

An equivalence in generalized almost-Jordan algebras

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Abstract

In this paper we work with the variety of commutative algebras satisfying the identity $\beta((x^2y)x - ((yx)x)x) + \gamma(x^3y - ((yx)x)x) = 0$, where β, γ are scalars. They are called generalized almost-Jordan algebras. We prove that this variety is equivalent to the variety of commutative algebras satisfying $(3\beta + \gamma)(G_y(x, z, t) - G_x(y, z, t)) + (\beta + 3\gamma)(J(x, z, t)y - J(y, z, t)x) = 0$, for all $x, y, z, t \in A$, where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ and $G_x(y, z, t) = (yz, x, t) + (yt, x, z) + (zt, x, y)$. Moreover, we prove that if A is a commutative algebra, then $J(x, z, t)y = J(y, z, t)x$, for all $x, y, z, t \in A$, if and only if A is a generalized almost-Jordan algebra for $\beta = 1$ and $\gamma = -3$, that is, A satisfies the identity $(x^2y)x + 2((yx)x)x - 3x^3y = 0$ and we study this identity. We also prove that if A is a commutative algebra, then $G_y(x, z, t) = G_x(y, z, t)$, for all $x, y, z, t \in A$, if and only if A is an almost-Jordan or a Lie Triple algebra.

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1. Introduction

In this work, F is a field of $\text{char} F \neq 2$ and A be a commutative not necessarily associative algebra over F .

The algebra A is called Jordan algebra if satisfies $(y^2, x, y) = 0$, for all $y, x \in A$. For properties of these algebras see [10]. It is know, see Osborn [7], that a Jordan algebra satisfies the identity

$$(1.1) \quad 3(x^2y)x - 2((yx)x)x - x^3y = 0.$$

Algebras satisfying identity (1.1), called *Lie Triple algebras or almost-Jordan algebras* have been studied by Hentzel, Peresi, Osborn, Peterson and Sidorov [5, 7, 8, 9, 11].

Identity (1.1) was generalized in 1988 by Carini, Hentzel and Piacentini-Cattaneo, see [3]. After that, Arenas and Labra call them generalized almost-Jordan algebras, see [1].

We say that A is a *generalized almost-Jordan algebra* if it satisfies:

$$(1.2) \quad \beta \left((x^2y)x - ((yx)x)x \right) + \gamma \left(x^3y - ((yx)x)x \right) = 0,$$

for all $x, y \in A$, where $\beta, \gamma \in F$ and $(\beta, \gamma) \neq (0, 0)$.

In the study of degree four identities not implied by commutativity, Osborn [8] classified those that were implied by the fact of possessing a unit element. Carini, Hentzel and Piacentini-Cattaneo [3] extended this work by dropping the restriction on the existence of the unit element. The identity defining a generalized almost-Jordan algebra with $\beta, \gamma \in F$ appears as one of these identities.

We have:

$$(x^2, y, x) = (x^2y)x - x^2(yx), (x^2, x, y) = x^3y - x^2(yx), (yx, x, x) = ((yx)x)x - (yx)x^2,$$

so

$$(x^2, y, x) - (yx, x, x) = (x^2y)x - ((yx)x)x, (x^2, x, y) - (yx, x, x) = x^3y - ((yx)x)x$$

and

$$0 = \beta \left((x^2y)x - ((yx)x)x \right) + \gamma \left(x^3y - ((yx)x)x \right) = \beta \left((x^2, y, x) - (yx, x, x) \right) + \gamma \left((x^2, x, y) - (yx, x, x) \right)$$

Therefore, in terms of associators a generalized almost-Jordan algebra satisfies,

$$(1.3) \quad \beta(x^2, y, x) + \gamma(x^2, x, y) = (\beta + \gamma)(yx, x, x)$$

If $\beta = 3$ and $\gamma = -1$, we obtain an almost-Jordan algebra, that is, A satisfies

$$3(x^2, y, x) = (x^2, x, y) + 2(yx, x, x).$$

Generalized almost-Jordan algebras A have been studied in [3] where the authors proved that for almost all the algebras, simplicity implies associativity, in [1], where the authors proved that these algebras always have a trace form in terms of the trace of right multiplication operators. They also prove that if A is finite-dimensional and solvable, then it is nilpotent. In [2] the author found the Wedderburn decomposition of A assuming that for every ideal I of A either I has a non zero idempotent or $I \subset R, R$ the solvable radical of A and the quotient A/R is separable, in [4] the authors give a characterization of representations and irreducibles modules of these algebras, and in [6] where, assuming that A also satisfies $((xx)x)x = 0$ the authors proved the existence of an ideal I of A such that $AI = IA = 0$ and the quotient algebra A/I is power-associative.

