Proyecciones Journal of Mathematics
Vol. 35, N^o 3, pp. 251-262, September 2016.
Universidad Católica del Norte
Antofagasta - Chile

Total edge irregularity strength of disjoint union of double wheel graphs

P. Jeyanthi
Govindammal Aditanar College for Women, India
and
A. Sudha
Wavoo Wajeeha Women's College of Arts & Science, India
Received : October 2015. Accepted : March 2016

Abstract

An edge irregular total k-labeling $f : V \cup E \to \{1, 2, 3, \ldots, k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of G in such a way that for any two different edges uv and $u'v'$ their weights $f(u) + f(uv) + f(v)$ and $f(u') + f(u'v') + f(v')$ are distinct. The total edge irregularity strength $tes(G)$ is defined as the minimum k for which the graph G has an edge irregular total k-labeling. In this paper, we determine the total edge irregularity strength of disjoint union of p isomorphic double wheel graphs and disjoint union of p consecutive non-isomorphic double wheel graphs.

Keywords: *Irregularity strength; total edge irregularity strength; edge irregular total labeling, disjoint union of double wheel graphs.*

AMS Classification (2010): *05C78.*

1. Introduction

The graphs in this paper are simple, finite and undirected. In [2] *Bača* et al. defined the notion of edge irregular total k-labeling of a graph G as a funtion $\phi : V \cup E \rightarrow \{1, 2, \ldots, k\}$ such that the edge weights $wt_{\phi}(uv) = \phi(u) + \phi(uv) + \phi(v)$ are distinct for all the edges. That is $wt_{\phi}(uv) \neq wt_{\phi}(u'v')$ for every pair of edges $uv, u'v' \in E$. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G, $tes(G)$. They found a lower bound for the total edge irregularity strength of a graph as

$$(1.1) \qquad tes(G) \geq max\left\{ \left\lceil \frac{(|E(G)|+2)}{3} \right\rceil, \left\lceil \frac{(\Delta(G)+1)}{2} \right\rceil \right\}$$

where $\Delta(G)$ is the maximum degree of G. *Ivančo* and *Jendroľ* [3] posed the following conjecture.

Conjecture:1.1 [3] Let G be an arbitrary graph different from K_5. Then

$$(1.2) \qquad tes(G) = max\left\{ \left\lceil \frac{(|E(G)|+2)}{3} \right\rceil, \left\lceil \frac{(\Delta(G)+1)}{2} \right\rceil \right\}$$

Conjecture(1.1) has been verified by several authors for several families of graphs. Motivated by the results in [1,4,5,6] we determine the total edge irregularity strength of the disjoint union of double wheel graphs. A *double wheel graph* DW_n of size n can be composed of $2C_n + K_1, n \geq 3$,that is it consists of two cycles of size n, where all the vertices of the two cycles are connected to a common hub.

2. Main Results

In this section, first we determine the total edge irregularity strength of the disjoint union of p isomorphic double wheel graphs with the vertex set $V(pDW_n) = \{v_j, v_i^j, u_i^j : 1 \leq i \leq n, 1 \leq j \leq p\}$ and the edge set $E(pDW_n) = \{v_j v_i^j, v_j u_i^j, v_i^j v_{i+1}^j, u_i^j u_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq p\}$ where the subscript i is taken modulo n.

