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On a sequence of functions $V_n^{(\alpha, \beta, \delta)}(x; a, k, s)$

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Abstract

In this paper, authors established various properties of a sequence of functions $\{V_n^{(\alpha, \beta, \delta)}(x; a, k, s)/n = 0, 1, 2, \dots\}$ such as generating relations, bilateral generating relations, finite summation formulae, generating functions involving Stirling number, explicit representation and integral transforms.

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1. Introduction

Differential operators plays important role in study of various kind of generalized Rodrigues formulae, generating relations, Finite summation formulae. In this paper, operators considered as follows

$$(1.1) \quad \theta \equiv x^a(s + xD) \text{ and } \theta_1 \equiv x^a(1 + xD),$$

where a and s are arbitrary and $D = \frac{d}{dx}$.

Shukla and Prajapati [6] studied some properties of a class of polynomials suggested by Mittal. A sequence of functions $\{V_n^{(\alpha, \beta, \delta)}(x; a, k, s); n = 0, 1, 2, \dots\}$ introduced by Prajapati and Ajudia [4] as,

$$(1.2) \quad V_n^{(\alpha, \beta, \delta)}(x; a, k, s) = \frac{1}{n!} x^{-\beta} W(\alpha, \delta; p_k(x)) \theta^n \left\{ x^\beta W(\alpha, \delta; -p_k(x)) \right\}$$

where $W(\alpha, \delta; z)$ is Wright function defined as

$$W(\alpha, \delta; z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\alpha n + \delta)}$$

Some noteworthy operational techniques required as follows

$$(1.3) \quad \theta^n \{x^{\alpha+k}\} = a^n \left(\frac{s + \alpha + k}{a} \right)_n x^{\alpha+k+na},$$

where k is an integer, n a non-negative integer and α is arbitrary.

$$(1.4) \quad \theta^n \{xuv\} = x \sum_{k=0}^n \binom{n}{k} \theta^k u \theta_1^{n-k} v$$

where n is non negative integer.

$$(1.5) \quad \theta^n \{x^\alpha f(x)\} = x^\alpha (\theta + \alpha x^a)^n \{f(x)\}$$

2. Generating Relations

Following generating relations (2.1) to (2.3) of equation (1.2) obtained by Prajapati and Ajudia [4] as

$$(2.1) \quad \sum_{n=0}^{\infty} x^{-an} V_n^{(\alpha, \beta, \delta)}(x; a, k, s) t^n =$$

$$(1 - at)^{-\left(\frac{\beta+s}{a}\right)} W(\alpha, \delta; p_k(x)) W\left(\alpha, \delta; -p_k\{x(1 - at)^{-\frac{1}{a}}\}\right)$$

with $|t| < |a|^{-1}; a \neq 0$

$$(2.2) \quad \sum_{n=0}^{\infty} x^{-an} V_n^{(\alpha, \beta-an, \delta)}(x; a, k, s) t^n =$$

$$(1 + at)^{\frac{\beta+s}{a}-1} W(\alpha, \delta; p_k(x)) W\left(\alpha, \delta; -p_k\{x(1 + at)^{\frac{1}{a}}\}\right)$$

with $|t| < |a|^{-1}; a \neq 0$

For $n \in \mathbf{N}$,

$$(2.3) \quad \sum_{m=0}^{\infty} \binom{m+n}{m} V_{n+m}^{(\alpha, \beta, \delta)}(x; a, k, s) x^{-am} t^m$$

$$= (1 - at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\{x(1 - at)^{-\frac{1}{a}}\}\right)} V_n^{(\alpha, \beta, \delta)}\left\{x(1 - at)^{-\frac{1}{a}}; a, k, s\right\}$$

with $|t| < |a|^{-1}; a \neq 0$.

In similar pattern, one can prove the following generating relation:

For $n \in \mathbf{N}$,

$$(2.4) \quad \sum_{m=0}^{\infty} \binom{n+m}{m} x^{-am} V_{n+m}^{(\alpha, \beta-an, \delta)}(x; a, k, s) t^m$$

$$= (1 + at)^{\frac{\beta+s}{a}-1} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\{x(1 + at)^{\frac{1}{a}}\}\right)} V_n^{(\alpha, \beta, \delta)}\left(x(1 + at)^{\frac{1}{a}}; a, k, s\right)$$

with $|t| < |a|^{-1}; a \neq 0$.

The following theorem plays an important role to get several generating relations.

