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Totally magic cordial labeling of mP_n and mK_n

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Abstract

A graph G is said to have a totally magic cordial labeling with constant C if there exists a mapping $f : V(G) \cup E(G) \rightarrow \{0,1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod{2}$ for all $ab \in E(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $n_f(i)$ ($i = 0,1$) is the sum of the number of vertices and edges with label i . In this paper we establish that mP_n and mK_n are totally magic cordial for various values of m and n .

Keywords : Binary magic total labeling; cordial labeling; totally magic cordial labeling; totally magic cordial deficiency of a graph.

AMS Subject Classification 05C78.

1. Introduction

All graphs in this paper are finite, simple and undirected. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we write p for $|V|$ and q for $|E|$. A general reference for graph theoretic notions is [3]. The concept of cordial labeling was introduced by Cahit [1]. A binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ induces an edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ defined by $f^*(uv) = |f(u) - f(v)|$. Such labeling is called cordial if the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ are satisfied, where $v_f(i)$ and $e_{f^*}(i)$ ($i = 0, 1$) are the number of vertices and edges with label i , respectively. A graph is called cordial if it admits cordial labeling. Also, Cahit [2] introduced the notion of totally magic cordial labeling (TMC) based on cordial labeling.

A graph G is said to have totally magic cordial labeling with constant C if there exists a mapping $f : V(G) \cup E(G) \rightarrow \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod{2}$ for all $ab \in E(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $n_f(i)$ ($i = 0, 1$) is the sum of the number of vertices and edges with label i .

In [4] it is proved that the complete graph K_n is TMC if and only if

$\sqrt{4k+1}$ has an integer value when $n = 4k$,

$\sqrt{k+1}$ or \sqrt{k} has an integer value when $n = 4k+1$,

$\sqrt{4k+5}$ or $\sqrt{4k+1}$ has an integer value when $n = 4k+2$,

$\sqrt{k+1}$ has an integer value when $n = 4k+3$.

Jeyanthi and Angel Benseera [5] established totally magic cordial labeling of one-point union of n -copies of cycles, complete graphs and wheels. In [7] we gave necessary condition for an odd graph to be not totally magic cordial.

In [6] we defined binary magic total labeling of a graph G as follows: A binary magic total labeling of a graph G is a function $f : V(G) \cup E(G) \rightarrow \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod{2}$ for all $ab \in E(G)$.

Also, in [6] we defined totally magic cordial deficiency of a graph as the minimum number of vertices taken over all binary magic total labeling of G , which it is necessary to add in order that G' become totally magic cordial is the totally magic cordial deficiency of G , denoted by $\mu_T(G)$. That is, $\mu_T(G) = \min \{|n_f(0) - n_f(1)| - 1\}$ such that f is a binary magic total labeling of G . Further, we determined totally magic cordial deficiency

of complete graphs, wheels and one-point union of complete graphs and wheels.

In this paper we establish the totally magic cordial labeling of mP_n , the disjoint union of m copies of path P_n and mK_n , the disjoint union of m copies of complete graph K_n .

2. Totally magic cordial labeling of mP_n

In this section, we give some sufficient conditions for mP_n to be TMC by means of the solution of a system which comprises an equation and an inequality.

Theorem 2.1. *Let G be the disjoint union of m copies of the path P_n of n vertices and for $i = 1, 2, \dots, k$, let f_i be the binary magic total labeling of P_n . Let $\gamma_i = n_{f_i}(0) - n_{f_i}(1)$ for $i = 1, 2, \dots, k$ then G is TMC if the system (2.1) has a nonnegative integral solution for x_i 's:*

$$(2.1) \quad \left| \sum_{i=1}^k \gamma_i x_i \right| \leq 1 \text{ and } \sum_{i=1}^k x_i = m$$

Proof. Suppose $x_i = \delta_i$, $i = 1, 2, \dots, k$ is a nonnegative integral solution of the system (2.1). We label δ_i copies of P_n with f_i ($i = 1, 2, \dots, k$). As each copy has the property $f_i(a) + f_i(b) + f_i(ab) \equiv C \pmod{2}$ for all $i = 1, 2, \dots, k$, the disjoint union of m copies of the path P_n of n vertices is TMC. \square

