Stability and boundedness in differential systems of third order with variable delay

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Abstract

In this paper, we consider a non-linear system of differential equations of third order with variable delay. We discuss the globally asymptotic stability/uniformly stability, boundedness and uniformly boundedness of solutions for the considered system. The technique of proofs involves defining an appropriate Lyapunov functional. The obtained results include and improve the results in literature.

Keywords : *Globally asymptotic stability, boundedness, Lyapunov functional, delay, differential system, third order.*

2010 Mathematics Subject Classifications : *34K12, 34K20.*

1. Introduction

In the last years, there is a good amount of interest in the qualitative behaviors of ordinary and functional differential equations of third order without and with delay, see the book of Reissig et al. [1] as a good survey for the works done by 1974 and the papers of Ademola and Arawomo ([2], [3], [4],[5]), Ademola et al. [6], Afuwape and Castellanos [7], Afuwape and Omeike [8], Ahmad and Rama Mohana Rao [9], Bai and Guo [10], Ezeilo [11], Ezeilo and Tejumola [12], Graef et al. ([13], [14]), Graef and Tunc [15], Korkmaz and Tunc [16], Mahmoud and Tunc [17], Ogundare [18], Ogundare et al. [19], Olutimo [20], Omeike [21], Qian [22], Remili and Oudjedi [23], Sadek [24], Swick [25], Tejumola and Tchegnani [26], Tunc ([27]-[44]), Tunc and Ates [45], Tunc and Gozen [46], Tunc and Mohammed [47], Tunc and Tunc [48], Zhang and Yu [49], Zhu [50] and theirs references. However, to the best of our knowledge from the literature, by this time, no attention was given to the investigation of the globally asymptotic stability/uniformly stability, boundedness and uniformly boundedness in the systems of nonlinear functional differential equations of third order with variable delay, except the recent work of Omeike [21].

Besides, it is well known that differential equations of third order play extremely important and useful roles in many scientific areas such as atomic energy, biology, chemistry, control theory, economy, engineering, information theory, mechanics, medicine, physics, etc.. Indeed, we can find applications such as nonlinear oscillations in Afuwape et al. [51], Andres [52], Fridedrichs [53], physical applications in Animalu and Ezeilo [54], non- resonant oscillations in Ezeilo and Onyia [55], prototypical examples of complex dynamical systems in a high-dimensional phase space, displacement in a mechanical system, velocity, acceleration in Chlouverakis and Sprott [56], Eichhorn et al. [57], Linz [58], the biological model and other models in Cronin-Scanlon [59], problems in biomathematics in Chlouverakis and Sprott [56], electronic theory in Rauch [60], and etc. Further, we refer the readers to the book of Smith [61] for some important applications of delay differential equation in sciences, biomathematics, engineering, and etc..

In 2015, Omeike [21] investigated the stability and boundedness of nonlinear differential system of third order with variable delay, $\tau(t)$:

$$(1.1) \qquad X''' + AX'' + BX' + H(X(t - \tau(t))) = P(t).$$

In this paper, we consider nonlinear differential system of the third

order with variable delay, $\tau(t)$:

$$X''' + AX'' + G(X'(t - \tau(t))) + H(X(t - \tau(t))) = F(t, X, X', X''),$$

(1.2)

where $X \in \mathcal{R}^n$, t$\in [0, \infty)$, $\mathcal{R}^+ = [0, \infty)$, $\tau(t)$ is a continuous differentiable function such that $0 \leq \tau(t) \leq \tau_0$ is a positive constant and $\tau'(t) \leq \tau_1$, $(0 < \tau_1 < 1)$, A is an $n \times n-$ constant symmetric matrix, $G, H : \mathcal{R}^n \to \mathcal{R}^n$ are continuous differentiable functions with $G(0) = H(0) = 0$ such that the Jacobian matrices $J_G(X')$ and $J_H(X)$ exist and are symmetric and continuous, that is,

$$\mathrm{J}_G(X') = \left(\frac{\partial g_i}{\partial x'_j} \right), J_H(X) = \left(\frac{\partial h_i}{\partial x_j} \right), (i, j = 1, 2, ..., n),$$

exist and are symmetric and continuous, where $(x_1, x_2, ..., x_n)$, $(x'_1, x'_2, ..., x'_n)$, (g_i) and (h_i) are components of X, X', G and H, respectively, $F : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ is a continuous function and the primes in Eq. (1.2) indicate differentiation with respect to t, $t \geq t_0 \geq 0$.