Theorem 2.1 (Srivastava's Theorem [8]). *If the functions $A(z), B_i(z)$ and $z^{-1}C_j(z)$ are analytic about the origin such that*

$$(2.5) \quad A(0), B_i(0), C_j(0) \neq 0, i = 1, 2, \dots, r; j = 1, 2, \dots, l$$

and if the sequence of function $\{g_n^{(p_1, p_2, \dots, p_r)}(x_1, x_2, \dots, x_l)\}_{n=0}^\infty$ defined by

$$A(z) \prod_{i=1}^r \{[B_i(z)]^{p_i}\} \exp \left(\sum_{j=1}^l x_j C_j(z) \right) = \sum_{n=0}^\infty g_n^{(p_1, p_2, \dots, p_r)}(x_1, x_2, \dots, x_l) \frac{z^n}{n!} \quad (2.6)$$

then, for arbitrary p_i, λ_i, x_j and $y_j (i = 1, 2, \dots, r; j = 1, 2, \dots, l)$ independent of z ,

$$(2.7) \quad \sum_{n=0}^\infty g_n^{(p_1 + \lambda_1 n, \dots, p_r + \lambda_r n)}(x_1 + ny_1, \dots, x_l + ny_l) \frac{t^n}{n!} = \frac{A(w) \prod_{i=1}^r \{[B_i(w)]^{p_i}\} \exp \left\{ \sum_{j=1}^l x_j C_j(w) \right\}}{1 - w \left[\sum_{i=1}^r \lambda_i \left(\frac{B_i'(w)}{B_i(w)} \right) + \sum_{j=1}^l y_j C_j'(w) \right]}$$

where

$$(2.8) \quad w = t \prod_{i=1}^r \{[B_i(w)]^{\lambda_i}\} \exp \left\{ \sum_{j=1}^l y_j C_j(w) \right\}$$

By considering $\alpha = 0, \delta = 1$ and $p_k(x) = px^k$ in generating relation (2.1) reduces into the form

$$(2.9) \quad \sum_{n=0}^\infty x^{-an} V_n^{(0, \beta, 1)}(x; a, k, s, p) t^n = (1 - at)^{-\left(\frac{\beta+s}{a}\right)} \exp\{px^k(1 - (1 - at)^{-\frac{k}{a}})\}$$

The above generating relation (2.9) is equivalently of the type (2.6), if we consider $r = 2, l = 1, A(z) = 1, B_1(z) = (1 - at)^{-\frac{1}{a}} = B_2(z), x_1 = p, C_1(z) = x^k(1 - (1 - at)^{-\frac{k}{a}})$ and $g_n^{(\beta, s)}(p) = n! V_n^{(0, \beta, 1)}(x; a, k, s, p)$.

By employing theorem (2.1) on (2.9) gives

$$(2.10) \quad \sum_{n=0}^\infty V_n^{(0, \beta + \lambda_1 n, 1)}(x; a, k, s + \lambda_2 n, p + ny) t^n =$$

$$\frac{(1 - aw)^{-\left(\frac{\beta+s}{a}\right)} \exp \left\{ px^k [1 - x(1 - aw)^{-\frac{1}{a}}] \right\}}{1 - w(1 - aw)^{-1} \left\{ \lambda_1 + \lambda_2 - ykx^k(1 - aw)^{-\frac{k}{a}} \right\}}$$

where,

$$w = t(1 - aw)^{-\left(\frac{\lambda_1 + \lambda_2}{a}\right)} \exp \left\{ yx^k [1 - (1 - aw)^{-\frac{k}{a}}] \right\}; a \neq 0$$

replacing aw by w in the generating relation (2.10), we get

$$(2.11) \quad \sum_{n=0}^{\infty} V_n^{(0, \beta + \lambda_1 n, 1)}(x; a, k, s + \lambda_2 n, p + ny)t^n = \frac{(1 - w)^{-\left(\frac{\beta+s}{a}\right)} \exp \left\{ px^k [1 - x(1 - w)^{-\frac{1}{a}}] \right\}}{1 - w \{ a(1 - w) \}^{-1} \left\{ \lambda_1 + \lambda_2 - ykx^k(1 - w)^{-\frac{k}{a}} \right\}}$$

where,

$$w = at(1 - w)^{-\left(\frac{\lambda_1 + \lambda_2}{a}\right)} \exp \left\{ yx^k [1 - (1 - w)^{-\frac{k}{a}}] \right\}; a \neq 0$$

On setting w by $w(1 + w)^{-1}$ in the generating relation (2.11), one can get

$$(2.12) \quad \sum_{n=0}^{\infty} V_n^{(0, \beta + \lambda_1 n, 1)}(x; a, k, s + \lambda_2 n, p + ny)t^n = \frac{(1 + w)^{\left(\frac{\beta+s}{a}\right)} \exp \left\{ px^k [1 - x(1 + w)^{\frac{1}{a}}] \right\}}{1 - \frac{w}{a} \left\{ \lambda_1 + \lambda_2 - ykx^k(1 + w)^{\frac{k}{a}} \right\}}$$

where,

$$w = at(1 + w)^{\left(\frac{\lambda_1 + \lambda_2}{a} + 1\right)} \exp \left\{ yx^k [1 - (1 + w)^{\frac{k}{a}}] \right\}; a \neq 0$$