In this paper we prove the equivalence between generalized almost-Jordan algebras, and commutative algebras satisfying the identity $(3\beta + \gamma)(G_y(x, z, t) - G_x(y, z, t)) + (\beta + 3\gamma)(J(x, z, t)y - J(y, z, t)x) = 0$, for all $x, y, z, t \in A$, where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ and $G_x(y, z, t) = (yz, x, t) + (yt, x, z) + (zt, x, y)$, Theorem 3.2. We prove that a Jordan algebra satisfies the identity $G_x(y, z, t) = 0$ for all $x, y, z, t \in A$. Conversely if $\text{char}F \neq 3$, then every commutative algebra $G_x(y, z, t) = 0$ for all $x, y, z, t \in A$ is a Jordan algebra, Proposition 3.1. Moreover, we prove that if A is a commutative algebra, then $J(x, z, t)y = J(y, z, t)x$ for all $x, y, z, t \in A$, if and only if A is a generalized almost-Jordan algebra for $\beta = 1$ and $\gamma = -3$, that is, A satisfies the identity $(x^2y)x + 2((yx)x)x - 3x^3y = 0$, Proposition 3.4. We also prove that if A is a commutative algebra, then $G_y(x, z, t) = G_x(y, z, t)$, for all $x, y, z, t \in A$, if and only if A is an almost-Jordan algebra, Proposition 3.5. Finally, we give some new identities, Theorem 3.13 and Proposition 3.15 for commutative algebras satisfying the identity $(x^2y)x + 2((yx)x)x - 3x^3y = 0$.

2. Preliminaries

In this section we found relationships among generalized almost-Jordan algebras and alternative algebras, Jordan algebras, baric algebras or b-algebras.

Proposition 2.1. *Let A be a commutative right alternative algebra. Then A is a generalized almost-Jordan algebra, for $\beta = \gamma = 1$.*

Proof: Since A is a right alternative algebra, then A is an alternative algebra and $(x, y, z) = -(x, z, y)$, so by (1.2) we have, $(x^2, y, x) + (x^2, x, y) = (x^2, y, x) - (x^2, y, x) = 0 = 2(yx, x, x)$. \square

If A is a F -algebra, then we will define a new algebra $A' = Fe \oplus A$, as vector space, and the multiplication given by:

$$(\alpha e + u)(\beta e + v) = \alpha\beta e + uv,$$

where e is an idempotent, $\alpha, \beta \in F$ and $u, v \in A$.

Proposition 2.2. *Let A be a generalized almost-Jordan algebra. Then A' is a generalized almost-Jordan algebra and $\omega: A' \rightarrow F$, given by $\omega(\alpha e + u) = \alpha$, is a nonzero homomorphism of algebras.*

Proof: Let $\alpha, \beta \in F, u, v \in A, x = \alpha e + u$ and $y = \beta e + v$. Since $ez = 0$ and $e^2 = e$ for all $z \in A$, then $(a, b, c) = 0$, if $a, b, c \in A \cup \{e\}$, and at least one of them is equal to e , so

$$\begin{aligned} x^2 &= \alpha^2 e + u^2, \quad yx = \alpha\beta e + vu, \\ (x^2, y, x) &= (\alpha^2 e + u^2, \beta e + v, \alpha e + u) = \alpha^3 \beta(e, e, e) + (u^2, v, u) = (u^2, v, u) \\ (x^2, x, y) &= (u^2, u, v) \quad \text{and} \quad (yx, x, x) = (vu, u, u). \end{aligned}$$

therefore

$$\beta(x^2, y, x) + \gamma(x^2, x, y) = \beta(u^2, v, u) + \gamma(u^2, u, v) = (\beta + \gamma)(vu, u, u) = (\beta + \gamma)(yx, x, x). \quad \square$$