It is more convenient to consider not Eq. (1.2) itself, but rather the system

$$X'_1 = X_2, X'_2 = X_3,$$
$$X'_3 = -AX_3 - G(X_2) - H(X_1) + \int_{t-\tau(t)}^t J_G(X_2(s))X_3(s)ds$$

(1.3)
$$+ \int_{t-\tau(t)}^t J_H(X_1(s))X_2(s)ds + F(t, X_1, X_2, X_3).$$

System (1.3) is obtained from Eq. (1.2) by setting $X = X_1$, $X' = X_2$ and $X'' = X_3$.

The continuity of the functions τ, H, G and F is a sufficient condition to guarantee the existence of solutions of Eq. (1.2). Besides, we assume that the functions H, G and F satisfy a Lipschitz condition with respect to their respective arguments, like X, X' and X''. In this case, the uniqueness of solutions of Eq. (1.2) is guaranteed.

The motivation of this paper comes from the results established by Omeike [21], the mentioned books, papers and theirs references. The main

purpose of this paper is to get new the globally asymptotic stability /uniformly stability, boundedness and uniformly boundedness results in Eq. (1.2) by defining a suitable new Lyapunov functional. By this paper, we extend and improve the stability and boundedness results of Omeike [21], and we give additional two results to that in Omeike [21], like uniformly stability and uniformly boundedness results. It follows that if we choose $G(X^{'}(t-\tau(t))) = BX^{'}(t)$, B is an $n \times n$-constant symmetric matrix, and $F(t, X, X^{'}, X^{''}) = P(t)$ in Eq. (1.2), then Eq. (1.2) reduces to Eq. (1.1), which is discussed by Omeike [21]. This means that instead of the linear term $BX^{'}(t)$ in Eq. (1.1), we take the non-linear term $G(X^{'}(t - \tau(t)))$ which includes a variable delay $\tau(t)$, and we also take a nonlinear generalization of the term $P(t)$ in Eq. (1.1) like $F(t, X, X^{'}, X^{''})$. Probably, these cases seem as similarity of Eq. (1.1) and Eq. (1.2).

However, till now, throughout all the papers published in the literature, no author discussed the stability and boundedness of the solutions when we take the possible second term in Eq. (1.1), $G(X^{'})$ or $BX^{'}(t)$ as a non-linear term with a deviating argument like $G(X^{'}(t - \tau(t)))$. To the best of our knowledge, it is not easy to discuss the topic for Eq. (1.2). The possible reason is that the construction or definition of a suitable Lyapunov function or functional for higher differential systems remains as an open problem in the literature by this time. This case is more difficult for the functional differential systems of higher order with variable delay. The choice of the second term in Eq.(1.2) as $G(X^{'}(t - \tau(t)))$ is important and the discussion of the problems for this case is some hard. This means that, in view of the whole mentioned discussion, it is worth to discuss the globally asymptotic stability /uniformly stability, boundedness and uniformly boundedness to Eq. (1.2).

Besides, this paper may be useful for researchers working on the qualitative behaviors of solutions of third differential equations and completes that in the literature. These cases show the novelty and originality of the present paper.

Consider general delay differential system

$$\dot{x} = f(x_t), x_t = x(t + \theta), -r \leq \theta < 0, t \geq 0.$$

Let $C = C([-r, 0], \mathcal{R}^n)$ denote the space of continuous function from $[-r, 0]$ into $\mathcal{R}^n$ and assume that $f : C \to \mathcal{R}^n$ is continuous function. We say that $V : C \to \mathcal{R}$ is a Lyapunov function on a set $G \subset C$ relative to f if

V is continuous on $\bar{G}$, the closure of G, $\dot{V}$ is defined on G and $\dot{V} \leq 0$ on G.

Lemma 1(Hale [62]). Suppose $f(0) = 0$. Let V be *a* continuous functional defined on $C_H = C$ with $V(0) = 0$ and let $u(s)$ be a function, non-negative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $s \to \infty$ with $u(0) = 0$. If for all $\varphi \in C$, $u(|\phi(0)|) \leq V(\phi)$, $\dot{V}(\varphi) \leq 0$, then the zero solution of $\dot{x} = f(x_t)$ is stable.