In particular, $\lambda_1 = -a - \lambda_2$ and $y = 0$, then generating relation (2.12) reduces as

$$(2.13) \quad \sum_{n=0}^{\infty} V_n^{(0, \beta - (a + \lambda_2)n, 1)}(x; a, k, s + \lambda_2 n, p + ny)t^n = (1 + at)^{\left(\frac{\beta+s}{a} - 1\right)} \exp \left\{ px^k [1 - x(1 + at)^{\frac{1}{a}}] \right\}$$

with $|t| < |a|^{-1}$; $a \neq 0$ and if $\lambda_2 = 0$ gives

$$\sum_{n=0}^{\infty} V_n^{(0, \beta - an, 1)}(x; a, k, s, p + ny)t^n = (1 + at)^{\left(\frac{\beta + s}{a} - 1\right)} \exp \left\{ px^k [1 - x(1 + at)^{\frac{1}{a}}] \right\}$$

(2.14)

with $|t| < |a|^{-1}$; $a \neq 0$.

3. Bilateral Generating Functions

A class of function $\{S_n(x), n = 0, 1, 2, \dots\}$ is generated by

$$(3.1) \quad \sum_{n=0}^{\infty} A_{m,n} S_{m+n} t^n = f(x, t) \{g(x, t)\}^{-m} S_m(h(x, t))$$

where m is non negative integer, the coefficient $A_{m,n}$ are arbitrary constant and f, g, h are suitable functions of x and t . Equation (3.1) is well-known in the literature as the Singhal-Srivastava generating function [7].

Theorem 3.1 (Singhal-Srivastava Theorem [12, 7]). For the sequence $\{S_n(x)\}$ generated by (3.1), let

$$(3.2) \quad F(x, t) = \sum_{n=0}^{\infty} a_n S_n(x) t^n$$

where $a_n \neq 0$ are arbitrary constants, then

$$(3.3) \quad f(x, t) F[h(x, t), yt \{g(x, t)\}^{-1}] = \sum_{n=0}^{\infty} S_n(x) \sigma_n(y) t^n$$

where $\sigma_n(y)$ is a polynomial (of degree n in y) defined by

$$(3.4) \quad \sigma_n(y) = \sum_{k=0}^n a_k A_{n-k} y^k$$

Theorem 3.1 play an important role to obtain a bilateral generating relation from (2.3).

Considering $S_n(x) = V_n^{(\alpha, \beta, \delta)}(x; a, k, s)$, $A_{m,n} = \binom{m+n}{n}$, $g(x, t) = (1 - at)$, $h(x, t) = x(1 - at)^{-\frac{1}{a}}$ and $f(x, t) = (1 - at)^{-\frac{\beta+s}{a}} \frac{W(\alpha, \delta; p_k(x))}{W(\alpha, \delta; p_k\{x(1 - at)^{-\frac{1}{a}}\})}$ in above theorem, produces

$$(3.5) \quad \sum_{n=0}^{\infty} V_n^{(\alpha, \beta, \delta)}(x; a, k, s) \sigma_n(y) t^n = (1 - at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W(\alpha, \delta; p_k\{x(1 - at)^{-\frac{1}{a}}\})} F\left[\frac{x}{(1 - at)^{\frac{1}{a}}}, \frac{yt}{(1 - at)}\right]$$

where $F(x, t) = \sum_{n=0}^{\infty} a_n S_n(x) t^n$

In 1962, Buchholz [1] given identity as

$$(3.6) \quad \sum_{n=0}^{\infty} (-1)^n \zeta_N^{(\mu+n)}(x) \frac{t^n}{n!} = e^{-t} \zeta_N^{(\mu)}(x + t).$$

Theorem 3.2 (Hubble and Srivastava [2]). For the sequence $\{S_n(x)\}$ generated by (3.1), let

$$(3.7) \quad \Phi_N(x, y, t) = \sum_{n=0}^{\infty} a_n S_n(x) \zeta_N^{(\mu+n)}(y) t^n, (a_n \neq 0)$$

where μ is an arbitrary (real or complex) parameter. Suppose that

$$(3.8) \quad \phi_{m,n}(u, v, x) = \sum_{r=0}^{\min\{m,n\}} \frac{(-1)^r}{r!} a_{m-r} A_{m-r, n-r} u^{m-r} v^{n-r} S_{m+n-2r}(x)$$

Then

$$(3.9) \quad \sum_{m,n=0}^{\infty} \phi_{m,n}(u, v, x) \zeta_N^{(\mu+n)}(y) t^m = e^{-t} f(x, v) \Phi_N\left[h(x, v), y + t, \frac{ut}{g(x, v)}\right]$$

provided that each member exists.