The following table shows the values of γ_i for distinct possible binary magic total labelings f_i of the path P_n :

i	$n_{f_i}(0)$	$n_{f_i}(1)$	γ_i
1	0	$2n - 1$	$-2n + 1$
2	2	$2n - 3$	$-2n + 5$
3	3	$2n - 4$	$-2n + 7$
4	4	$2n - 5$	$-2n + 9$
.	.	.	.
.	.	.	.
.	.	.	.
$n - 1$	$n - 1$	n	-1
n	n	$n - 1$	1

Corollary 2.2. *The graph mP_2 is TMC if $m \not\equiv 2 \pmod{4}$.*

Proof. The two values of γ_i corresponding to the binary magic total labelings of P_2 are -3 and 1. Therefore, the system (2.1) in Theorem 2.1 becomes $|-3x_1 + x_2| \leq 1$ such that $x_1 + x_2 = m$. When $m = 4t$, then $x_1 = t$ and $x_2 = 3t$ is a solution. When $m = 4t + 1$, then $x_1 = t$ and $x_2 = 3t + 1$ is a solution. When $m = 4t + 2$, then the system has no solution. When $m = 4t + 3$, then $x_1 = t + 1$ and $x_2 = 3t + 2$ is a solution. Hence, mP_2 is TMC if $m \not\equiv 2 \pmod{4}$. \square

Corollary 2.3. *The graph mP_n is TMC for all $m \geq 1$ and $n \geq 3$.*

Proof. For any $n \geq 3$, using the binary magic total labelings with γ_i values as -1 and 1, we get the system $|-x_1 + x_2| \leq 1$, $x_1 + x_2 = m$. Thus for any $m \geq 1$, the above system has solution. Hence, mP_n is TMC for all $m \geq 1$ and $n \geq 3$. \square

3. Totally magic cordial labeling of mK_n

In this section, we establish the TMC labeling of the disjoint union of m copies of complete graph K_n using the solution of a system which comprises an equation and an inequality.

Let f_i be a TMC labeling of the i^{th} copy of mK_n . Without loss of generality, we assume that $C = 1$. Then for any edge $e = uv \in E(K_n)$, we have either $f_i(e) = f_i(u) = f_i(v) = 1$ or $f_i(e) = f_i(u) = 0$ and $f_i(v) = 1$ or $f_i(e) = f_i(v) = 0$ and $f_i(u) = 1$ or $f_i(u) = f_i(v) = 0$ and $f_i(e) = 1$. Hence, under the labeling f_i , the complete graph can be decomposed as $K_n = K_p \cup K_r \cup K_{p,r}$ where K_p is the subgraph whose vertices and edges are labeled with 1, K_r is the subgraph whose vertices are labeled with 0 and its edges are labeled with 1 and $K_{p,r}$ is the subgraph of K_n with the bipartition $V(K_p) \cup V(K_r)$ in which the edges are labeled with 0. Thus, we have $n_{f_i}(0) = r + pr$ and $n_{f_i}(1) = p + \frac{p(p-1)}{2} + \frac{r(r-1)}{2}$.

Theorem 3.1. *Let G be the disjoint union of m copies of the complete graph K_n of n vertices and for $i = 1, 2, \dots, k$, f_i be the binary magic total labeling of the i^{th} copy of K_n . Let $n_{f_i}(0) = \alpha_i$ for $i = 1, 2, \dots, k$, then G is TMC if the system (3.1) has a nonnegative integral solution for x_i 's:*

$$(3.1) \quad \left| \sum_{i=1}^k \left[2\alpha_i - \frac{n^2 + n}{2} \right] x_i \right| \leq 1 \text{ and } \sum_{i=1}^k x_i = m$$

Proof. Suppose $x_i = \delta_i, i = 1, 2, \dots, k$ is a nonnegative integral solution of the system (3.1), then we label the δ_i copies of K_n with $f_i (i = 1, 2, \dots, k)$. We have, $n_{f_i}(1) = \frac{n^2+n}{2} - \alpha_i$. Thus, $n_{f_i}(0) - n_{f_i}(1) = 2\alpha_i - \frac{n^2+n}{2}$. As each copy has the property $f_i(a) + f_i(b) + f_i(ab) \equiv C(\text{mod } 2)$, the disjoint union of m copies of the complete graph K_n is TMC. \square

