C(u_i) + 1\} (\bmod\ 4);$
if ($i = n - 1$)
$e'_i = 2;$
if ($i = n$)
$e'_i = 3;$
$v_i = \{C(e'_i) + 1\} (\bmod\ 4);$
if ($i = n - 1$)
$v_i = 4;$
if ($i = n$)
$v_i = 2;$
}
end for
for $i = 1$ to n
{
$e_i = C(u_i);$
}
end for
if ($n \equiv 2 \bmod\ 4$)
for $i = 1$ to $n - 1$
{
if ($i = n$)
$u_i = 4;$
$e'_i = \{C(u_i) + 1\} (\bmod\ 4);$
if ($i = n$)
$e'_i = 3;$

$v_i = \{C(e_i') + 1\}(\text{mod } 4)$;
if $(i = n)$
$v_i = 2$;
}
end for
for $i = 1$ to n
{
$e_i = C(u_i)$;
}
end for
if $(n \equiv 3 \text{ mod } 4)$
for $i = 1$ to $n - 1$
{
if $(i = n)$
$u_i = 4$;
$e_i' = \{C(u_i) + 1\}(\text{mod } 4)$;
if $(i = n)$
$e_i' = 3$;
$v_i = \{C(e_i') + 1\}(\text{mod } 4)$;
if $(i = n)$
$v_i = 1$;
}
end for
for $i = 1$ to n
{
$e_i = C(u_i)$;
}
end for
return f ;

Theorem 3.2. *For Wheel graph W_n with $n \geq 4$, $\chi''_{=}(W_n) = n$.*

Proof. The Wheel graph W_n consists of n vertices and $2(n-1)$ edges.

$$\text{Let } V(W_n) = \{v_0\} \bigcup \{v_i : 1 \leq i \leq n-1\} \textit{ and}$$

$$E(W_n) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e_i' : 1 \leq i \leq n-1\}$$

where e_i is the edge $v_0 v_i \, (1 \leq i \leq n-1)$ and e_i' is the edge $v_i v_{i+1} \, (1 \leq i \leq n-1)$.

We define an equitable total coloring f, such that $f : S \to C$ where $S = V(W_n) \cup E(W_n)$ and $C = \{1, 2, \ldots, n\}$. In this coloration, $C(e_i)$

means the color of the i^{th} edge e_i and when the value mod n is equal to 0 it is replaced by n. The equitable total coloring is obtianed by coloring the vertices and edges as follows:

$$f\left(v_0\right) = 1$$

$$f\left(v_1\right) = n$$

$$f\left(v_i\right) = i, \text{ for } 2 \leq i \leq n-1$$

$$f\left(e_i\right) = i+1, \text{ for } 1 \leq i \leq n-1$$

$$f\left(e_i'\right) = \begin{cases} \{C(e_i)+2\}(\text{mod } n), & \text{for } 1 \leq i \leq n-2 \\ 3, \text{ for } i = n-1 \end{cases}$$

It is clear from the above rule of coloring W_n is equitably total colorable with n colors. The color class of W_n are grouped as $T\left(W_n\right) = \{T_1, T_2, \ldots, T_n\}$, which are independent sets with no vertices and edges in common and $||T_i| - |T_j|| \leq 1$, for any $i \neq j$. For example consider the case $n = 7$ (See Figure 2), for which $|T_1| = |T_2| = 2$ and $|T_3| = |T_4| = |T_5| = |T_6| = |T_7| = 3$, such that it satisfies the condition $||T_i| - |T_j|| \leq 1$, for $i \neq j$. So it is equitably total colorable with n colors. Hence $\chi''_{=}\left(W_n\right) \leq n$. Further, since $\Delta = n - 1$, we have $\chi''_{=}\left(W_n\right) \geq \chi''\left(W_n\right) \geq \Delta + 1\ (= n)$. Therefore $\chi''_{=}\left(W_n\right) = n$. Similarly it holds the inequality $||T_i| - |T_j|| \leq 1$ if $i \neq j$ for all other values of $n \geq 4$. Hence $\chi'_{=}\left(W_n\right) = n$. $\square$