Lemma 2.1. $tes(2DW_n) = \left\lceil \frac{8n+2}{3} \right\rceil, n \geq 3.$

Proof. Since $|E(2DW_n)| = 8n$, $tes(2DW_n) \geq \left\lceil \frac{8n+2}{3} \right\rceil$ by (1.1). Let $k = \left\lceil \frac{8n+2}{3} \right\rceil$. To prove the reverse inequality we define $f : V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ as follows:
$f(v_1) = 1$;
$f(v_2) = n$;
$f(v_i^1) = i, 1 \leq i \leq n$;
$f(v_i^2) = k, 1 \leq i \leq n$;
$f(u_i^1) = n, 1 \leq i \leq n$;
$f(u_i^2) = k, 1 \leq i \leq n$;
$f(v_1 v_i^1) = 1, 1 \leq i \leq n$;
$f(v_1 u_i^1) = 1 + i, 1 \leq i \leq n$;
$f(v_i^1 v_{i+1}^1) = 2n + 1 - i, 1 \leq i \leq n$;
$f(u_i^1 u_{i+1}^1) = n + 2 + i, 1 \leq i \leq n$;
$f(v_2 v_i^2) = 3n + 2 - k + i, 1 \leq i \leq n$;
$f(v_2 u_i^2) = 4n + 2 - k + i, 1 \leq i \leq n$;
$f(v_i^2 v_{i+1}^2) = 6n + 2 - 2k + i, 1 \leq i \leq n$;
$f(u_i^2 u_{i+1}^2) = 7n + 2 - 2k + i, 1 \leq i \leq n$.
We observe that,
$wt(v_1 v_i^1) = 2 + i, 1 \leq i \leq n$;
$wt(v_1 u_i^1) = n + 2 + i, 1 \leq i \leq n$;
$wt(v_i^1 v_{i+1}^1) = 2n + 2 + i, 1 \leq i \leq n$;
$wt(u_i^1 u_{i+1}^1) = 3n + 2 + i, 1 \leq i \leq n$;
$wt(v_2 v_i^2) = 4n + 2 + i, 1 \leq i \leq n$;
$wt(v_2 u_i^2) = 5n + 2 + i, 1 \leq i \leq n$;
$wt(v_i^2 v_{i+1}^2) = 6n + 2 + i, 1 \leq i \leq n$;
$wt(u_i^2 u_{i+1}^2) = 7n + 2 + i, 1 \leq i \leq n$. □

Theorem 2.2. *Let $n \geq 3$ and $p \geq 3$ be two integers. Then the total edge irregularity strength of disjoint union of p isomorphic double wheel graphs is $\left\lceil \frac{4pn+2}{3} \right\rceil$.*

Proof. Since $|E(pDW_n)| = 4pn$, by (1.1) we have $tes(pW_n) \geq \left\lceil \frac{4pn+2}{3} \right\rceil$. Let $k = \left\lceil \frac{4pn+2}{3} \right\rceil$. To prove the reverse inequality, we define the total edge irregular k-labeling f for $1 \leq i \leq n$ and $1 \leq j \leq p$ as follows:
$f(v_j) = f(v_i^j) = f(u_i^j) = \min\{(j-1)2n + 1, k\}$.
For $1 \leq j \leq p$ and $(j-1)2n + 1 \leq k$,
$f(v_j v_i^j) = i$;
$f(v_j u_i^j) = n + i$;

$f(v_i^j v_{i+1}^j) = 2n + i$;
$f(u_i^j u_{i+1}^j) = 3n + i$.
For $1 \leq j \leq p$ and $(j-1)2n + 1 > k$,
$f(v_j v_i^j) = 4(j-1)n + 2 - 2k + i$;
$f(v_j u_i^j) = 4(j-1)n + 2 - 2k + n + i$;
$f(v_i^j v_{i+1}^j) = 4(j-1)n + 2 - 2k + 2n + i$;
$f(u_i^j u_{i+1}^j) = 4(j-1)n + 2 - 2k + 3n + i$.

We observe that,
$wt(v_j v_i^j) = 4(j-1)n + 2 + i, 1 \leq i \leq n, 1 \leq j \leq p$;
$wt(v_j u_i^j) = 4(j-1)n + 2 + n + i, 1 \leq i \leq n, 1 \leq j \leq p$;
$wt(v_i^j v_{i+1}^j) = 4(j-1)n + 2 + 2n + i, 1 \leq i \leq n, 1 \leq j \leq p$;
$wt(u_i^j u_{i+1}^j) = 4(j-1)n + 2 + 3n + i, 1 \leq i \leq n, 1 \leq j \leq p$.