Following corollaries followed by theorem (3.2)

Corollary 3.3. *If*

$$(3.10) \quad \Phi_N(x, y, t) = \sum_{n=0}^{\infty} a_n V_n^{(\alpha, \beta, \delta)}(x; a, k, s) \zeta_N^{(\mu+n)}(y) t^n, (a_n \neq 0)$$

where μ is an arbitrary (real or complex) parameter
and

$$(3.11) \quad \phi_{m,n}(u, v, x) = \sum_{r=0}^{\min\{m,n\}} \frac{(-1)^r}{r!} \binom{m+n-2r}{n-r} a_{m-r} u^{m-r} v^{n-r} V_{m+n-2r}^{(\alpha, \beta, \delta)}(x; a, k, s)$$

then

$$(3.12) \quad \sum_{m,n=0}^{\infty} \phi_{m,n}(u, v, x) \zeta_N^{(\mu+n)}(y) t^m \\ = e^{-t} (1-av)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\left\{x(1-av)^{-\frac{1}{a}}\right\}\right)} \\ \Phi_N \left[x(1-av)^{-\frac{1}{a}}, y+t, \frac{ut}{(1-av)} \right]$$

Corollary 3.4. *If*

$$(3.13) \quad \Psi_N(x, y, t) = \sum_{n=0}^{\infty} a_n V_n^{(\alpha, \beta-an, \delta)}(x; a, k, s) \zeta_N^{(\mu+n)}(y) t^n, (a_n \neq 0)$$

where μ is an arbitrary (real or complex) parameter

and

$$(3.14) \quad \psi_{m,n}(u, v, x) = \sum_{r=0}^{\min\{m,n\}} \frac{(-1)^r}{r!} \binom{m+n-2r}{n-r} a_{m-r} u^{m-r} v^{n-r} V_{m+n-2r}^{(\alpha, \beta-a(m+n-2r), \delta)}(x; a, k, s)$$

Then

$$\sum_{m,n=0}^{\infty} \psi_{m,n}(u, v, x) \zeta_N^{(\mu+n)}(y) t^m = e^{-t} (1 + av)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\left\{x(1 + av)^{-\frac{1}{a}}\right\}\right)} \tag{3.15}$$

$$\Psi_N \left[x(1 + av)^{-\frac{1}{a}}, y + t, \frac{ut}{(1 + av)} \right]$$

Srivastava and Lavoie have generalized McBride’s theorem [3] in 1975 as follows

Theorem 3.5 (Srivastava and Lavoie [11, 12]). *If sequence $\{\zeta_{\mu}(x) : \mu \text{ is a complex number}\}$ is generated by*

$$\sum_{n=0}^{\infty} A_{\mu,n} \zeta_{\mu+n}(x) t^n = f(x, t) \{g(x, t)\}^{-\mu} \zeta_{\mu}(h(x, t)) \tag{3.16}$$

where $A_{\mu,n}$ arbitrary constants, and f, g, h are arbitrary functions of x and t .

Let

$$\Phi_{q,\nu}(x, t) = \sum_{n=0}^{\infty} a_{\nu,n} \zeta_{\mu+qn}(x) t^n, (a_{\nu,n} \neq 0), \tag{3.17}$$

where q is positive integer and ν is an arbitrary complex parameter.

Then

$$\sum_{n=0}^{\infty} \zeta_{\mu+n}(x) P_{n,\nu}^q(y) t^n = f(x, t) \{g(x, t)\}^{-\mu} \Phi_{q,\nu} [h(x, t), yt^q \{g(x, t)\}^{-q}], \tag{3.18}$$

where $P_{n,\nu}^q(y)$ is a polynomial of degree $\left[\frac{n}{q}\right]$ in y , which is defined as,

$$P_{n,\nu}^q(y) = \sum_{r=0}^{\left[\frac{n}{q}\right]} A_{\nu+qr, n-qr} a_{\nu,n} y^r$$