Definition 2.3. *Let A be a F -algebra. If $\omega: A \rightarrow F$ is a nonzero algebra homomorphism, then the ordered pair (A, ω) is called a baric algebra or b-algebra. When a b-algebra (A, ω) is a generalized almost-Jordan algebra, then we call it generalized almost-Jordan b-algebra.*

Corollary 2.4. *Let A be a generalized almost-Jordan algebra. Then (A', ω) is a generalized almost-Jordan b-algebra.*

If A is a F -algebra, then we will define a new algebra $A^\# = F \oplus A$, as vector space, and the multiplication given by:

$$(\alpha + u)(\beta + v) = \alpha\beta + \alpha v + \beta u + uv,$$

where $\alpha, \beta \in F$ and $u, v \in A$, $A^\#$ has unit element $1 + 0 = 1$.

Theorem 2.5. *Let A be a generalized almost-Jordan algebra. Then $A^\#$ is a generalized almost-Jordan algebra if and only if $\beta + 3\gamma = 0$ or A is an alternative algebra.*

Proof: Let $\alpha, \beta \in F, u, v \in A, x = \alpha + u$ and $y = \beta + v$. We note that $(1, a, b) = (a, 1, b) = (a, b, 1) = 0 = (a, b, a)$, for all $a, b \in A^\#$, so

$$\begin{aligned} x^2 &= \alpha^2 + 2\alpha u + u^2, yx = \alpha\beta + \alpha v + \beta u + vu, \\ (x^2, y, x) &= (\alpha^2 + 2\alpha u + u^2, \beta + v, \alpha + u) = 2\alpha(u, v, u) + (u^2, v, u) = (u^2, v, u), \\ (x^2, x, y) &= (\alpha^2 + 2\alpha u + u^2, \alpha + u, \beta + v) = 2\alpha(u, u, v) + (u^2, u, v), \\ (yx, x, x) &= (\alpha\beta + \alpha v + \beta u + vu, \alpha + u, \alpha + u) = \alpha(v, u, u) + (vu, u, u), \\ (u, u, v) &= u^2v - u(uv) = -\left((vu)u - vu^2\right) = -(v, u, u). \end{aligned}$$

therefore

$$\begin{aligned} \beta(x^2, y, x) + \gamma(x^2, x, y) - (\beta + \gamma)(yx, x, x) &= \beta(u^2, v, u) + 2\alpha\gamma(u, u, v) + \\ \gamma(u^2, u, v) - \alpha(\beta + \gamma)(v, u, u) - (\beta + \gamma)(vu, u, u) &= 2\alpha\gamma(u, u, v) - \alpha(\beta + \\ \gamma)(v, u, u) = -2\alpha\gamma(v, u, u) - \alpha(\beta + \gamma)(v, u, u) &= -\alpha(3\gamma + \beta)(v, u, u). \end{aligned}$$

Since α is arbitrary, then the Theorem follows. \square

Corollary 2.6. *Let A be an almost-Jordan algebra. Then $A^\#$ is an almost-Jordan algebra.*

Corollary 2.7. *If A is an almost-Jordan algebra and $\omega: A^\# \rightarrow F$ is given by $\omega(\alpha + u) = \alpha$. Then $(A^\#, \omega)$ is an almost-Jordan b -algebra.*

Example 2.8. *Let F be a field of characteristic not 2 and A be a commutative F -algebra of basis $\{s, t\}$ with the multiplication:*

	s	t
s	$s + t$	$\frac{1}{2}t$
t	$\frac{1}{2}t$	0

This algebra is an almost-Jordan algebra, but is not a Jordan algebra, see [7]. Moreover, it is a b -algebra and the only idempotent is zero.

In fact, let $\omega: A \rightarrow F$ given by $\omega(as + bt) = a$, where $a, b \in F$, then

$$\omega((as + bt)(a's + b't)) = \omega\left(aa's + \frac{1}{2}(2aa' + ab' + a'b)t\right) = aa' = \omega(as + bt)\omega(a's + b't), \text{ for all } a, a', b, b' \in F,$$

so, this algebra is a b -algebra.

