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On the toral rank conjecture and some consequences

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Abstract

The aim of this work is to improve the lower bound of the Puppe inequality. His theorem [15, Theorem 1.1] states that the sum of all Betti numbers of a well-behaved space X is at least equal to $2n$, where n is rank of an n-torus T^n acting almost freely on X.

1. Introduction

The well-known Halperin conjecture [8, p. 271] about torus actions on topological spaces is behind many works in mathematics like the Hilali conjecture [12] and the inequality of Puppe [15 Theorem 1.1]: If X is a space on which an n-torus acts, we say the action is almost-free if each isotropy subgroup is finite. The largest integer n ≥ 1 for which X admits an almost free n-torus is called the toral rank of X and denoted $rk(X)$. If X does not admit any almost free torus action, then $rk(X) = 0$. Unfortunately $rk(X)$ is not a homotopy invariant and is quite difficult to compute. To obtain a homotopy invariant, we introduce the rational toral rank, $rk_0(X)$ that is, the maximum of $rk(Y)$ among all finite CW complexes Y in the same rational homotopy type as X.

Conjecture (The Toral rank conjecture).
If X is simply connected, then $\dim H^*(X;\mathbf{Q}) \geq 2^{rk_0(X)}$.

Conjecture (The Hilali conjecture).
If X is elliptic and simply connected, then $\dim(\pi_*(X) \bigotimes \mathbf{Q}) \leq \dim(H^*(X;\mathbf{Q}))$.

Theorem 1.1 (Puppe Inequality).
If X is simply connected, then $\dim H^*(X;\mathbf{Q}) \geq 2rk_0(X)$.

In the present paper we give in section 2 an optimised proof of the theorem 1.1, and in section 3 we establish a new theorem with an improvement of the lower bound of the Puppe inequality.

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2. New proof of the theorem 1.1

Let X be a simply connected topological space with an almost free T^n-action. We denote $rk_0(X) = n$ and we suppose that $n \neq 0$.

The Sullivan minimal model of the classifying space B_{T^n} of the Lie group T^n is a polynomial ring denoted here R, with the following form:

$R = (\Lambda(t_1, \cdots, t_n), 0)$ with $deg(t_i) = 2$.

The associated Borel fibration [5, p. 53] of this action is:

$$X \longrightarrow X_{T^n} \longrightarrow B_{T^n}$$

According to Brown [6], there exists a complex of differential R-modules $(R \otimes H^*(X;\mathbf{Q}), \Delta)$ with a quasi-isomorphism of R-modules :

$$\varrho : (R \otimes (H^*(X;\mathbf{Q}), \Delta) \rightarrow A_{PL}(X_{T^n})$$

Let $\beta = \{\alpha_1, \cdots, \alpha_p, \alpha_{p+1}, \cdots, \alpha_{2p}\}$ be a basis of $H^*(X, \mathbf{Q})$ such that:

$$|\alpha_i| \quad \text{is odd for :} \quad 1 \leq i \leq p$$

$$|\alpha_i| \text{ is even for :} \quad p+1 \leq i \leq 2p$$

The differential Δ can be written for i, $1 \leq i \leq p$

$$\Delta(1 \otimes \alpha_i) = P_i \otimes 1 + \sum_{j=1}^{p} t_{ij}\ \alpha_{j+p}$$

where P_i is a homogeneous polynomial in $(t_1, \cdots, t_n)$.

Lets consider the $p \times p$ matrix over R, $M = (t_{ij})_{1 \leq i,j \leq p}$, and lets denote

$$I = \{\sum_{j=1}^{p} a_i P_i / M \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = 0\} \tag{2.1}$$

I is an ideal of R, and we have the following diagram:

$$\begin{array}{ccc} (R,0) & \xrightarrow{i} & (R \otimes H^*(X,\mathbf{Q}), \Delta) \\ \downarrow p & & \overline{i} \nearrow \\ (\overline{R},0) & & \end{array}$$

where i is the canonical injection and p is the canonical surjection, $\overline{i}$ is induced by passing i to the quotient.