In the generating relation (2.3), consider
 $\mu = m, A_{\mu,n} = \binom{m+n}{n}, \zeta_n(x) = V_n^{(\alpha,\beta,\delta)}(x; a, k, s),$
 $f(x, t) = (1 - at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\left\{x(1-at)^{-\frac{1}{a}}\right\}\right)}, g(x, t) = 1, h(x, t)$
 $= x(1 - at)^{-\frac{1}{a}}.$ Then

$$(3.19) \quad \sum_{n=0}^{\infty} V_{m+n}^{(\alpha,\beta,\delta)}(x; a, k, s) P_{n,\nu}^q(y) t^n \\ = (1 - at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\left\{x(1-at)^{-\frac{1}{a}}\right\}\right)} \Phi_{q,\nu}\left[x(1-at)^{-\frac{1}{a}}, yt^q\right]$$

where,

$$(3.20) \quad \Phi_{q,\nu}(x, t) = \sum_{n=0}^{\infty} a_{\nu,n} V_{m+qn}^{(\alpha,\beta,\delta)}(x; a, k, s) t^n, \text{ with } a_{\nu,n} \neq 0,$$

$$\text{and } P_{n,\nu}^q(y) = \sum_{r=0}^{\left[\frac{n}{q}\right]} \binom{\nu+n}{n-qr} a_{\nu,n} y^r$$

is a polynomial of degree $\left[\frac{n}{q}\right]$ in y , q is positive integer and ν is an arbitrary complex number.

Theorem 3.6. (Srivastava [9]) For the function $\zeta_{\mu}(x)$ defined by (3.16), let

$$(3.21) \quad \Theta_{q,\mu}^{p,\nu}[x; y_1, \dots, y_l; t] = \sum_{n=0}^{\infty} C_n^{\mu,\nu} \zeta_{\mu+qn}(x) \Omega_{\nu+pn}(y_1, \dots, y_l) t^n, C_n^{\mu,\nu} \neq 0,$$

where μ and ν are arbitrary complex numbers, p and q are positive integers, and $\Omega_{\nu}(y_1, \dots, y_l)$ is non-vanishing function of l variables $y_1, \dots, y_l, l \geq 1$.

Then

$$(3.22) \quad \sum_{n=0}^{\infty} \zeta_{\mu+n}(x) Q_{n,q,\mu}^{p,\nu}(y_1, \dots, y_l; z) t^n \\ = f(x, t) \{g(x, t)\}^{-\mu} \Theta_{q,\mu}^{p,\nu}\left[h(x, t); y_1, \dots, y_l; z \left\{\frac{t}{g(x, t)}\right\}^q\right],$$

where $Q_{n,q,\mu}^{p,\nu}(y_1, \dots, y_l; z)$ is a polynomial of degree $\left[\frac{n}{q}\right]$ in z (with coefficient's dependent on y_1, \dots, y_l) defined by

$$(3.23) \quad Q_{n,q,\mu}^{p,\nu}(y_1, \dots, y_l; z) = \sum_{r=0}^{\left[\frac{n}{q}\right]} A_{\mu+qr, n-qr} C_r^{\mu,\nu} \Omega_{\nu+pr}(y_1, \dots, y_l) z^r$$

On considering

$$\Theta_{q,m}^{p,\nu}[x; y_1, \dots, y_l; t] = \sum_{n=0}^{\infty} C_n^{m,\nu} V_{m+qn}^{(\alpha, \beta, \delta)}(x; a, k, s) \Omega_{\nu+pn}(y_1, \dots, y_l) t^n, C_n^{m,\nu} \neq 0,$$

$$(3.24)$$

where ν is arbitrary complex number, p and q are positive integers and $\Omega_\nu(y_1, \dots, y_l)$ is non-vanishing function of l variables $y_1, \dots, y_l, l \geq 1$.

Taking $A_{m,n} = \binom{m+n}{n}, \zeta_n(x) = V_n^{(\alpha, \beta, \delta)}(x; a, k, s), f(x, t) = (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\left\{x(1-at)^{-\frac{1}{a}}\right\}\right)}, g(x, t) = 1, h(x, t) = x(1-at)^{-\frac{1}{a}}$

in above theorem, and generating relation (2.3) yields bilateral generating relation as

$$(3.25) \quad \sum_{n=0}^{\infty} V_{m+n}^{(\alpha, \beta, \delta)}(x; a, k, s) Q_{n,q,\mu}^{p,\nu}(y_1, \dots, y_l; z) t^n = (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\left\{x(1-at)^{-\frac{1}{a}}\right\}\right)} \Theta_{q,\mu}^{p,\nu}\left[x(1-at)^{-\frac{1}{a}}; y_1, \dots, y_l; zt^q\right]$$

where $Q_{n,q,m}^{p,\nu}(y_1, \dots, y_l; z)$ is a polynomial of degree $\left[\frac{n}{q}\right]$ in z (with coefficient's dependent on y_1, \dots, y_l) defined by

$$(3.26) Q_{n,q,m}^{p,\nu}(y_1, \dots, y_l; z) = \sum_{r=0}^{\left[\frac{n}{q}\right]} \binom{m+n}{n-qr} C_r^{m,\nu} \Omega_{\nu+pr}(y_1, \dots, y_l) z^r$$