If $e = as + bt \in A$, such that $e^2 = e$, then

$$as + bt = a^2s + \frac{1}{2}(2a^2 + 2ab)t = a^2s + (a^2 + ab)t,$$

so $a = a^2$ and $b = a^2 + ab$, therefore, $a = b = 0$, then $e = 0$ is the only idempotent of A .

Example 2.9. Let F be a field and A be a commutative F -algebra of basis $\{x_1, x_2, x_3, x_4\}$ with the multiplication:

	x_1	x_2	x_3	x_4
x_1	x_2	x_3	0	0
x_2	x_3	x_3	0	x_3
x_3	0	0	0	0
x_4	0	x_3	0	$x_2 + x_3$

In [1] the authors prove that this algebra is a generalized almost-Jordan algebra for all $\beta, \gamma \in F$, because $(x^2y)x = ((yx)x)x = x^3y = 0$ for all $x, y \in A$. Since $(x_1, x_1, x_2) = x_1^2x_2 - x_1(x_1x_2) = x_3$, then A is not alternative algebra.

We will to prove that this algebra is not a b -algebra.

Let $\omega: A \rightarrow F$ be an algebra homomorphism, since $x_3^2 = 0, x_2^2 = x_3, x_1^2 = x_2$ and $x_4^2 = x_2 + x_3$, then $\omega(x_3) = \omega(x_2) = \omega(x_1) = \omega(x_4) = 0$, so A is not a b -algebra.

3. Main Results

Let A be a generalized almost-Jordan algebra.

Linearising (1.3) we have,

$$(3.1) \quad \begin{aligned} & \beta\left((x^2, y, z) + 2(xz, y, x)\right) + \gamma\left((x^2, z, y) + 2(xz, x, y)\right) = \\ & = (\beta + \gamma)\left((yx, x, z) + (yx, z, x) + (yz, x, x)\right) \end{aligned}$$

$$\begin{aligned}
 (3.2) \quad & 2\beta \left((tx, y, z) + (xz, y, t) + (tz, y, x) \right) + \\
 & + 2\gamma \left((tx, z, y) + (xz, t, y) + (tz, x, y) \right) = \\
 & = (\beta + \gamma) \left((yx, t, z) + (yx, z, t) + \right. \\
 & \left. + (yt, x, z) + (yt, z, x) + (yz, x, t) + (yz, t, x) \right)
 \end{aligned}$$

Let $G_x: A \times A \times A \rightarrow A$ given by

$$G_x(y, z, t) = (yz, x, t) + (yt, x, z) + (zt, x, y)$$

It is easy to see that, G_x is 3-lineal function and symmetric in every two variables. Moreover, the complete linearization of the (x^2, y, x) is $2G_x(y, z, t)$. If A is a Jordan algebra, then $G_x(y, z, t) = 0$, for all $x, y, z, t \in A$. Conversely we have.

Proposition 3.1. *Let A be a commutative algebra over a field of characteristic not 3, such that*

$$G_x(y, z, t) = 0,$$

for all $x, y, z, t \in A$. Then A is a Jordan algebra.

Proof: Setting $z = t = y$ in $G_x(y, z, t) = (yz, x, t) + (yt, x, z) + (zt, x, y) = 0$, we get $(y^2, x, y) + (y^2, x, y) + (y^2, x, y) = 0$, so $3(y^2, x, y) = 0$, then A is a Jordan algebra. \square

Theorem 3.2. *A is a generalized almost-Jordan algebra, if and only if A is a commutative algebra satisfying*

$$(3\beta + \gamma) \left(G_y(x, z, t) - G_x(y, z, t) \right) + (\beta + 3\gamma) \left(J(x, z, t)y - J(y, z, t)x \right) = 0,$$

for all $x, y, z, t \in A$, where $J(a, b, c) = (ab)c + (bc)a + (ca)b$.