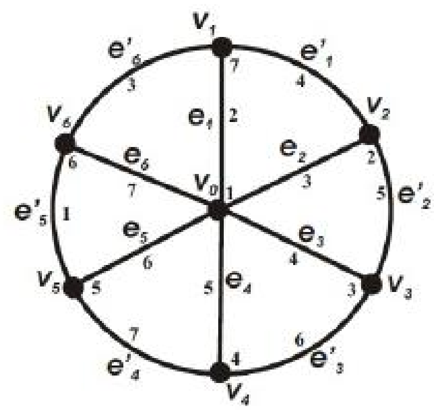

Figure 2: Wheel W_7.

Algorithm : Equitable total coloring of Wheel graph
Input: n, the number of vertices of W_n
Output: Equitably total colored W_n

Initialize W_n with n vertices, the center vertices by v_0 and rim vertices by $v_1, v_2, v_3, \ldots, v_{n-1}$.

Initialize the adjacent edges on the center by $e_1, e_2, e_3, \ldots, e_{n-1}$ and adjacent edges on the rim by $e'_1, e'_2, e'_3, \ldots, e'_{n-1}$.

Let f be the coloring of vertices and edges in W_n such that $f : S \rightarrow \{1, 2, \ldots, n\}$ where $S = V(W_n) \cup E(W_n)$.

Apply the coloring rules of Theorem 3.2 for each of the following cases

for $i = 0$ to $n - 1$
{
if $(i = 0)$
$v_i = 1;$
if $(i = 1)$
$v_i = n;$
else
$v_i = i;$
}

end for

for $i = 1$ to $n - 1$
{
$e_i = i + 1$;
if $(i = n - 1)$
$e_i' = 3$;
else
$e_i' = \{C(e_i) + 2\}(\text{mod } n)$;
}
end for
return f ;

Theorem 3.3. *For Helm graph H_n with $n \geq 4$, $\chi''_{=}(H_n) = n$.*

Proof. The Helm graph H_n consists of $2n - 1$ vertices and $3(n - 1)$ edges.

Let $V(H_n) = \{v_0\} \bigcup \{v_i : 1 \leq i \leq n - 1\} \bigcup \{u_i : 1 \leq i \leq n - 1\}$ and

and $E(H_n) = \{e_i : 1 \leq i \leq n - 1\} \bigcup \{e_i' : 1 \leq i \leq n - 2\} \bigcup \{e_{n-1}'\} \bigcup \{e_i'' : 1 \leq i \leq n - 1\}$
where e_i is the edge $v_0 v_i$ $(1 \leq i \leq n - 1)$, e_i' is the edge $v_0 v_{i+1}$ $(1 \leq i \leq n - 2)$, e_{n-1}' is the edge $v_{n-1} v_1$ and e_i'' is the edge $v_i u_i$ $(1 \leq i \leq n - 1)$.

Define a function $f : S \to C$ where $S = V(H_n) \cup E(H_n)$ and $C = \{1, 2, \ldots, n\}$. The equitable total coloring pattern is as follows:

$$f(v_0) = 1$$

$$f(v_1) = n - 1$$

$$f(v_2) = n$$

$$f(v_i) = i - 1, \text{ for } 3 \leq i \leq n - 1$$

$$f(e_i) = i + 1, \text{ for } 1 \leq i \leq n - 1$$

$$f(e_i') = \begin{cases} i + 3(\text{mod } n), \text{ for } 1 \leq i \leq n - 2 \\ 3, \quad \text{for } i = n - 1 \end{cases}$$

$$f\left(e_i''\right) = \begin{cases} i + 4(\text{mod } n), \text{ for } 1 \leq i \leq n-2 \\ 4, \quad \text{for } i = n-1 \end{cases}$$