4. Generating Functions involving Stirling Number

Riordan [5], defined Stirling number of second kind as

$$(4.1) \quad S(n, m) = \frac{1}{m!} \sum_{j=1}^m (-1)^{m-j} \binom{m}{j} j^n$$

so that

$$S(n, 1) = S(n, n) = 1, S(n, n-1) = \binom{n}{2} \text{ and } S(n, 0) = \begin{cases} 1, & n = 0 \\ 0 & n \in \mathbf{N} \end{cases}$$

The following theorem is useful to get some generating relation for $V_n^{(\alpha, \beta, \delta)}(x; a, k, s)$;

Theorem 4.1 (Srivastava [10]). Let the sequence $\{\zeta_n(x)\}_{n=0}^{\infty}$ be generated by

$$(4.2) \quad \sum_{m=0}^{\infty} \binom{n+m}{m} \zeta_{m+n}(x) t^m = f(x, t) \{g(x, t)\}^{-n} \zeta_n(h(x, t))$$

where f, g and h are suitable function of x and t .

Then the following family of generating function

$$(4.3) \quad \sum_{m=0}^{\infty} m^n \zeta_m(h(x, -z)) z^m \{g(x, -z)\}^{-m} = \{f(x, -z)\}^{-1} \sum_{m=0}^n m! S(n, m) \zeta_m(x) z^m$$

holds true provided that each member of (4.3) exists.

The generating functions (2.3) and (2.4) related to the family given by (4.2).

By comparing (2.3) and (4.2); $\zeta_m(x) = V_m^{(\alpha, \beta, \delta)}(x; a, k, s)$, $f(x, t) = (1 - ax^a t)^{-\frac{\beta+s}{a}} \frac{W(\alpha, \delta; p_k(x))}{W(\alpha, \delta; p_k\{x(1 - ax^a t)^{-\frac{1}{a}}\})}$, $h(x, t) = x(1 - ax^a t)^{-\frac{1}{a}}$, and $g(x, t) = (1 - ax^a t)$, we get

$$(4.4) \quad \sum_{m=0}^{\infty} m^n V_m^{(\alpha, \beta, \delta)}(x(1 + ax^a z)^{-\frac{1}{a}}; a, k, s) (1 + ax^a z)^{-m} z^m$$

$$= (1 + ax^a z)^{\left(\frac{\beta+s}{a}\right)} \frac{W\left(\alpha, \delta; p_k\{x(1 + ax^a z)^{-\frac{1}{a}}\}\right)}{W(\alpha, \delta; p_k(x))} \sum_{m=0}^n m! S(n, m) V_m^{(\alpha, \beta, \delta)}(x; a, k, s) z^m$$

Replacing $x^a z$ by $\frac{z}{1-az}$ and x by $\frac{x}{(1-az)^{\frac{1}{a}}}$ above equation gives

$$\sum_{m=0}^{\infty} m^n V_m^{(\alpha, \beta, \delta)}(x; a, k, s) z^m = (1 - az)^{\left(-\frac{\beta+s}{a}\right)} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\{x(1 - az)^{-\frac{1}{a}}\}\right)} \tag{4.5}$$

$$\sum_{m=0}^n m! S(n, m) V_m^{(\alpha, \beta, \delta)}\left(x(1 - az)^{-\frac{1}{a}}; a, k, s\right) \left(\frac{z}{1 - az}\right)^m$$

with $|t| < |a|^{-1}; a \neq 0$.

Similarly, theorem 4.1, applied to generating relation (??) gives generating relation as

$$\begin{aligned} (4.6) \quad & \sum_{m=0}^{\infty} m^n V_m^{(\alpha, \beta - am, \delta)}(x(1 - ax^a z)^{\frac{1}{a}}; a, k, s) (1 - ax^a z)^{-m} z^m \\ &= (1 - ax^a z)^{1 - \left(\frac{\beta+s}{a}\right)} \frac{W\left(\alpha, \delta; p_k\{x(1 - ax^a z)^{\frac{1}{a}}\}\right)}{W(\alpha, \delta; p_k(x))} \\ & \quad \sum_{m=0}^n m! S(n, m) V_m^{(\alpha, \beta - am, \delta)}(x; a, k, s) z^m \end{aligned}$$

Replacing $x^a z$ by $\frac{z}{1+az}$ and x by $\frac{x}{(1+az)^{\frac{1}{a}}}$ above equation gives

$$\sum_{m=0}^{\infty} m^n V_m^{(\alpha, \beta - am, \delta)}(x; a, k, s) z^m = (1 + az)^{\left(\frac{\beta+s}{a}\right) - 1} \frac{W(\alpha, \delta; p_k(x))}{W\left(\alpha, \delta; p_k\{x(1 + az)^{\frac{1}{a}}\}\right)} \tag{4.7}$$

$$\sum_{m=0}^n m! S(n, m) V_m^{(\alpha, \beta - am, \delta)} \left(x(1 + az)^{\frac{1}{a}}; a, k, s \right) \left(\frac{z}{1 + az} \right)^m$$

with $|t| < |a|^{-1}; a \neq 0$.