Proof: By (3.2) we have,

$$\begin{aligned}
 (3.3) \quad & 2\beta G_y(x, z, t) + 2\gamma \left((tx, z, y) + (xz, t, y) + (tz, x, y) \right) = (\beta + \gamma) \left(G_t(x, y, z) + \right. \\
 & \left. G_x(y, z, t) + G_z(x, y, t) \right) - (\beta + \gamma) \left((xz, t, y) + (tz, x, y) + (tx, z, y) \right), \quad \text{so} \\
 & 2\beta G_y(x, z, t) + (\beta + 3\gamma) \left((xz, t, y) + (tz, x, y) + (tx, z, y) \right) = \\
 & (\beta + \gamma) \left(G_t(x, y, z) + G_x(y, z, t) + G_z(x, y, t) \right).
 \end{aligned}$$

In (3.3), replacing x by y and y by x , we have

$$(3.4) \quad \begin{aligned} & 2\beta G_x(y, z, t) + (\beta + 3\gamma) \left((yz, t, x) + (tz, y, x) + (ty, z, x) \right) = \\ & (\beta + \gamma) \left(G_t(x, y, z) + G_y(x, z, t) + G_z(x, y, t) \right). \end{aligned}$$

By (3.3) and (3.4),

$$\begin{aligned} & 2\beta \left(G_y(x, z, t) - G_x(y, z, t) \right) + \\ & (\beta + 3\gamma) \left((xz, t, y) + (tz, x, y) + (tx, z, y) - \right. \\ & \quad \left. - (yz, t, x) - (tz, y, x) - (ty, z, x) \right) = \\ & (\beta + \gamma) \left(G_x(y, z, t) - H_y(x, z, t) \right), \quad \text{so} \\ & (3\beta + \gamma) \left(G_y(x, z, t) - G_x(y, z, t) \right) + \\ & (\beta + 3\gamma) \left((xz, t, y) + (tz, x, y) + (tx, z, y) \right. \\ & \quad \left. - (yz, t, x) - (tz, y, x) - (ty, z, x) \right) = 0, \quad \text{but} \end{aligned}$$

$$\begin{aligned} & (xz, t, y) + (tz, x, y) + (tx, z, y) - (yz, t, x) - (tz, y, x) - (ty, z, x) = \\ & ((xz)t)y - (xz)(ty) + ((tz)x)y - (tz)(xy) + ((tx)z)y - (tx)(zy) - \\ & ((yz)t)x + (yz)(tx) - ((tz)y)x + (tz)(yx) - ((ty)z)x + (ty)(zx) = \\ & \left((xz)t + (tz)x + (tx)z \right) y - \left((yz)t + (tz)y + (ty)z \right) x = J(x, z, t)y - J(y, z, t)x, \end{aligned}$$

where $J(a, b, c) = (ab)c + (bc)a + (ca)b$. Therefore,

$$(3.5) \quad (3\beta + \gamma) \left(G_y(x, z, t) - G_x(y, z, t) \right) + (\beta + 3\gamma) \left(J(x, z, t)y - J(y, z, t)x \right) = 0.$$

Conversely, setting $z = t = x$ in (3.5) we have

$$(3\beta + \gamma) \left(G_y(x, x, x) - G_x(y, x, x) \right) + (\beta + 3\gamma) \left(J(x, x, x)y - J(y, x, x)x \right) = 0. \quad (*)$$

Then, using the definition of G_x, G_y and the commutativity of the algebra we obtain:

$$\begin{aligned} G_y(x, x, x) - G_x(y, x, x) &= 3(x^2, y, x) - 2(yx, x, x) - (x^2, x, y) \\ &= 3(x^2y)x - 3x^2(yx) - 2((yx)x)x + 2(yx)x^2 - x^3y + x^2(yx) \\ &= 3(x^2y)x - 2((yx)x)x - x^3y. \end{aligned}$$

Moreover, $J(x, z, t)y - J(y, z, t)x = 3x^3y - 2((yx)x)x - (x^2y)x$.