It can be easily verified that all the vertex and edge labels are at most k and the weights of the edges are pair-wise distinct. Thus the resulting total labeling is the edge irregular total k-labeling. Figure 1 illustrates the edge irregular total labeling of the disjoint union of 4 copies of double wheel graphs. □

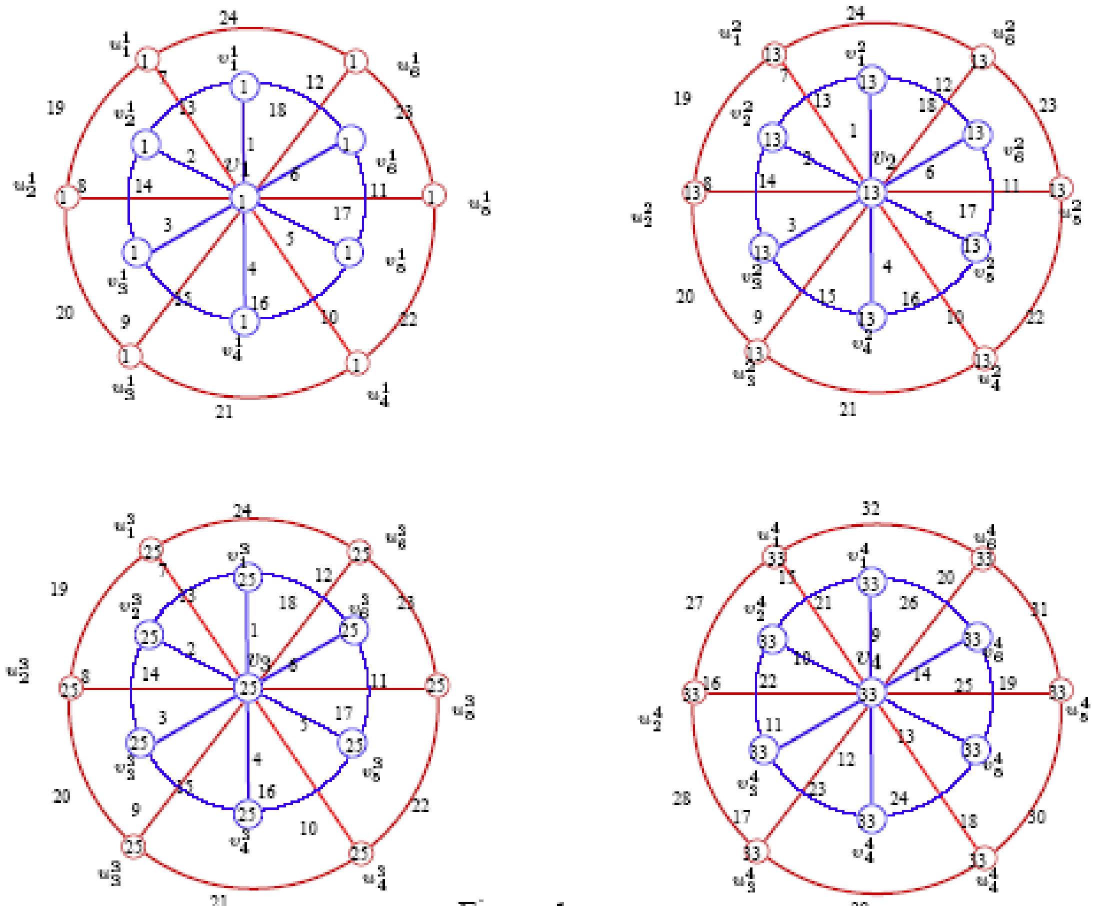

Figure 1

Total edge irregularity strength of disjoint union of 4 isomorphic double Wheel graphs.

$tes(4DW_6) = 33$

Now we determine the total edge irregularity strength of the disjoint union of p consecutive $(n_{i+1} = n_i + 1, i \geq 1)$ non-isomorphic double wheel graphs with the vertex set $V\left(\bigcup_{j=1}^{p} DW_{n_j}\right)$ and the edge set $E\left(\bigcup_{j=1}^{p} DW_{n_j}\right)$ where $V\left(\bigcup_{j=1}^{p} DW_{n_j}\right) = \{v_j, v_i^j, u_i^j : 1 \leq i \leq n_j, 1 \leq j \leq p\}$ and $E\left(\bigcup_{j=1}^{p} DW_{n_j}\right) = \{v_j v_i^j, v_j u_i^j, v_i^j v_{i+1}^j, u_i^j u_{i+1}^j : 1 \leq i \leq n_j, 1 \leq j \leq p\}$ where the subscript i is taken modulo n.