5. Finite Summation Formulae

Finite summation formulae for (1.2) obtained as follows,

$$(5.1) \quad V_n^{(\alpha, \sigma, \delta)}(x; a, k, s) = \sum_{m=0}^n \frac{(ax^a)^m}{m!} \left(\frac{\sigma}{a} \right)_m (ax^a)^m V_{n-m}^{(\alpha, \beta, \delta)}(x; a, k, s - \beta)$$

$$(5.2) \quad V_n^{(\alpha, \beta + \mu + 1, \delta)}(x; a, k, s) = \sum_{m=0}^n \frac{(ax^a)^m}{m!} \left(\frac{\mu + 1}{a} \right)_m V_{n-m}^{(\alpha, \gamma, \delta)}(x; a, k, s)$$

Proof. From (1.2) and (1.5) it follows that,

$$V_n^{(\alpha, \beta, \delta)}(x; a, k, s) = \frac{1}{n!} W(\alpha, \delta; p_k(x)) (\theta + \beta x^a)^n \{W(\alpha, \delta; -p_k(x))\}$$

replacing s by $s - \beta$, above equation can be written as

$$(5.3) \quad \theta^n \{W(\alpha, \delta; -p_k(x))\} = \frac{n!}{W(\alpha, \delta; p_k(x))} V_n^{(\alpha, \beta, \delta)}(x; a, k, s - \beta)$$

employing (1.4) in (1.2) gives

$$(5.4) \quad V_n^{(\alpha, \sigma, \delta)}(x; a, k, s) = \frac{x^{-\sigma+1}}{n!} W(\alpha, \delta; p_k(x)) \sum_{m=0}^n \binom{n}{m} \theta_1^m \{x^{\sigma-1}\} \theta^{n-m} \{W(\alpha, \delta; -p_k(x))\}$$

Using (5.3), simplification of (5.4) reduces to (??). \square

Proof. Using (1.2), equation (1.4) can be written as

$$(5.5) \quad V_n^{(\alpha, \beta + \mu + 1, \delta)}(x; a, k, s) = \frac{x^{-(\beta + \mu)}}{n!} W(\alpha, \delta; p_k(x)) \sum_{m=0}^n \binom{n}{m} \theta_1^m \{x^\mu\} \theta^{n-m} \{x^\beta W(\alpha, \delta; -p_k(x))\}$$

this leads to

$$(5.6) \quad \begin{aligned} & V_n^{(\alpha,\beta+\mu+1,\delta)}(x; a, k, s) \\ &= \sum_{m=0}^n \binom{\mu+1}{a}_m V_{(n-m)}^{(\alpha,\beta,\delta)}(x; a, k, s) \frac{(ax^a)^m}{m!} \end{aligned}$$

□

6. Explicit representation of $V_n^{(\alpha,\beta,\delta)}(x; a, k, s)$

Now consider $p_k(x) = x^k$ in (1.1), reduces as

$$V_n^{(\alpha,\beta,\delta)}(x; a, k, s) = \frac{1}{n!} x^{-\beta} W(\alpha, \delta; x^k) \theta^n [x^\beta W(\alpha, \delta; -x^k)]$$

this can be written as

$$(6.1) \quad V_n^{(\alpha,\beta,\delta)}(x; a, k, s) = \frac{a^n}{n!} W(\alpha, \delta; x^k) \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(\alpha r + \delta)} \left(\frac{s+\beta+kr}{a}\right)_n x^{kr+an}$$

Further simplification of above expression leads to

$$V_n^{(\alpha,\beta,\delta)}(x; a, k, s) = \frac{a^n}{n!} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{s+\beta+kr}{a}\right)_n}{m! r! \Gamma(\alpha m + \delta) \Gamma(\alpha r + \delta)} x^{kr+km+an}$$

Using, Series Manipulation technique, we get

$$V_n^{(\alpha,\beta,\delta)}(x; a, k, s) = \frac{a^n x^{an}}{n!} \sum_{m=0}^{\infty} \sum_{r=0}^m \frac{(-1)^{m-r} \left(\frac{s+\beta+kr}{a}\right)_n}{(m-r)! r! \Gamma(\alpha m - \alpha r + \delta) \Gamma(\alpha r + \delta)} x^{km}$$

This can be expressed as

$$(6.2) \quad V_n^{(\alpha,\beta,\delta)}(x; a, k, s) = \frac{a^n x^{an}}{n!} \sum_{m=0}^{\infty} C_{m,n}^{(\alpha,\beta,\delta,k,a,s)} \frac{x^{km}}{m!}$$

where

$$(6.3) \quad \begin{aligned} & C_{m,n}^{(\alpha,\beta,\delta,k,a,s)} \\ &= \sum_{r=0}^m (-1)^{m-r} \binom{m}{r} \left(\frac{s+\beta+kr}{a}\right)_n \frac{1}{\Gamma(\alpha m - \alpha r + \delta) \Gamma(\alpha r + \delta)} \end{aligned}$$

Equation (6.1) and (6.2) are explicit representation of $V_n^{(\alpha,\beta,\delta)}(x; a, k, s)$.