Replacing these values in (*) we get

$$(3\beta + \gamma) \left(3(x^2y)x - 2((yx)x)x - x^3y \right) + (\beta + 3\gamma) \left(3x^3y - 2((yx)x)x - (x^2y)x \right) = 0$$

Reordering these terms we obtain

$$8\gamma x^3y - (8\beta + 8\gamma)((yx)x)x + 8\beta(x^2y)x = 0.$$

Since characteristic of the field is different of 2 we get

$$\gamma x^3y - (\beta + \gamma)((yx)x)x + \beta(x^2y)x = 0,$$

and by identity (2), A is a generalized almost-Jordan algebra. □

In [7], Osborn introduced two mappings,

$$\begin{aligned} H(y; x, z, t) &= (y(xz))t + (y(zx))x + (y(tx))z \text{ and} \\ K(y, x, z, t) &= (xy)(zt) + (yz)(xt) + (yt)(xz), \text{ so} \end{aligned}$$

$$\begin{aligned} G_y(x, z, t) &= (xz, y, t) + (xt, y, z) + (zt, y, x) = ((xz)y)t + ((xt)y)z + \\ &+ ((zt)y)x - (xz)(yt) - (xt)(yz) - (zt)(yx) = H(y; x, z, t) - K(y, x, z, t), \end{aligned}$$

but $K(x, y, z, t) = (yx)(zt) + (xz)(yt) + (xt)(yz) = K(y, x, z, t)$, then

$$(3.6) \quad G_y(x, z, t) - G_x(y, z, t) = H(x; y, z, t) - H(y; x, z, t),$$

for all $x, y, z, t \in A$.

Corollary 3.3. *If A satisfies the identity $(x^2)^2 = x^4$ for all $x \in A$ and $\beta + 3\gamma \neq 0$, then $J(x, z, t)y = J(y, z, t)x$.*

Proof: By [7], we have $H(y; x, z, t) = H(x; y, z, t)$ for all $x, y, z, t \in A$ and by Theorem 3.2, $J(x, z, t)y = J(y, z, t)x$. □

Proposition 3.4. *Let A be a commutative algebra. Then the following identities are equivalent:*

1. $(x^2y)x + 2((yx)x)x - 3x^3y = 0$,
2. $J(x, z, t)y = J(y, z, t)x$.

Proof: Since A satisfies the identity $(x^2y)x + 2((yx)x)x - 3x^3y = 0$, then A is a generalized almost-Jordan algebra for $\beta = 1, \gamma = -3$ and $\beta + 3\gamma \neq 0$, so by (3.5), $J(x, z, t)y = J(y, z, t)x$.

Conversely, setting $z = t = x$ in $J(x, z, t)y = J(y, z, t)x$, we get $J(x, x, x)y = J(y, x, x)x$, that is $3x^3y = ((yx)x)x + (x^2y)x + ((yx)x)x$, so A satisfies the identity $(x^2y)x + 2((yx)x)x - 3x^3y = 0$. \square

Proposition 3.5. *Let A be a commutative algebra. Then A is an almost-Jordan algebra if and only if*

$$G_y(x, z, t) = G_x(y, z, t),$$

for all $x, y, z, t \in A$.

Proof: Since A is an almost-Jordan algebra, $\beta + 3\gamma = 0$ so $3\beta + \gamma \neq 0$, and by (3.5), $\left(G_y(x, z, t) - G_x(y, z, t)\right) = 0$, so $G_y(x, z, t) = G_x(y, z, t)$.

Conversely, if A satisfies the identity, $G_y(x, z, t) = G_x(y, z, t)$, then developing the associators we have

$$[(yz)x - (xz)y]t + [(yt)x - (xt)y]z + ((zt)x)y - ((zt)y)x = 0.$$

Since $(y, z, x) = (yz)x - y(zx)$ and $(y, t, x) = (yt)x - y(tx)$, we get

$$(y, z, x)t + (y, t, x)z + ((zt)x)y - ((zt)y)x = 0.$$

Replacing $(zt, x, y) = ((zt)x)y - (zt)(xy)$ and $(zt, y, x) = ((zt)y)x - (zt)(xy)$ in the above expression we obtain

$$(3.7) \quad (y, z, x)t + (y, t, x)z + (zt, x, y) - (zt, y, x) = 0.$$

Since A is a commutative algebra, then $(a, b, c) = -(c, b, a)$ and (3.7) becomes