$$f\left(u_i'\right) = i, \text{ for } 1 \leq i \leq n-1$$

With this pattern we can equitably total color the graph H_n with n colors. The color classes of H_n are grouped as $T\left(H_n\right) = \{T_1, T_2, \ldots, T_n\}$ which are independent sets and satisfies the condition $||T_i| - |T_j|| \leq 1$, $i \neq j$. For example consider the case $n = 7$ (See Figure 3), for which $|T_1| = |T_2| = |T_3| = |T_7| = 4$ and $|T_4| = |T_5| = |T_6| = 5$. This implies $||T_i| - |T_j|| \leq 1$, for $i \neq j$ and so it is equitably total colorable with n colors. Hence $\chi''_{=}\left(H_n\right) \leq n$. Since $\Delta = n - 1$, we have $\chi''_{=}\left(H_n\right) \geq \chi''\left(H_n\right) \geq \Delta + 1\ (= n)$. Therefore $\chi''_{=}\left(H_n\right) = n$. Similarly this is true for all other values of $n \geq 4$. Hence $\chi''_{=}\left(H_n\right) = n$. □

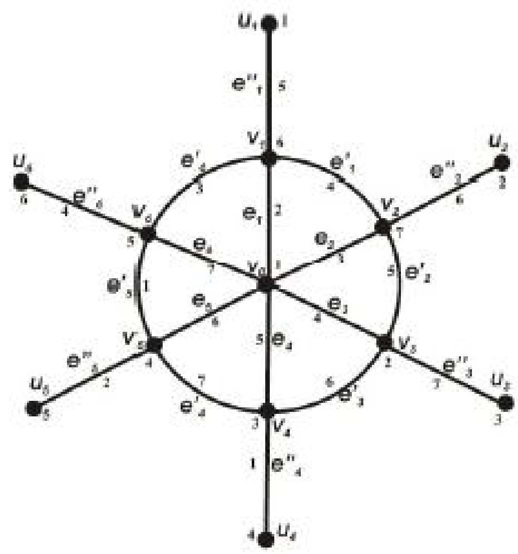

Figure 3: Helm H_7.

Algorithm : Equitable total coloring of Helm graph
Input: n, the number of vertices of H_n
Output: Equitably total colored H_n

Initialize H_n with $2n-1$ vertices, the center vertices by v_0, the rim vertices by $v_1, v_2, v_3, \ldots, v_{n-1}$ and the pendant vertices by $u_1, u_2, u_3, \ldots, u_{n-1}$.

Initialize the $3(n-1)$ edges, the adjacent edges on the center by $e_1, e_2, e_3, \ldots, e_{n-1}$, the adjacent edges on the rim by $e'_1, e'_2, e'_3, \ldots, e'_{n-1}$ and the pendant edges by $e''_1, e''_2, e''_3, \ldots, e''_{n-1}$.

Let f be the coloring of vertices and edges in H_n such that $f : S \rightarrow \{1, 2, \ldots, n\}$ where $S = V(H_n) \cup E(H_n)$.

Apply the coloring rules of Theorem 3.3 for each of the following cases

for $i = 0$ to $n-1$
{
if $(i = 0)$
$v_i = 1;$
if $(i = 1)$
$v_i = n-1;$
if $(i = 2)$
$v_i = n;$
else
$v_i = i-1;$
}
end for

for $i = 1$ to $n-1$
{
$u_i = i;$
$e_i = i+1;$
if $(i = n-1)$
$e'_i = 3;$
else
$e'_i = i + 3 (\text{mod } n);$
if $(i = n-1)$
$e''_i = 4;$

else
$e''_i = i + 4(\text{mod } n)$;
}
end for
return f ;

Theorem 3.4. *For Gear graph G_n with $n \geq 4$, $\chi''_{=}(G_n) = n$.*

Proof. The Gear graph G_n consists of $2n-1$ vertices and $3(n-1)$ edges. Let $V(G_n) = \{v_0\} \bigcup \{v_i : 1 \leq i \leq n-1\} \bigcup \{v'_i : 1 \leq i \leq n-1\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e'_i : 1 \leq i \leq n-1\} \bigcup \{e''_i : 1 \leq i \leq n-2\} \bigcup \{e''_{n-1}\}$ where e_i is the edge $v_0 v_i \, (1 \leq i \leq n-1)$, e'_i is the edge $v_i v'_i \, (1 \leq i \leq n-1)$, e''_i is the edge $v'_i v_{i+1} \, (1 \leq i \leq n-2)$ and e'_{n-1} is the edge $v'_{n-1} v_1$.