Theorem 3.2. *If $\sqrt{n+1}$ has an integer value then the disjoint union of m copies of K_n, mK_n is TMC for all $m \geq 1$.*

Proof. The system (3.1) has solution when $\alpha_i = \frac{n^2+n}{4}$. Thus if there exists a positive integer $t, 1 \leq t \leq n$ such that $t(n-t+1) = \frac{n^2+n}{4}$, then mK_n is TMC. By solving the above equation we get, $t = \frac{n+1}{2} \pm \frac{\sqrt{n+1}}{2}$. Hence, if $\sqrt{n+1}$ has an integer value then mK_n is TMC for all $m \geq 1$. \square

The following table shows the values of α_i and β_i for distinct possible binary magic total labelings f_i of the complete graph K_n :

i	p	r	α_i	β_i
1	0	n	n	$\frac{n^2-n}{2}$
2	1	$n-1$	$2 \times (n-1)$	$\frac{n^2-3n+4}{2}$
3	2	$n-2$	$3 \times (n-2)$	$\frac{n^2-5n+12}{2}$
4	3	$n-3$	$4 \times (n-3)$	$\frac{n^2-7n+24}{2}$
.
.
.
$\lfloor \frac{n+1}{2} \rfloor$	$\lfloor \frac{n-1}{2} \rfloor$	$\lceil \frac{n+1}{2} \rceil$	$\lfloor \frac{n-1}{2} \rfloor \times \lceil \frac{n+1}{2} \rceil$	$\frac{[\lfloor \frac{n-1}{2} \rfloor]^2 + (\lceil \frac{n+1}{2} \rceil)^2 + \lfloor \frac{n-1}{2} \rfloor + \lceil \frac{n+1}{2} \rceil}{2}$

Corollary 3.3. *Let f_1 and f_2 be binary magic total labelings of mK_n . Let $n_{f_i}(0) = \alpha_i, i = 1, 2$ be such that $\alpha_1 + \alpha_2 = \frac{n^2+n}{2}$, then mK_n is TMC if and only if m is even.*

Proof. Let $m = 2t$. We assume that $\alpha_1 + \alpha_2 = \frac{n^2+n}{2}$. Then

$\left| \sum_{i=1}^k \left[2\alpha_i - \frac{n^2+n}{2} \right] x_i \right| \leq 1$ and $\sum_{i=1}^k x_i = m$ implies that
 $\left| \left[2\alpha_1 - \frac{n^2+n}{2} \right] x_1 + \left[\frac{n^2+n}{2} - 2\alpha_1 \right] x_2 \right| \leq 1$ and $x_1 + x_2 = 2t$. Clearly, $x_1 = t$
 and $x_2 = t$ satisfy the above system. Also, if m is odd there is no solution.
 Hence, mK_n is TMC if and only if m is even. \square

Corollary 3.4. *The graph mK_{j^2} ($j \geq 1$) is TMC if and only if m is even.*

Proof. Let $j^2 = n$. We consider the labelings f_1 and f_2 with $\alpha_1 = r(j^2 - r + 1)$ and $\alpha_2 = \alpha_1 + j$ where $r = \frac{j(j-1)}{2}$. Clearly, $\alpha_1 + \alpha_2 = \frac{n^2+n}{2}$. Hence by Corollary 3.3, mK_{j^2} is TMC if and only if m is even. \square

Illustration 3.5. *We consider the graph mK_4 . Clearly, $j = 2$ and $r = 1$. Thus $\alpha_1 = n_{f_1}(0) = 4$ and $\alpha_2 = n_{f_2}(0) = 6$. Therefore, $\alpha_1 + \alpha_2 = \frac{4^2+4}{2} = 10$. Hence, under the labeling f_1 , all the four vertices of K_4 can be labeled with 0 and under the labeling f_2 , only one vertex can be labeled with 1 and the remaining vertices can be labeled with 0. Therefore, by Corollary 3.3, mK_4 is TMC if and only if m is even.*

Corollary 3.6. *The graph mK_{j^2+3} ($j \geq 2$) is TMC if and only if m is even.*