If we define $Z = \{\varphi \in C_H : \dot{V}(\varphi) = 0\}$, then the zero solution of $\dot{x} = f(x_t)$ is asymptotically stable, provided that the largest invariant set in Z is $Q = 0$.

Besides, we consider the general non-autonomous delay differential system

$$\dot{x} = g(t, x_t), x_t = x(t + \theta), -r \leq \theta < 0, t \geq 0,$$

where $g : [0, \infty) \times C_\rho \to \mathcal{R}^n$ is a continuous mapping, $g(t, 0) = 0$, and we suppose that G takes closed bounded sets into bounded sets of $\mathcal{R}^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi : [-r, 0] \to \mathcal{R}^n$ and for $\phi \in C$, $\|\phi\| = sup_{-r \leq \theta \leq 0} |\phi(\theta)|, r > 0$, C_ρ is the open ball in C of radius ρ; $C_\rho := \{\varphi \in C([-r, 0], \mathcal{R}^n) : \|\varphi\| < \rho\}$.

Theorem A(Yoshizawa [63 pp.191]). Assume that there is a Lyapunov functional $V_0(t, x)$ for $\dot{x} = g(t, x_t)$, and wedges satisfying;

(i) $W_1(|\phi(0)|) \leq V_0(t, \phi) \leq W_2(\|\phi\|)$, (where $W_1(r)$ and $W_2(r)$ are wedges),

(ii) $\dot{V}_0(t, \phi) \leq 0$.

Then the zero solution of $\dot{x} = g(t, x_t)$ is uniformly stable.

Let S be the set of $\phi \in C$ such that $\|\varphi\| > \rho$, denote by S^* the set of all functions $\phi \in C$ such that $|\phi(0)| \geq \rho$, where ρ may be large (Yoshizawa [63, pp.202]).

Theorem B (Yoshizawa [63, pp.202]). Suppose that there exists a continuous Lyapunov functional $V_0(t, \phi)$ defined for all $t \in \mathcal{R}^+$ and $\phi \in S^*$ which

satisfies the following conditions;

(i) $a(|\phi(0)|) \leq V_0(t, \phi) \leq b_1(|\phi(0)|) + b_2(\|\phi\|)$,
where $a(r), b_1(r), b_2(r) \in CI$, ($CI$ denotes the families of continuous increasing functions), and are positive for $r > H$ and $a(r) - b_2(r) \to \infty$ as $r \to \infty$,

(ii) $\dot{V}_0(t, \phi) \leq 0$. Then the solutions of $\dot{x} = g(t, x_t)$ are uniformly bounded.

Lemma 2. (Bellman [64, pp.288]). Let M be a real symmetric $n \times n$ -matrix. Then for any $X \in \mathcal{R}^n$

$$\delta_M \|X\|^2 \leq \langle MX, X \rangle \leq \Delta_M \|X\|^2,$$

where δ_M and Δ_M are, respectively, positive and simple, the least and greatest eigenvalues of the matrix M.

2. Stability

We introduce some basic assumptions needed in the proofs.
(A1) A is an $n \times n$- symmetric constant matrix,

$$\delta_a \leq \lambda_i(A) \leq \triangle_a,$$

where δ_a and $\triangle_a$ are positive constants.
(A2) $H(0) = 0$, J_H exists and is an $n \times n$-symmetric matrix,

$$H(X_1) \neq 0, \quad when \quad X_1 \neq 0,$$

$$\delta_h \leq \lambda_i(J_H(X_1)) \leq \triangle_h \quad for \quad X_1 \in \mathcal{R}^n,$$

where δ_h and $\triangle_h$ are positive constants.
(A3) $G(0) = 0$, J_G exists and is an $n \times n$- symmetric matrix,

$$\delta_b \leq \lambda_i(J_G(X_2)) \leq \triangle_b \quad for \quad X_2 \in \mathcal{R}^n,$$

where δ_b and $\triangle_b$ are positive constants, and the matrices J_G and J_H commute with each other.

Let $F(.) \equiv 0$. The stability result of this paper is given by the following theorem.

Theorem 1. We assume that assumptions $(A1) - (A3)$ hold. If

$$\tau_0 < min \left\{ \frac{2(\delta_b - \beta \triangle_h)(1 - \tau_1)}{(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_b}, \frac{2(\beta \delta_b - 1)(1 - \tau_1)}{\beta(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_h} \right\},$$

then the zero solution of Eq. (1.2) is globally asymptotic stable, where τ_1 and β are positive constants with $0 < \tau_1 < 1$.