6.1. Particular cases:

Setting particular values $\alpha = 0, \delta = 1$, Wright function reduces to exponential function

and

$$C_{m,n}^{(0,\beta,1,k,a,s)} = \sum_{r=0}^m (-1)^{m-r} \binom{m}{r} \left(\frac{s + \beta + kr}{a} \right)_n$$

which is m^{th} difference of a polynomial of degree n , which vanishes whenever m exceeds n .

Consider Δ_k is forward difference with common difference k in argument define as

$$(6.4) \quad \Delta_k f(z) = f(z + k) - f(z)$$

hence

$$(6.5) \quad \Delta_k^m f(z) = \sum_{r=0}^m (-1)^{m-r} \binom{m}{r} f(z + kr)$$

and equation (6.2) can be written as

$$(6.6) \quad V_n^{(0,\beta,1)}(x; a, k, s) = \frac{a^n x^{an}}{n!} \sum_{m=0}^n \frac{x^{km}}{m!} \Delta_k^m \left(\frac{s + \beta}{a} \right)_n$$

7. Integral Transforms on the sequence of functions

Equation (6.2) follows that

$$(7.1) \quad \int_0^t x^{p-1} (t-x)^{q-1} V_n^{(\alpha,\beta,\delta)}(x; a, k, s) dx \\ = \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}}{m!} \int_0^t x^{km+an+p-1} (t-x)^{q-1} dx$$

Therefore,

$$(7.2) \quad \int_0^t x^{p-1} (t-x)^{q-1} V_n^{(\alpha,\beta,\delta)}(x; a, k, s) dx$$

$$= \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{C_{m,n}^{(\alpha, \beta, \delta, k, a, s)}}{m!} \beta(km + an + p, q) t^{km+an+p+q-1}$$

Beta (Euler) Transform

Setting $t = 1$ in (7.2) gives Beta(Euler) transform of $V_n^{(\alpha, \beta, \delta)}(x; a, k, s)$

$$B \left(V_n^{(\alpha, \beta, \delta)}(x; a, k, s); p, q \right) = \beta(p, q) \frac{a^n (p)_{an}}{n! (p+q)_{an}} \sum_{m=0}^{\infty} \frac{C_{m,n}^{(\alpha, \beta, \delta, k, a, s)} (an+p)_{km}}{m! (an+p+q)_{km}}$$

(7.3)

The following integral transforms follows from equation (6.2).

Finite Laplace Transform

$$L_T \left\{ V_n^{(\alpha, \beta, \delta)}(x; a, k, s) \right\} = \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{C_{m,n}^{(\alpha, \beta, \delta, k, a, s)}}{m!} \frac{\gamma(an + km + 1, pT)}{p^{km+an+1}}$$

(7.4)

Laplace Transform

$$(7.5) \quad L \left\{ V_n^{(\alpha, \beta, \delta)}(x; a, k, s) \right\} = \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{C_{m,n}^{(\alpha, \beta, \delta, k, a, s)}}{m!} \frac{(an+1)_{km} (a)_n \Gamma a}{p^{km+an+1}}$$

Laguerre Transform

$$(7.6) \quad L \left\{ V_n^{(\alpha, \beta, \delta)}(x; a, k, s) \right\} \\ = \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{C_{m,n}^{(\alpha, \beta, \delta, k, a, s)}}{m!} \frac{\Gamma(km + an + \mu + 1) \Gamma(r - km - an)}{r! \Gamma(-km - an)}$$

provided each gamma function exist.

Generalized Stieltjes Transform

$$(7.7) \quad S_g \left\{ V_n^{(\alpha, \beta, \delta)}(x; a, k, s) \right\} \\ = \frac{a^n x^{an+1-\rho}}{n! \Gamma(\rho)} \sum_{m=0}^{\infty} \frac{C_{m,n}^{(\alpha, \beta, \delta, k, a, s)}}{m!} \Gamma(km + an + 1) \Gamma(\rho - km - an - 1) x^{km}$$

where $Re(km + an + 1) > 0$, $|arg(x)| < \pi$.

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