Proof. Let $j^2 + 3 = n$. We consider the labelings f_1 and f_2 with $\alpha_1 = r(j^2 - r + 4)$ and $\alpha_2 = \alpha_1 + 2j$ where $r = \frac{j(j-1)}{2} + 1$. Clearly, $\alpha_1 + \alpha_2 = \frac{n^2+n}{2}$. Thus by Corollary 3.3, mK_{j^2+3} is TMC if and only if m is even. \square

Corollary 3.7. *The graph mK_{j^2+8} ($j \geq 1$) is TMC if and only if m is even.*

Proof. Let $j^2 + 8 = n$. We consider the labelings f_1 and f_2 with $\alpha_1 = \frac{3j}{2} + \frac{n^2+n}{4}$ and $\alpha_2 = \frac{n^2+n}{4} - \frac{3j}{2}$. Clearly, $\alpha_1 + \alpha_2 = \frac{n^2+n}{2}$. Hence by Corollary 3.3, mK_{j^2+8} is TMC if and only if m is even. \square

Corollary 3.8. *Let f_i , $i = 1, 2, 3, 4$ be the binary magic total labelings of mK_n . Let $n_{f_i}(0) = \alpha_i$, $i = 1, 2, 3, 4$ be such that $\sum_{i=1}^4 \alpha_i = n^2 + n$, then mK_n is TMC if and only if $m \equiv 0 \pmod{4}$.*

Proof. Let $m = 4t$. We assume that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = n^2 + n$. Then $\left| \sum_{i=1}^4 \left[2\alpha_i - \frac{n^2+n}{2} \right] x_i \right| \leq 1$ such that $\sum_{i=1}^4 x_i = 4t$. Clearly, $x_1 = x_2 = x_3 = x_4 = t$ is a solution of the above system. Also when $m \not\equiv 0 \pmod{4}$ the system has no solution. Thus mK_n is TMC if and only if $m \equiv 0 \pmod{4}$. \square

Corollary 3.9. *The graph mK_{j^2+2} ($j \geq 2$) is TMC if and only if $m \equiv 0 \pmod{4}$.*

Proof. Let $j^2 + 2 = n$. We consider the labelings f_1 and f_2 with $\alpha_1 = r(n - r + 1)$ and $\alpha_2 = \alpha_1 + 2j - 2$ where $r = \frac{j^2-j+2}{2}$. We can easily prove that $\alpha_1 + \alpha_2 = n^2 + n$. Hence, $x_1 = 3t$ and $x_2 = t$ is a solution of the system (3.1). Thus, mK_{j^2+2} is TMC if $m \equiv 0 \pmod{4}$. \square

Theorem 3.10. *The graph mK_{j^2+4} ($j \geq 2$) is TMC if and only if $m \equiv 0 \pmod{4}$.*

Proof. Let $j^2 + 4 = n$. We consider the labelings f_1, f_2, f_3 and f_4 with $\alpha_1 = r(n - r + 1)$, $\alpha_2 = \alpha_1 + j + 2$, $\alpha_3 = \alpha_1 + 2j + 2$ and $\alpha_4 = \alpha_1 + 3j$. We can easily prove that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = n^2 + n$. Hence by Corollary 3.8, mK_{j^2+4} is TMC if and only if $m \equiv 0 \pmod{4}$. \square

Theorem 3.11. *If mK_n and $m'K_n$ are TMC and m or m' is even, then $(m + m')K_n$ is TMC.*

Proof. Let f and f' be the TMC labelings of mK_n and $m'K_n$ respectively with $C = 1$. We assume that m is even. Then, $n_f(0) = n_f(1)$. For $m'K_n$, we have $n_{f'}(0) = n_{f'}(1)$, $n_{f'}(0) = n_{f'}(1) + 1$ or $n_{f'}(0) = n_{f'}(1) - 1$. Let f'' be a binary magic total labeling of $(m + m')K_n$ with $C = 1$. Clearly, $n_{f''}(0) = n_f(0) + n_{f'}(0)$ and $n_{f''}(1) = n_f(1) + n_{f'}(1)$. Therefore, $n_{f''}(0) = n_{f''}(1)$ or $n_{f''}(0) = n_{f''}(1) + 1$ or $n_{f''}(0) = n_{f''}(1) - 1$ are derived from $n_{f'}(0) = n_{f'}(1)$ or $n_{f'}(0) = n_{f'}(1) + 1$ or $n_{f'}(0) = n_{f'}(1) - 1$ respectively. Hence, $(m + m')K_n$ is TMC with $C = 1$. \square