Proof. Define a functional $W_0 = W_0(t) = W_0(X_1(t), X_2(t), X_3(t))$ by
$$W_0 = 2 \int_0^1 \langle H(\sigma X_1), X_1 \rangle d\sigma + \langle AX_2, X_2 \rangle + 2\beta \int_0^1 \langle G(\sigma X_2), X_2 \rangle d\sigma$$

$$+ \beta \langle X_3, X_3 \rangle + 2 \langle X_2, X_3 \rangle + 2\beta \langle X_2, H(X_1) \rangle$$

$$(2.1) \quad + 2\lambda \int_{-\tau(t)}^0 \int_{t+s}^t ||X_2(\theta)||^2 d\theta ds + 2\eta \int_{-\tau(t)}^0 \int_{t+s}^t ||X_3(\theta)||^2 d\theta ds,$$

where β, λ and η are positive constants, the constants λ and η will be determined later in the proof.

From assumption (A_3), Lemma 2 and

$$G(0) = 0, \frac{\partial}{\partial \sigma} G(\sigma X_2) = J_G(\sigma X_2) X_2$$

We obtain

$$2 \int_0^1 \langle G(\sigma X_2), X_2 \rangle d\sigma = 2 \int_0^1 \int_0^1 \sigma_1 \langle J_G(\sigma_1 \sigma_2 X_2) X_2, X_2 \rangle d\sigma_1 d\sigma_2 \geq \delta_b ||X_2||^2.$$

Similarly, it follows from

$$H(0) = 0, \frac{\partial}{\partial \sigma_1} H(\sigma_1 X_1) = J_H(\sigma_1 X_1) X_1$$

that

$$\frac{\partial}{\partial \sigma_1} \langle H(\sigma_1 X_1), H(\sigma_1 X_1) \rangle = 2 \langle J_H(\sigma_1 X_1) X_1, H(\sigma_1 X_1) \rangle.$$

Integrating, from $\sigma_1 = 0$ to $\sigma_1 = 1$, the both sides of the last estimates, respectively, we get

$$H(X_1) = \int_0^1 J_H(\sigma_1 X_1) X_1 d\sigma_1$$

and

$$\langle H(X_1), H(X_1) \rangle = 2 \int_0^1 \langle J_H(\sigma_1 X_1) X_1, H(\sigma_1 X_1) \rangle d\sigma_1.$$

It can also be seen that

$$2 \int_0^1 \langle H(\sigma_1 X_1), X_1 \rangle d\sigma_1 = 2 \int_0^1 \int_0^1 \sigma_1 \langle J_H(\sigma_1 \sigma_2 X_1) X_1, X_1 \rangle d\sigma_1 d\sigma_2.$$

Further, it is obvious that

$$\frac{\partial}{\partial \sigma_2} \langle H(\sigma_1 \sigma_2 X_1), J_H(\sigma_1 X_1) X_1 \rangle = \langle \sigma_1 J_H(\sigma_1 X_1) X_1, J_H(\sigma_1 X_1) X_1 \rangle.$$

Integrating the both sides of the last equality from $\sigma_2 = 0$ to $\sigma_2 = 1$, we obtain

$$\langle H(\sigma_1 X_1), J_H(\sigma_1 X_1) X_1 \rangle = \int_0^1 \langle \sigma_1 J_H(\sigma_1 X_1) X_1, J_H(\sigma_1 X_1) X_1 \rangle d\sigma_2.$$

Then, we have

$$\langle H(X_1), H(X_1) \rangle = 2 \int_0^1 \int_0^1 \langle \sigma_1 J_H(\sigma_1 X_1) X_1, J_H(\sigma_1 X_1) X_1 \rangle d\sigma_1 d\sigma_2.$$

From (2.1), the above discussion, Lemma 2 and the assumptions of Theorem 1, we can obtain

$2W_0 \geq 2\int_0^1 \langle H(\sigma X_1), X_1 \rangle d\sigma + \langle AX_2, X_2 \rangle + \beta \delta_b ||X_2||^2$