By passing to cohomology the diagram hereinabove induces the following diagram:

$$\begin{array}{ccc} (R,0) & \xrightarrow{i^*} & H(R \otimes H^*(X,\mathbf{Q}), \Delta) \\ \downarrow & & \overline{i^*} \nearrow \\ (\overline{R},0) & & \end{array}$$

$\overline{i^*}$ is injective because if $\overline{a} \in \overline{R}$ such that $\overline{i^*}(\overline{a}) = 0 = i^*(a)$ then

$$\begin{aligned} a \otimes 1 &= \Delta\left(\sum_{i=1}^{p} a_i\alpha_i\right) \\ &= \sum_{i=1}^{p} a_iP_i \otimes 1 + \sum_{i=1}^{p} a_i\left(\sum_{i=1}^{p} t_{ij}\alpha_{j+p}\right) \\ &= \sum_{i=1}^{p} a_iP_i \otimes 1 + \sum_{i=1}^{p} a_i\left(\left(\sum_{i=1}^{p} a_it_{ij}\right)\alpha_{j+p}\right) \end{aligned}$$

Then $a = \sum_{i=1}^{p} a_i\, t_{ij}$ and $\sum_{i=1}^{p} at_{ij} = 0\ \forall i, 1 \le i \le p$.

Hence $a \in I$ so $\overline{a} = 0$.

Therefore $\dim \overline{R} \le \dim H^*(X_{T^n}; \mathbf{Q}) < \infty$.

And $\dim \overline{R} \ge n$ since $\{\overline{t_1}, \cdots, \overline{t_n}\}$ is a free family in $\overline{R}$.

In another hand since M is a square matrix of order p over the ring R then I is a free R-module of dimension $N \le p$.

We have $\dim R/I < \infty$ so $\dim I = N \ge n$ witch implies that $\dim H^*(X, \mathbf{Q}) \ge 2n$.

3. An improvement of the lower bound of Puppe inequality

3.1. The main theorem

Let X be a simply connected topological space with an almost free T^n-action, such that $\dim H^*(X; \mathbf{C}) < \infty$.

The associated Borel fibration [5, p. 53] of this action is:

$$X \longrightarrow X_{T^n} \longrightarrow B_{T^n}$$

X_{T^n} has the Hirsch-Brown minimal model
$D_{T^n} = (H^*(B_{T^n}; \mathbf{C}) \otimes H^*(X; \mathbf{C}); \tilde{d})$ as a $H^*(B_{T^n}; \mathbf{C})$-Module [2, Section 1.3].

We define an increasing filtration F_q on $H^*(X; \mathbf{C})$ by:
$F_{-1} = 0$
$F_q = (\tilde{d}(x)|_{H^*(X;\mathbf{C})})^{-1}(H^*(B_{T^n}; \mathbf{C}) \otimes F_{q-1})$

The length $l(X)$ of $H^*(X, \mathbf{C})$ is defined by :

$$l = \inf\left\{q \in \mathbf{N}/F_q = H^*(X, \mathbf{Q})\right\}.$$

We use the evaluation at $\alpha = (\alpha_1, \cdots, \alpha_n)$ to define the new space:

$$D_{T^n}(X)^\alpha = \mathbf{C} \otimes_{H^*(B_{T^n};\mathbf{C})} D_{T^n}(X)$$

where the structure of this $H^*(X;\mathbf{C})$-module is defined by the map:

$$\begin{array}{ccc} H^*(B_{T^n};\mathbf{C}) & \rightarrow & \mathbf{C} \\ t_i & \mapsto & \alpha_i \end{array}$$

we know by [3, theorem 4-1], that for $\alpha \neq (0, \cdots, 0)$ we have:
$H^*(D_{T^n}(X)^\alpha, \tilde{d}_\alpha) = 0$.