Lemma 2.3. *Let $n_1 \geq 3$ be an integer and $n_2 = n_1 + 1$. Then $tes(DW_{n_1} \cup DW_{n_2}) = \left\lceil \frac{8n_1+6}{3} \right\rceil$.*

Proof. Let $k = \left\lceil \frac{8n_1+6}{3} \right\rceil$. Then by (1.1), $tes(DW_{n_1} \cup W_{n_2}) \geq k$. Now to prove the reverse inequality, we define an edge irregular k-labeling of f as follows:
$f(v_1) = 1;$
$f(v_i^1) = i, 1 \leq i \leq n_1;$
$f(v_i^2) = f(u_i^2) = k, 1 \leq i \leq n_2;$
$f(u_i^1) = n_1, 1 \leq i \leq n_1;$
$f(v_2) = n_1 + 1;$
$f(v_1 v_i^1) = 1, 1 \leq i \leq n_1;$
$f(v_1 u_i^1) = 1 + i, 1 \leq i \leq n_1;$
$f(v_2 v_i^2) = 3n_1 + 1 - k + i, 1 \leq i \leq n_2;$
$f(v_i^1 v_{i+1}^1) = 2n_1 + 1 - i, 1 \leq i \leq n_1;$
$f(u_i^1 u_{i+1}^1) = n_1 + 2 + i, 1 \leq i \leq n_1;$
$f(v_2 u_i^2) = 4n_1 + 2 - k + i, 1 \leq i \leq n_2;$
$f(v_i^2 v_{i+1}^2) = 6n_1 + 4 - 2k + i, 1 \leq i \leq n_2;$
$f(u_i^2 u_{i+1}^2) = 7n_1 + 5 - 2k + i, 1 \leq i \leq n_2.$

We observe that,
for $1 \leq i \leq n_j, j = 1, 2$
$wt(v_1 v_i^1) = 2 + i;$
$wt(v_1 u_i^1) = n_1 + 2 + i;$
$wt(v_i^1 v_{i+1}^1) = 2n_1 + 2 + i;$
$wt(u_i^1 u_{i+1}^1) = 3n_1 + 2 + i;$
$wt(v_2 v_i^2) = 4n_1 + 2 + i;$
$wt(v_2 u_i^2) = 5n_1 + 3 + i;$
$wt(v_i^2 v_{i+1}^2) = 6n_1 + 4 + i;$

$wt(u_i^2 u_{i+1}^2) = 7n_1 + 5 + i.$

It can be easily verified that all the vertex and edge labels are at most k and the weights of the edges are pair-wise distinct. Thus the resulting total labeling is the edge irregular total k-labeling. □

Lemma 2.4. *Let n_1, n_2, n_3 be integers and $n_1 \geq 3$. Then $tes(DW_{n_1} \cup DW_{n_2} \cup DW_{n_3}) = \left\lceil \frac{2(6n_1+7)}{3} \right\rceil$.*

Proof. Since $|E(DW_{n_1} \cup DW_{n_2} \cup DW_{n_3})| = 12(n_1 + 1)$ by (1.1), $tes(DW_{n_1} \cup DW_{n_2} \cup DW_{n_3}) \geq \left\lceil \frac{12(n_1+1)+2)}{3} \right\rceil = \left\lceil \frac{2(6n_1+7)}{3} \right\rceil$.Let $k = \left\lceil \frac{2(6n_1+7)}{3} \right\rceil$.