$$(3.8) \quad (x, z, y)t + (x, t, y)z + (y, x, zt) - (x, y, zt) = 0.$$

Setting $z = t = x$ in (3.8), we obtain

$$2(x, x, y)x + (y, x, x^2) - (x, y, x^2) = 0.$$

Developing the associators and using the commutativity we get

$$3(x^2y)x - 2(x(xy))x - yx^3 = 0.$$

By identity (1.1), A is an almost-Jordan algebra. \square

By identity (3.6), we have

Corollary 3.6. *Let A be a commutative algebra. Then A is an almost-Jordan algebra if and only if*

$$H(y; x, z, t) = H(x; y, z, t),$$

for all $x, y, z, t \in A$.

By [7], we have

Proposition 3.7. *If A satisfies the identity $(x^2)^2 = x^4$ for all $x \in A$, then A is an almost-Jordan algebra.*

Remark 3.8. *The converse of the Proposition 3.7 is not true. Let A be the algebra of Example 2.8, so $s^4 = s + \frac{7}{4}t$ and $(s^2)^2 = s + 2t \neq s^4$.*

An algebra A is called *power-associative algebra* if for all $x \in A$, the subalgebra $A(x)$ of A generated by x is associative algebra.

Corollary 3.9. *If A is a commutative power-associative algebra, then A is an almost-Jordan algebra.*

By Corollaries 3.3 and 3.9, we have

Proposition 3.10. *If A is a commutative power-associative algebra and $\beta + 3\gamma \neq 0$, then A is an almost-Jordan algebra and $J(x, z, t)y = J(y, z, t)x$.*

Corollary 3.11. *If A is a commutative power-associative algebra and $\beta + 3\gamma \neq 0$, then the following identities hold,*

1. $3(x^2y)x - 2((yx)x)x - x^3y = 0,$
2. $(x^2y)x + 2((yx)x)x - 3x^3y = 0.$

for all $x, y \in A$.

Remark 3.12. *The converse of the Corollary 3.11 is not true. Let A be the algebra of Example 2.9, so A satisfies both identities, but A is not power-associative algebra, because $x_1^4 = 0$ and $(x_1^2)^2 = x_3$.*

Let A be a commutative algebra which satisfies the identities of Corollary 3.11, then $(x^2y)x = ((yx)x)x = x^3y$ for all $x, y \in A$. In this case the converse is true.

Next, let A be a commutative non necessarily power-associative algebra, so identities (1) and (2) of the above Corollary are not equivalent. Since identity (1), a Lie triple or almost-Jordan algebra has been largely studied we will study an algebra A satisfying

$$(3.9) \quad (x^2y)x + 2((yx)x)x - 3x^3y = 0,$$

for all $x, y \in A$, which is identity (2) of Corollary 3.11.

It is known (see [1]) that every finite dimensional solvable algebra satisfying (3.9) is nilpotent. If R is radical of A and A/R is solvable, then A has Wedderburn decomposition, (see [2]). Moreover, if A has an idempotent element, then $A = A_0 \oplus A_1 \oplus A_{-\frac{3}{2}}$, where $A_i = \{x \in A \mid ex = ix\}$, $i = 0, 1, -\frac{3}{2}$, is the Peirce decomposition of A . The subspaces A_i satisfies the relations, (see [2]):

$$A_0^2 \subseteq A_0, \quad A_1^2 \subseteq A_1, \quad A_0A_1 = \{0\} = A_{-\frac{3}{2}}A_0 = A_{-\frac{3}{2}}^2, \quad A_{-\frac{3}{2}}A_1 \subseteq A_{-\frac{3}{2}}.$$

In this work we give some new identities.

Substituting $y = x^k$ in (3.9), we get $(x^2x^k)x + 2x^{k+3} - 3x^3x^k = 0$, so

$$(3.10) \quad 2x^{k+3} = 3x^3x^k - (x^2x^k)x, \quad k \geq 2$$

The identity (3.9) is equivalent to

$$(3.11) \quad (x^2, y, x) + 2(yx, x, x) - 3(x^2, x, y) = 0,$$

for all $x, y \in A$.