Theorem 3.12. [7] *If G is an odd graph with $p + q \equiv 2 \pmod{4}$, then G is not TMC.*

Corollary 3.13. *If $m \equiv 1 \pmod{2}$ and $n \equiv 4 \pmod{8}$ then mK_n is not TMC.*

Proof. Proof follows from Theorem 3.12. \square

Theorem 3.14. *Let $f_i, i = 1, 2, 3$ be the binary magic total labelings of mK_n . Let $n_{f_i}(0) = \alpha_i, i = 1, 2, 3$ be such that $\alpha_1 + 2\alpha_2 + \alpha_3 = n^2 + n$, then mK_n is TMC if and only if $m \equiv 0 \pmod{4}$.*

Proof. Let $m = 4t$. We assume that $\alpha_1 + 2\alpha_2 + \alpha_3 = n^2 + n$. Then

$$\left| \sum_{i=1}^3 \left[2\alpha_i - \frac{n^2 + n}{2} \right] x_i \right| \leq 1$$

such that $\sum_{i=1}^3 x_i = 4t$. Clearly, $x_1 = t, x_2 = 2t$ and $x_3 = t$ is a solution of the above system. When $m \not\equiv 0 \pmod{4}$ the system has no solution. Thus mK_n is TMC if and only if $m \equiv 0 \pmod{4}$. \square

Corollary 3.15. *The graph mK_{j^2+1} ($j \geq 2$) is TMC if and only if $m \equiv 0 \pmod{4}$.*

Proof. Let $j^2 + 1 = n$. Consider the labelings f_1, f_2 and f_3 with $\alpha_1 = r(n - r + 1), \alpha_2 = \alpha_1 + j + 1$ and $\alpha_3 = \alpha_1 + 2j$ where $r = \frac{j^2 - j}{2}$. We can easily prove that $\alpha_1 + 2\alpha_2 + \alpha_3 = n^2 + n$. Hence by Theorem 3.14, mK_n is TMC if and only if $m \equiv 0 \pmod{4}$. \square

Theorem 3.16. *Let $f_i, i = 1, 2, 3$ be binary magic total labelings of mK_n . Let $n_{f_i}(0) = \alpha_i, i = 1, 2, 3$ be such that $\alpha_1 + \alpha_2 + 2\alpha_3 = n^2 + n$, then mK_n is TMC if and only if $m \equiv 0 \pmod{4}$.*

Proof. Let $m = 4t$. We assume that $\alpha_1 + \alpha_2 + 2\alpha_3 = n^2 + n$. Then

$$\left| \sum_{i=1}^3 \left[2\alpha_i - \frac{n^2 + n}{2} \right] x_i \right| \leq 1$$

such that $\sum_{i=1}^3 x_i = 4t$. Clearly, $x_1 = t, x_2 = t$ and $x_3 = 2t$ is a solution of the above system. Also when $m \not\equiv 0 \pmod{4}$ the system has no solution. Thus mK_n is TMC if and only if $m \equiv 0 \pmod{4}$. \square

Corollary 3.17. *The graph mK_{j^2+5} ($j \geq 3$) is TMC if and only if $m \equiv 0 \pmod{4}$.*

Proof. Let $j^2 + 5 = n$. Consider the labelings f_1 , f_2 and f_3 with $\alpha_1 = r(n - r + 1)$, $\alpha_2 = \alpha_1 + 2j + 4$ and $\alpha_3 = \alpha_1 + 3j + 6$ where $r = \frac{j^2 - j + 2}{2}$. We can easily prove that $\alpha_1 + 2\alpha_2 + 2\alpha_3 = n^2 + n$. Hence by Theorem 3.16, mK_n is TMC if and only if $m \equiv 0 \pmod{4}$. \square

We conclude this paper with the following conjecture:

Conjecture 3.18. *The graphs mK_{j^2+k} ($j \geq 5$) for $k = 6, 7, 9, 10, \dots, 2j - 1$ and $m \geq 1$ admit totally magic cordial labeling.*

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