$+\beta \langle X_3, X_3 \rangle + 2\langle X_2, X_3 \rangle + 2\beta \langle X_2, H(X_1) \rangle$

$= 2\int_0^1 \langle H(\sigma X_1), X_1 \rangle d\sigma - \beta \delta_b^{-1} \langle H(X_1), H(X_1) \rangle$

$+\beta ||\delta_b^{\frac{1}{2}} X_2 + \delta_b^{-\frac{1}{2}} H(X_1)||^2 + \beta ||X_3 + \beta^{-1} X_2||^2$

$+\langle (A - \beta^{-1} I) X_2, X_2 \rangle$

$\geq 2\int_0^1 \langle H(\sigma X_1), X_1 \rangle d\sigma - \beta \delta_b^{-1} \langle H(X_1), H(X_1) \rangle$

$+\beta ||X_3 + \beta^{-1} X_2||^2 + \langle (A - \beta^{-1} I) X_2, X_2 \rangle$

$\geq 2\int_0^1 \int_0^1 \sigma_1 \langle J_H(\sigma_1 \sigma_2 X_1) X_1, X_1 \rangle d\sigma_1 d\sigma_2 + \beta ||X_3 + \beta^{-1} X_2||^2$

$-2\beta \delta_b^{-1} \int_0^1 \int_0^1 \langle \sigma_1 J_H(\sigma_1 X_1) X_1, J_H(\sigma_1 X_1) X_1 \rangle d\sigma_1 d\sigma_2$

$+\langle (A - \beta^{-1} I) X_2, X_2 \rangle$

$\geq (1 - \beta \delta_b^{-1} \Delta_h) \delta_h) ||X_1||^2 + \beta ||X_3 + \beta^{-1} X_2||^2 + (\delta_a - \beta^{-1}) ||X_2||^2.$

It is clear all the coefficients in the last inequality are positive and hence there exists a positive constant K such that

$$W_0 \geq K(||X_1||^2 + ||X_2||^2 + ||X_3||^2)$$

and

(2.2) $$||X_1||^2 + ||X_2||^2 + ||X_3||^2 \leq K^{-1} W_0.$$

For the time derivative the functional W_0 by a straightforward calculation from (1.3) and (2.1), we obtain

$$\dot{W}_0(t) = -\langle(\beta A - I)X_3, X_3\rangle - \langle G(X_2), X_2\rangle + \beta\langle J_H(X_1)X_2, X_2\rangle$$

$$+\beta \int_{t-\tau(t)}^t \langle X_3(t), J_G(X_2(s))X_3(s)\rangle ds + \int_{t-\tau(t)}^t \langle X_2(s), J_G(X_2(s))X_3(s)\rangle ds$$

$$+\beta \int_{t-\tau(t)}^t \langle X_3(t), J_H(X_1(s))X_2(s)\rangle ds + \int_{t-\tau(t)}^t \langle X_2(t), J_H(X_1(s))X_2(s)\rangle ds$$

$$+\lambda\tau(t)||X_2||^2 + \eta\tau(t)||X_3||^2 - \lambda(1-\tau'(t))\int_{t-\tau(t)}^t ||X_2(\theta)||^2 d\theta$$

$$-\eta(1-\tau'(t))\int_{t-\tau(t)}^t ||X_3(\theta)||^2 d\theta.$$

The assumptions of Theorem 1, $0 \le \tau(t) \le \tau_0$ and $\tau'(t) \le \tau_1$, $0 < \tau_1 < 1$, imply that

$$\langle(\beta A - I)X_3, X_3\rangle \ge (\beta\delta_a - 1)||X_3||^2,$$

$$\langle G(X_2), X_2\rangle = \int_0^1 \langle J_G(\sigma X_2)X_2, X_2\rangle d\sigma$$

$$\ge \int_0^1 \langle\delta_b X_2, X_2\rangle d\sigma = \delta_b||X_2||^2,$$

$$\beta\langle J_H(X_1)X_2, X_2\rangle \le \beta\Delta_h||X_2||^2,$$

$$\beta \int_{t-\tau(t)}^t \langle X_3(t), J_G(X_2(s))X_3(s)\rangle ds \le \beta||X_3(t)|| \int_{t-\tau(t)}^t ||J_G(X_2(s))||||X_3(s)||ds$$