Linearising (3.11), we have

$$(3.12) \quad 2(xz, y, x) + (x^2, y, z) + 2(yz, x, x) + 2(yx, z, x) + 2(yx, x, z) - 6(xz, x, y) - 3(x^2, z, y) = 0$$

Interchanging y and z , we have

$$2(xy, z, x) + (x^2, z, y) + 2(yz, x, x) + 2(zx, y, x) + 2(zx, x, y) - 6(xy, x, z) - 3(x^2, y, z) = 0$$

Subtracting both identities and canceling out by the factor 4, we obtain

$$(3.13) \quad (x^2, y, z) - (x^2, z, y) + 2(yx, x, z) - 2(zx, x, y) = 0,$$

so $(x^2, y, z) + 2(yx, x, z) = (x^2, z, y) + 2(zx, x, y)$, and substituting in identity (3.11), we get

$$(3.14) \quad (xz, y, x) + (yz, x, x) + (yx, z, x) - 2(xz, x, y) - (x^2, z, y) = 0$$

Theorem 3.13. *Let A be an algebra which satisfies identity (3.9). Then A satisfies the following identities for $i, j \geq 2, i \neq j$:*

1. $(x^2, x^i, x^j) = 2(x^{j+2}x^i - x^{i+2}x^j)$,
2. $2x^{j+2}x^i = (x^{j+1}x^i)x + (x^{i+1}x^j)x + ((x^i x^j)x)x - (x^2 x^j)x^i$.

Proof: Setting $y = x^i, z = x^j$ and then $y = x^j, z = x^i$ in identity (3.14), we have,

$$\begin{aligned} (x^{j+1}, x^i, x) + (x^i x^j, x, x) + (x^{i+1}, x^j, x) - 2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) &= 0, \\ (x^{i+1}, x^j, x) + (x^j x^i, x, x) + (x^{j+1}, x^i, x) - 2(x^{i+1}, x, x^j) - (x^2, x^i, x^j) &= 0. \end{aligned}$$

Subtracting both identities we obtain

$$-2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) + 2(x^{i+1}, x, x^j) + (x^2, x^i, x^j) = 0,$$

Developing the associators, we obtain

$$-2x^{j+2}x^i + 2x^{i+2}x^j + (x^2 x^i)x^j - x^2(x^i x^j) = 0,$$

This is identity (1).

To get identity (2) we use the commutativity and we will develop the associator in the identity: $(x^{j+1}, x^i, x) + (x^i x^j, x, x) + (x^{i+1}, x^j, x) - 2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) = 0$. \square

Remark 3.14. *Setting $y = z = x^i$ in identity (3.14), we have*

$$2x^{i+2}x^i = 2(x^{i+1}x^i)x + ((x^i)^2 x)x - (x^i)^2 x^2 - (x^2 x^i)x^i + x^2(x^i)^2.$$

Proposition 3.15. *Let A be an algebra which satisfies identity (3.9). Then A satisfies the following identities for $k \geq 1$:*

1. $2x^4 x^k - 2x^{k+2} x^2 + (x^2)^2 x^k - x^2(x^2 x^k) = 0,$

2. $4x^{k+4} = 4(x^3x^k)x + 3x^3x^{k+1} - 2x^{k+2}x^2 - x^2(x^2x^k),$
3. $4x^{k+4} = 4(x^3x^k)x + 3x^3x^{k+1} - 2x^4x^k - (x^2)^2x^k.$

Proof: Setting $i = 2, j = k$ in (1) of Theorem 3.13, we obtain identity (1).

Setting $i = 2, j = k$ in (2) of Theorem 3.13, we obtain

$$2x^{k+2}x^2 = (x^{k+1}x^2)x + (x^3x^k)x + ((x^2x^k)x)x - (x^2x^k)x^2,$$

Using the identity (3.10), we get

$$2x^{k+2}x^2 = (3x^{k+1}x^3 - 2x^{k+4}) + (x^3x^k)x + (3x^kx^3 - 2x^{k+3})x - (x^2x^k)x^2 = \\ 3x^{k+1}x^3 - 4x^{k+4} + 4(x^3x^k)x - (x^2x^k)x^2,$$

which is identity (2).

Finally identity (3) follows from identities (1) and (2). \square

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