Now to prove the reverse inequality,we define an edge irregular k-labeling f as follows:
$f(v_1) = 1;$
$f(v_i^1) = f(u_i^1) = 1, 1 \leq i \leq n_1;$
$f(v_2) = f(v_i^2) = f(u_i^2) = 2n + 1, 1 \leq i \leq n_2;$
$f(v_3) = 3n_1 + 3;$
$f(u_i^3) = f(v_i^3) = k, 1 \leq i \leq n_3;$
$f(v_1 v_i^1) = i, 1 \leq i \leq n_1;$
$f(v_1 u_i^1) = n_1 + i, 1 \leq i \leq n_1;$
$f(v_2 v_i^2) = i, 1 \leq i \leq n_2;$
$f(v_2 u_i^2) = n_1 + 1 + i, 1 \leq i \leq n_2;$
$f(v_3 v_i^3) = 5n_1 + 3 - k + i, 1 \leq i \leq n_3;$
$f(v_3 u_i^3) = 6n_1 + 5 - k + i, 1 \leq i \leq n_3;$
$f(v_i^1 v_{i+1}^1) = 2n_1 + i, 1 \leq i \leq n_1;$
$f(u_i^1 u_{i+1}^1) = 3n_1 + i, 1 \leq i \leq n_1;$
$f(v_i^2 v_{i+1}^2) = 2n_1 + 2 + i, 1 \leq i \leq n_2;$
$f(u_i^2 u_{i+1}^2) = 3n_1 + 3 + i, 1 \leq i \leq n_2;$
$f(v_i^3 v_{i+1}^3) = 10n_1 + 10 - 2k + i, 1 \leq i \leq n_3;$
$f(u_i^3 u_{i+1}^3) = 11n_1 + 12 - 2k + i.1 \leq i \leq n_3.$

We observe that,
for$1 \leq i \leq n_j; j = 1, 2, 3$
$wt(v_1 v_i^1) = 2 + i;$
$wt(v_1 u_i^1) = n_1 + 2 + i;$
$wt(v_2 v_i^2) = 4n_1 + 2 + i;$
$wt(v_2 u_i^2) = 5n_1 + 3 + i;$
$wt(v_3 v_i^3) = 8n_1 + 6 + i;$

$wt(v_3u_i^3) = 9n_1 + 8 + i;$
$wt(v_i^1v_{i+1}^1) = 2n_1 + 2 + i;$
$wt(u_i^1u_{i+1}^1) = 3n_1 + 2 + i;$
$wt(v_i^2v_{i+1}^2) = 6n_1 + 4 + i;$
$wt(u_i^2u_{i+1}^2) = 7n_1 + 5 + i;$
$wt(v_i^3v_{i+1}^3) = 10n_1 + 10 + i;$
$wt(u_i^3u_{i+1}^3) = 11n_1 + 12 + i.$

It can be easily verified that all the vertex and edge labels are at most k and the weights of the edges are pair-wise distinct. Thus the resulting total labeling is the edge irregular total k-labeling. □

Theorem 2.5. *The total edge irregularity strength of the disjoint union of* $p\,(p \geq 4)$ *consecutive non-isomorphic double wheel graphs is* $\left\lceil \frac{2p(2n_1+p-1)+2}{3} \right\rceil$.

Proof. Since $|E\left(\bigcup_{j=1}^{p} DW_{n_j}\right)| = 2p(2n_1+p-1)$ by (1.1) $tes\left(\bigcup_{j=1}^{p} DW_{n_j}\right) \geq \left\lceil \frac{2p(2n_1+p-1)+2}{3} \right\rceil$. Let $k = \left\lceil \frac{2p(2n_1+p-1)+2}{3} \right\rceil$. To prove the reverse inequality we define the total edge irregular k-labeling f for $1 \leq i \leq n_j$ and $1 \leq j \leq p$ as follows:
$f(v_1) = f(v_i^1) = f(u_i^1) = 1, 1 \leq i \leq n_1;$
$f(v_1v_i^1) = i, 1 \leq i \leq n_1;$
$f(v_1u_i^1) = n_1 + i, 1 \leq i \leq n_1;$
$f(v_i^1v_{i+1}^1) = 2n_1 + i, 1 \leq i \leq n_1;$
$f(u_i^1u_{i+1}^1) = 3n_1 + i, 1 \leq i \leq n_1;$
$f(v_j) = f(v_i^j) = f(u_i^j) = \min\left\{2(\sum_{s=1}^{j-1} n_s) + 1, k\right\}, 1 \leq i \leq n_j$ and $2 \leq j \leq p$.