The coboundary $\tilde{d}_\alpha$ is given by $\tilde{d}$ evaluated at $\alpha \in \mathbf{C}$ ([2, p. 26]),

Now for every $q, 0 \leq q \leq l$, we define A_q to be a complement of F_{q-1} in F_q :

$$F_q = A_q \oplus F_{q-1}$$

The main result of this article is an improvement of the lower bound of Puppe inequality expressed in the following theorem:

Theorem 3.1. *Let X be a simply connected topological space with an almost free T^n-action, we denote $n = rk_0(X)$ for $n \geq 4$ we always have* $\dim H^*(X;\mathbf{Q}) \geq 3n - 2$

The proof of this theorem is based on the following lemma:

Lemma 3.2. *Under the same conditions as the theorem above one has:* $\dim A_1 \geq n$.

Definition 3.3.[17,vol 1, p. 90] *By $\mathbf{P}^n$ we denote n-dimensional projective space over $\mathbf{C}$. A projective algebraic variety V is an algebraic subset of $\mathbf{P}^n$, that is, the zero-set of some homogeneous polynomials $f_i, i \in I$, in the homogeneous coordinates $(x_0, \cdots, x_n)$ of $\mathbf{P}^n$: $V = \{(x_0, \cdots, x_n) | f_i(x_0, \cdots, x_n) = 0, i \in I\}$.*

Proof of the lemma 3.2 According to Puppe [16, p. 7], we know that $l \geq n$, and $\dim A_q \geq 2$, for every q, $1 \leq q \leq l - 1$.

Let $\{a_1, \cdots, a_r\}$ and $\{b_1, \cdots, b_s\}$ be two bases of A_1 and A_0. For each i, $1 \leq i \leq r$ we can write:

$$\tilde{d}(a_i) = p_i b_1 + \omega_i;$$

where p_i are homogeneous polynomials on $t_1, \cdots, t_n$ and ω_i is a linear composition of $b_2, \cdots, b_s$ over $\mathbf{Q}[t_1, \cdots, t_n]$. If we suppose that $r < n$, then the

algebraic variety $V(p_1, \cdots, p_r)$ is different from $\{(0, \cdots, 0)\}$. Hence we can take $\alpha \in V(p_1, \cdots, p_r) \backslash \{(0, \cdots, 0)\}$ such that:
$\tilde{d}_\alpha b_1 = 0$ (because $F_0 = A_0 = ker(\tilde{d})$).
$\tilde{d}_\alpha a_i = \omega_i$ for $1 \leq i \leq r$.

This shows us that the $\tilde{d}_\alpha$- cocyle $b_1$is not a zero in $H^*(D_{T^n}(X)^\alpha, \tilde{d}_\alpha)$ which is absurd.

Proof of the Theorem 3.1 Let's denote $m_q = \dim A_q, 0 \leq q \leq l$. Then we have $\dim H^*(X; \mathbf{Q}) = \sum_{q=0}^{l} m_q$.

One has:
$\dim H^*(X; \mathbf{Q}) \geq m_0 + m_1 + m_l + \sum_{q=2}^{l-1} m_q$
$\geq 2 + n + 2(n-1)$.
$\geq 3n - 2$.

3.2. Examples

Remark 3.4. The theorem 3.1 gives a measurement of the obstruction of a manifold to have an almost free T^n-action, for example a compact simply connected manifold M with the sum of it's Betti numbers $< 3n - 2$ can't have an almost free T^n-action.

Example 3.5. The toral rank of the manifold $M = (\mathbf{S}^{2n+1})^r$ is equal to r and the sum of it's Betti numbers is equal to 2^r [8, p. 284], We have $\dim H^*((\mathbf{S}^{2n+1})^4; \mathbf{Q}) = 16 < 3 \times 7 - 2 = 19$ so $(\mathbf{S}^{2n+1})^4$ can't have an almost free T^7-action.

Remark 3.6. In 2012 M. Amann [4, Theorem A] established the following result:

Theorem A. *If an n-torus T acts almost freely on a finite-dimensional paracompact Hausdorf space X, then* $\dim H^*(X; \mathbf{Q}) \geq 2(n + [n/3])$

X may be taken to be a finite CW-complex or a compact manifold.

It's clear that starting from n=7 the theorem 3.1 gives a greater lower bound than theorem A .

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