$$\le \beta\Delta_b||X_3(t)|| \int_{t-\tau(t)}^t ||X_3(s)||ds$$

$$\le \tfrac{1}{2}\beta\Delta_b \int_{t-\tau(t)}^t \left\{||X_3(t)||^2 + ||X_3(s)||^2\right\}ds$$

$$= \tfrac{1}{2}\beta\Delta_b\tau(t)||X_3||^2 + \tfrac{1}{2}\beta\Delta_b \int_{t-\tau(t)}^t ||X_3(s)||^2 ds$$

$$\le \tfrac{1}{2}\beta\Delta_b\tau_0||X_3||^2 + \tfrac{1}{2}\beta\Delta_b \int_{t-\tau(t)}^t ||X_3(s)||^2 ds,$$

$$\int_{t-\tau(t)}^{t} \langle X_2(t), J_G(X_2(s))X_3(s)\rangle ds \leq \|X_2(t)\| \int_{t-\tau(t)}^{t} \|J_G(X_2(s))\|\|X_3(s)\| ds$$

$$\leq \Delta_b \|X_2(t)\| \int_{t-\tau(t)}^{t} \|X_3(s)\| ds$$

$$\leq \tfrac{1}{2}\Delta_b \int_{t-\tau(t)}^{t} \left\{ \|X_2(t)\|^2 + \|X_3(s)\|^2 \right\} ds$$

$$= \tfrac{1}{2}\Delta_b \tau(t)\|X_2\|^2 + \tfrac{1}{2}\Delta_b \int_{t-\tau(t)}^{t} \|X_3(s)\|^2 ds$$

$$\leq \tfrac{1}{2}\Delta_b \tau_0 \|X_2\|^2 + \tfrac{1}{2}\Delta_b \int_{t-\tau(t)}^{t} \|X_3(s)\|^2 ds,$$

$$\beta \int_{t-\tau(t)}^{t} \langle X_3(t), J_H(X_1(s))X_2(s)\rangle ds \leq \beta \|X_3(t)\| \int_{t-\tau(t)}^{t} \|J_H(X_1(s))\|\|X_2(s)\| ds$$

$$\leq \beta \Delta_h \|X_3(t)\| \int_{t-\tau(t)}^{t} \|X_2(s)\| ds$$

$$\leq \tfrac{1}{2}\beta \Delta_h \int_{t-\tau(t)}^{t} \left\{ \|X_3(t)\|^2 + \|X_2(s)\|^2 \right\} ds$$

$$= \tfrac{1}{2}\beta \Delta_h \tau_0 \|X_3\|^2 + \tfrac{1}{2}\beta_h \int_{t-\tau(t)}^{t} \|X_2(s)\|^2 ds,$$

$$\int_{t-\tau(t)}^{t} \langle X_2(t), J_H(X_1(s))X_2(s)\rangle ds \leq \|X_2(t)\| \int_{t-\tau(t)}^{t} \|J_H(X_1(s))\|\|X_2(s)\| ds$$

$$\leq \Delta_h \|X_2(t)\| \int_{t-\tau(t)}^{t} \|X_2(s)\| ds$$

$$\leq \tfrac{1}{2}\Delta_h \int_{t-\tau(t)}^{t} \left\{ \|X_2(t)\|^2 + \|X_2(s)\|^2 \right\} ds$$

$$= \tfrac{1}{2}\Delta_h \tau(t)\|X_2\|^2 + \tfrac{1}{2}\Delta_h \int_{t-\tau(t)}^{t} \|X_2(s)\|^2 ds$$

$$\leq \tfrac{1}{2}\Delta_h \tau_0 \|X_2\|^2 + \tfrac{1}{2}\Delta_h \int_{t-\tau(t)}^{t} \|X_2(s)\|^2 ds,$$

$$\lambda \tau(t)\|X_2\|^2 \leq \lambda \tau_0 \|X_2\|^2,$$

$$\eta \tau(t)\|X_3\|^2 \leq \eta \tau_0 \|X_3\|^2,$$

$$-\lambda(1-\tau'(t)) \int_{t-\tau(t)}^{t} \|X_2(\theta)\|^2 d\theta \leq -\lambda(1-\tau_1) \int_{t-\tau(t)}^{t} \|X_2(\theta)\|^2 d\theta,$$

$$-\eta(1-\tau'(t)) \int_{t-\tau(t)}^{t} \|X_3(\theta)\|^2 d\theta \leq -\eta(1-\tau_1) \int_{t-\tau(t)}^{t} \|X_3(\theta)\|^2 d\theta.$$