For $1 \leq i \leq n_j$ and $2 \leq i \leq p$, $2\left[\sum_{s=1}^{j-1} n_s\right] + 1 \leq k$,
$f(v_jv_i^j) = i;$
$f(v_ju_i^j) = n_j + i;$
$f(v_i^jv_{i+1}^j) = 2n_j + i;$
$f(u_i^ju_{i+1}^j) = 3n_j + i.$

We observe that,
$wt(v_jv_i^j) = 2\left[2\left(\sum_{s=1}^{j-1} n_s)\right] + 1\right] + i;$
$wt(v_ju_i^j) = 2\left[2\left(\sum_{s=1}^{j-1} n_s\right) + 1\right] + n_j + i;$

$wt(v_i^j v_{i+1}^j) = 2\left[2\left(\sum_{s=1}^{j-1} n_s\right) + 1\right] + 2n_j + i;$
$wt(u_i^j u_{i+1}^j) = 2\left[2\left(\sum_{s=1}^{j-1} n_s\right) + 1\right] + 3n_j + i.$
For $1 \leq i \leq n_j$ and $2 \leq i \leq p$, $2\left[\sum_{s=1}^{j-1} n_s\right] + 1 > k$,
$f(v_j v_i^j) = 4n_{j-1} + 2\left[2\left(\sum_{s=1}^{j-2} n_s\right) + 1\right] - 2k + i;$
$f(v_j u_i^j) = 4n_{j-i} + 2\left[2\left(\sum_{s=1}^{j-2} n_s\right) + 1\right] + n_j - 2k + i;$
$f(v_i^j v_{i+1}^j) = 4n_{j-i} + 2\left[2\left(\sum_{s=1}^{j-2} n_s\right) + 1\right] + 2n_j - 2k + i;$
$f(u_i^j u_{i+1}^j) = 4n_{j-i} + 2\left[2\left(\sum_{s=1}^{j-2} n_s\right) + 1\right] + 3n_j - 2k + i.$

We observe that,
$wt(v_j v_i^j) = 4n_{j-1} + 2\left[2\left(\sum_{s=1}^{j-2} n_s\right) + 1\right] + i;$
$wt(v_j u_i^j) = 4n_{j-i} + 2\left[2\left(\sum_{s=1}^{j-2} n_s\right) + 1\right] + n_j + i;$
$wt(v_i^j v_{i+1}^j) = 4n_{j-i} + 2\left[2\left(\sum_{s=1}^{j-2} n_s\right) + 1\right] + 2n_j + i;$
$wt(u_i^j u_{i+1}^j) = 4n_{j-i} + 2\left[2\left(\sum_{s=1}^{j-2} n_s\right) + 1\right] + 3n_j + i.$

It can be easily verified that all the vertex and edge labels are at most k and the weights of the edges are pair-wise distinct .Thus the resulting total labeling is the edge irregular k-labeling. Figure 2 illustrates the edge irregular total labelings of the disjoint union of 4 Consecutive non-isomorphic double wheel graphs $DW_3 \cup DW_4 \cup DW_5 \cup DW_6$. $\square$