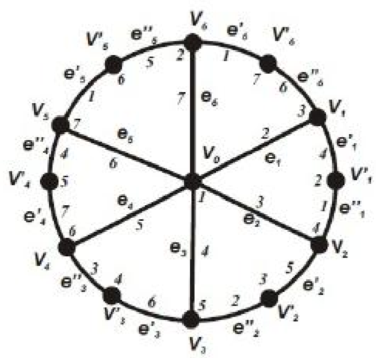

Figure 4: Gear G_7.

Define a function $f : S \to C$ where $S = V(G_n) \cup E(G_n)$ and $C = \{1, 2, \ldots, n\}$. The coloring pattern is as follows:

$$f(v_0) = 1$$

$$f(v_i) = \begin{cases} i + 2(\text{mod } n), & \text{for } 1 \leq i \leq n-2 \\ 2, & \text{for } i = n-1 \end{cases}$$

$$f(v'_i) = i + 1, \text{ for } 1 \leq i \leq n-1$$

$$f\left(e_i\right) = i+1, \text{ for } 1 \le i \le n-1$$

$$f\left(e_i'\right) = \begin{cases} C(e_i) + 2(\text{mod } n), \text{ for } 1 \le i \le n-2 \\ 1, \quad \text{for } i = n-1 \end{cases}$$

$$f\left(e_i''\right) = i, \ 1 \le i \le n-1$$

Based on the above procedure, the graph G_n is equitably total colored with n colors and by sustituting differnet values for n, it is inferred that no adjacent vertices and edges receives the same color. The color classes can be classified as $T\left(G_n\right) = \{T_1, T_2, \ldots, T_n\}$ and satisfies $||T_i| - |T_j|| \le 1$, for any $i \ne j$. For example consider the case $n = 7$ (See Figure 4), for which $|T_1| = |T_2| = |T_3| = |T_7| = 4$ and $|T_4| = |T_5| = |T_6| = 5$. This implies $||T_i| - |T_j|| \le 1$, for $i \ne j$ and so it is equitably total colorable with n colors. Hence $\chi''_{=}\left(G_n\right) \le n$. Further, since $\Delta = n-1$, we have $\chi''_{=}\left(G_n\right) \ge \chi''\left(G_n\right) \ge \Delta + 1\ (= n)$. Therefore $\chi''_{=}\left(G_n\right) = n$. □

Algorithm : Equitable edge coloring of Gear graph

Input: n, the number of vertices of G_n

Output: Equitably edge colored G_n

Initialize G_n with $2n-1$ vertices, the center vertices by v_0, the rim vertices by $v_1, v_2, v_3, \ldots, v_{n-1}$ and $v_1', v_2', v_3', \ldots, v_{n-1}'$.

Initialize the $3(n-1)$edges, the adjacent edges on the center by

$e_1, e_2, e_3 \ldots, e_{n-1}$, the adjacent edges on the rim by $e_1', e_2', e_3', \ldots, e_{n-1}'$ and $e_1'', e_2'', e_3'', \ldots, e_{n-1}''$.

Let $f : S \to \{1, 2, \ldots, n\}$ where $S = V\left(G_n\right) \cup E\left(G_n\right)$.

Apply the coloring rules of Theorem 3.4 for each of the following cases

for $i = 0$ to n
{
if $(i = 0)$

$v_i = 1;$
if $(i = n - 1)$
$v_i = 2;$
else
$v_i = i + 2;$
}
end for

for $i = 1$ to $n - 1$
{
$v'_i = i + 1;$
$e_i = i + 1;$
if $(i = n - 1)$
$e'_i = 1;$
else
$e'_i = [C(e_i) + 2] \pmod{n};$
$e''_i = i;$
}
end for
return f ;

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