On gathering the obtained inequalities into $\dot{W}_0(t)$, we arrive at

$$\dot{W}_0(t) \leq -\left\{ \delta_b - \beta\Delta_h - \tfrac{1}{2}(\Delta_b + \Delta_h + 2\lambda)\tau_0 \right\}\|X_2\|^2$$

$$-\left\{ (\beta\delta_a - 1) - \tfrac{1}{2}(\beta(\Delta_b + \Delta_h) + 2\eta)\tau_0 \right\}\|X_3\|^2$$

$$+\left\{ \tfrac{1}{2}(\beta + 1)\Delta_h - \lambda(1 - \tau_1) \right\} \int_{t-\tau(t)}^{t} \|X_2(\theta)\|^2 ds$$

$$+\left\{ \tfrac{1}{2}(\beta + 1)\Delta_b - \eta(1 - \tau_1) \right\} \int_{t-\tau(t)}^{t} \|X_3(\theta)\|^2 ds.$$

Let

$$\lambda = \frac{(\beta + 1)\Delta_h}{2(1 - \tau_1)}$$

and

$$\eta = \frac{(\beta + 1)\Delta_b}{2(1 - \tau_1)}.$$

Hence

$$\dot{W}_0(t) \leq -\left\{ \delta_b - \beta\Delta_h - \tfrac{1}{2}\left(\frac{(\Delta_b + \Delta_h)(1 - \tau_1) + (\beta + 1)\Delta_h}{1 - \tau_1} \right)\tau_0 \right\}\|X_2\|^2$$

$$-\left\{ (\beta\delta_b - 1) - \tfrac{1}{2}\left(\frac{\beta(\Delta_b + \Delta_h)(1 - \tau_1) + (\beta + 1)\Delta_b}{1 - \tau_1} \right)\tau_0 \right\}\|X_3\|^2.$$

If

$$\tau_0 < \min\left\{ \frac{2(\delta_b - \beta\triangle_h)(1 - \tau_1)}{(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_b}, \frac{2(\beta\delta_b - 1)(1 - \tau_1)}{\beta(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_h} \right\},$$

$0 < \tau_1 < 1,$

then

$$\dot{W}_0(t) \leq -K_1\|X_2\|^2 - K_2\|X_3\|^2 \leq 0$$

for some positive constants K_1 and K_2. In addition, we can easily see that

$$W_0 \ (X_1, X_2, X_3) \to \infty \quad as \quad \|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \to \infty.$$

Consider the set defined by

$$Q \equiv \{(X_1, X_2, X_3) : \dot{W}_0(X_1, X_2, X_3) = 0\}.$$

When we apply the LaSalle's invariance principle, we observe that $(X_1, X_2, X_3) \in Q$ implies that $X_2 = X_3 = 0$ and hence $X_1 = \mu$, $(\mu \neq 0$ is a constant vector). From the last estimate and system (1.3), we have $H(\mu) = 0$, which necessarily implies that $\mu = 0$ since $H(0) = 0$. Therefore

$$X_1 = X_2 = X_3 = 0.$$

In fact, this result implies that the largest invariant set contained in Q is $(0, 0, 0) \in Q$. By Lemma 1, we conclude that the zero solution of system (1.3) is asymptotically stable. Hence, the zero solution of Eq. (1.2) is the globally asymptotic stable. This completes the proof of Theorem 1.

Theorem 2. If assumptions $(A1) - (A3)$ and

$$\tau_0 < min\left\{ \frac{2(\delta_b - \beta\triangle_h)(1 - \tau_1)}{(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_b}, \frac{2(\beta\delta_b - 1)(1 - \tau_1)}{\beta(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_h} \right\},$$

hold, then the zero solution of Eq. (1.2) is uniformly stable, where τ_1 and β are positive constant with $0 < \tau_1 < 1$.

Proof. To prove Theorem 2, our main tool is the functional W_0 given by (2.1). It is clear from the proof of Theorem 1 that the functional W_0 and its time derivative satisfy the assumptions of Theorem A, except $W_2(\|\phi\|)$;

$$W_1(|\phi(0)|) \leq W_0 \leq W_2(\|\phi\|), \dot{W}(\phi) \leq 0.$$

Besides, subject to the assumptions of Theorem 2, it can be easily obtained that $W_0 \leq W_2(\|\phi\|)$ We omit the detail of the proof. This completes the proof of Theorem 2.