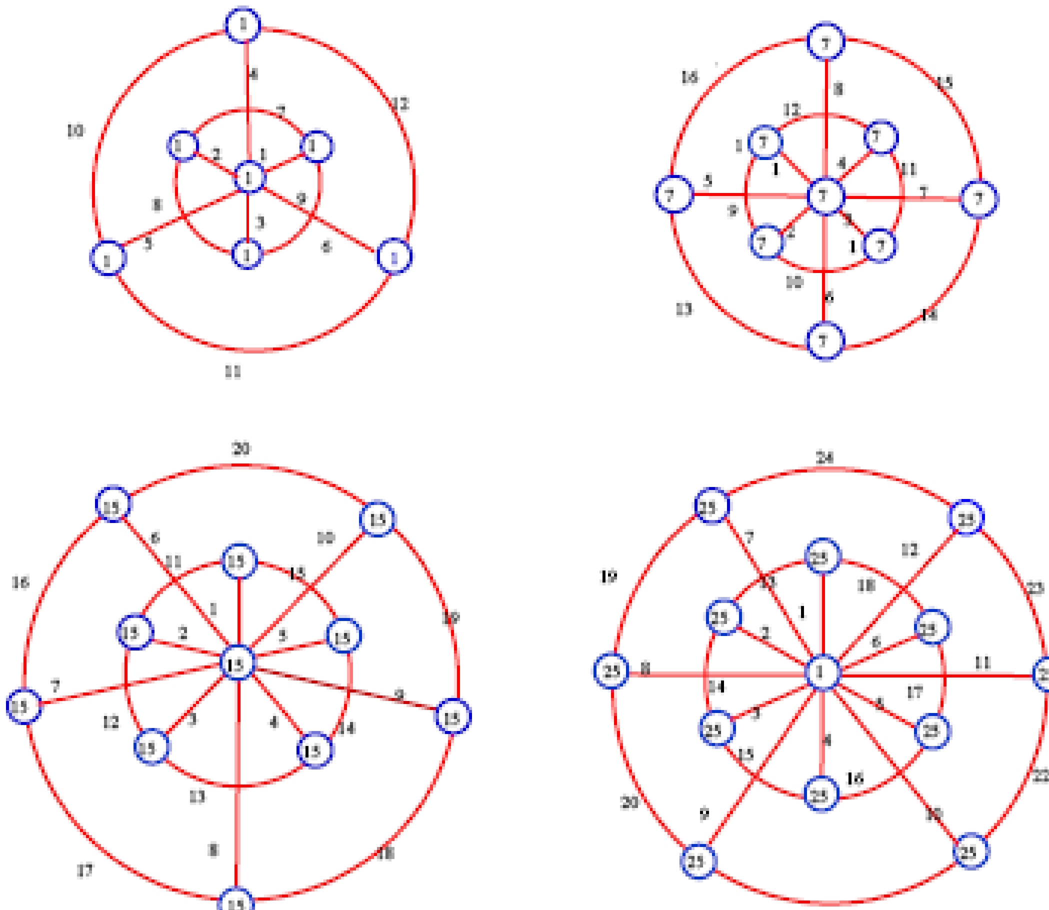

Figure 2

Total edge irregularity strength of disjoint union of 4 non- isomorphic double Wheel graphs.

$$tes(DW_3 \bigcup DW_4 \bigcup DW_5 \bigcup DW_6) = 25$$

3. Conclusion

In this paper we determine the total edge irregularity strength of the disjoint union of p isomorphic double wheel graphs and disjoint union of p consecutive non isomorphic double wheel graphs. We conclude this paper by stating the following open problem.

Open problem:
For $m \geq 2$,find the exact value of the total edge irregularity strength of a disjoint union of m arbitrary double wheel graphs.

References

[1] A. Ahmad and *M.Bača*, and Muhammad Numan, *On irregularity strength of the disjoint union of friendship graphs,* Electronic Journal of Graph Theory and Applications, **11** (2), pp. 100–108, (2013).

[2] *M.Bača,S.Jendroľ*, M. Miller and J. Ryan, *On irregular total labellings,* Discrete Math., 307, pp. 1378-1388, (2007).

[3] *J.Ivančo,S.Jendroľ*, *Total edge irregularity strength of trees,* Discussiones Math. Graph Theory, 26, pp. 449-456, (2006).

[4] M.K.Siddiqui,A.Ahmad,M.F.Nadeem,Y.Bashir, *Total edge irregularity strength of the disjoint union of sun graphs,* International Journal of Mathematics and Soft Computing 3 (1), pp. 21-27, (2013).

[5] P. Jeyanthi and A. Sudha, *Total Edge Irregularity Strength of Disjoint Union of Wheel Graphs,* Electron. Notes in Discrete Math., 48, pp. 175-182, (2015).

P. Jeyanthi
Research Centre,
Department of Mathematics
Govindammal Aditanar College for Women
Tiruchendur-628 215, Tamil Nadu,
India
e-mail: jeyajeyanthi@rediffmail.com

and

A. Sudha
Department of Mathematics
Wavoo Wajeeha Women's College of Arts & Science,
Kayalpatnam -628 204,Tamil Nadu, India
e-mail: sudhathanalakshmi@gmail.com