3. Boundedness

Let $F(.) \neq 0$. The boundedness results of this paper are given by the following theorems.

Theorem 3. We assume that all the assumptions of Theorem 1 hold, except $F(.) \equiv 0$. Further, we suppose that there exists a non-negative and continuous function $P = P(t)$ such that

$$\|F(t, X_1, X_2, X_3)\| \leq P(t) \quad for \ all \ \ t \geq 0, \quad max \ P(t) < \infty \ \ and \ \ P \in L^1(0, \infty),$$

where $L^1(0, \infty)$ denotes the space of Lebesgue integrable functions.
If

$$\tau_0 < min\left\{\frac{2(\delta_b - \beta\triangle_h)(1 - \tau_1)}{(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_b}, \frac{2(\beta\delta_b - 1)(1 - \tau_1)}{\beta(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_h}\right\},$$

$$0 < \tau_1 < 1,$$

then there exists a constant $M > 0$ such that any solution $(X_1(t), X_2(t), X_3(t))$
of system (1.3) determined by

$$X_1(0) = X_{10}, X_2(0) = X_{20}, X_3(0) = X_{30}$$

satisfies

$$\|X_1(t)\| \le M, \|X_2(t)\| \le M, \|X_3(t)\| \le M$$

for all $t \in \mathcal{R}^+$.

Proof. Let $F(.) = F(t, X_1, X_2, X_3)$. For the case of $F(.) \ne 0$, it can be
concluded that

$$\dot{W}_0(t) \le -K_1\|X_2\|^2 - K_2\|X_3\|^2 + \langle X_2, F(.)\rangle + \langle \beta X_3, F(.)\rangle.$$

Then

$$\dot{W}_0(t) \le (\|X_2\| + \beta\|X_3\|)\|F(.)\|$$

$$\le K_3(\|X_2\| + \|X_3\|)\|F(.)\|$$

$$\le K_3(2 + \|X_2\|^2 + \|X_3\|^2)P(t)$$

$$\le 2K_3P(t) + K^{-1}K_3W_0(t)P(t),$$

by the assumptions of Theorem 3 and (2.2), where

$$K_3 = max\{1, \beta\}.$$

The integration of both sides of the last inequality, between 0 to t,
$(t \ge 0)$, leads that

$$W_0(t) \le W_0(0) + 2K_3\int_0^t P(s)ds + K^{-1}K_3\int_0^t W_0(s)P(s)ds.$$

Let

$$M = W_0(0) + 2K_3\int_0^\infty P(s)ds.$$

Then

$$W_0(t) \le M + K^{-1}K_3 \int_0^\infty W_0(s)P(s)ds.$$

By noting the Gronwall-Bellman inequality (see Ahmad and Rama Mohana Rao [9, pp.41]), we can get

$$W_0(t) \le Mexp(K^{-1}K_3 \int_0^\infty P(s)ds).$$

By the estimate $\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \le K^{-1}W_0$ and the assumption $P \in L^1(0, \infty)$, we can conclude that all solutions of system (1.3) are bounded. This completes the proof of Theorem 3.

Theorem 4. If the assumptions Theorem 3 and

$$\tau_0 < min\left\{ \frac{2(\delta_b - \beta\triangle_h)(1 - \tau_1)}{(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_b}, \frac{2(\beta\delta_b - 1)(1 - \tau_1)}{\beta(\triangle_b + \triangle_h)(1 - \tau_1) + (\beta + 1)\triangle_h} \right\},$$

hold, then the zero solution of Eq. (1.2) is uniformly bounded, where τ_1 is positive constant with $0 < \tau_1 < 1$.

Proof. To complete the poof of Theorem 4, the main tool is the functional W_0 given by (2.1). When we benefit from the functional W_0 and the assumptions of Theorem 4, we can easily complete the poof of Theorem 4. Therefore, we omit the details of the proof.

4. Conclusion

We consider a functional differential system of third order with variable delay. We investigate the globally asymptotic stability/uniformly stability/boundedness/ uniformly boundedness of solutions. The technique of proofs involves defining an appropriate Lyapunov functional. Our results include, improve and complete some recent results in the literature.

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