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Unicyclic graphs with equal domination and complementary tree domination numbers

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Abstract

Let $G = (V, E)$ be a simple graph. A set $D \subseteq V(G)$ is a dominating set if every vertex in $V(G) \setminus D$ is adjacent to a vertex of D. A dominating set D of a graph G is a complementary tree dominating set if induced sub graph $\langle V \setminus D \rangle$ is a tree. The domination (complementary tree domination, respectively) number of G is the minimum cardinality of a dominating (complementary tree dominating, respectively) set of G. We characterize all unicyclic graphs with equal domination and complementary tree domination numbers.

Keywords : *Domination, complementary tree domination, unicyclic graphs.*

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1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We denote $L(G)$ to be the set of leaves of the graph G and $S(G)$ is the set of all support vertices of G. The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a complementary tree dominating set, abbreviated CTDS, of G if the induced sub graph $\langle V \setminus D\rangle$ is a tree. The domination (complementary tree domination, respectively) number of a graph G, denoted by $\gamma(G)$ ($\gamma_{ctd}(G)$, respectively), is the minimum cardinality of a dominating (complementary tree dominating, respectively) set of G. A complementary tree dominating set of G of minimum cardinality is called a $\gamma_{ctd}(G)$-set. The complementary tree domination in graphs was studied in [5]. For a comprehensive survey of domination in graphs, see [1, 2].

A unicyclic graph is a graph that contains exactly one cycle. In this paper we provide a constructive characterization of all unicyclic graphs with equal domination number and complementary tree domination number. In [3], unicyclic graphs with equal total and total outer-connected domination numbers are characterized.

2. Preliminary results

We begin with the following straightforward observation.

Observation 1. *[5] Every leaf of a graph G is in every $\gamma_{ctd}(G)$-set.*

In [4] trees with equal domination number and complementary tree domination numbers are characterized. For this purpose the family $\mathcal{T}$ of trees $T = T_k$ is defined. Let T_1 be a path P_4. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a path P_2 by joining its any vertex to a vertex of T_k, which is not a leaf and is adjacent to a support vertex of degree two.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining its any vertex to a support vertex of T_k.

Theorem 1. *[4] Let T be a tree. Then $\gamma_{ctd}(T) = \gamma(T)$ if and only if $T \in \mathcal{T}$.*

3. Unicyclic graphs

We characterize all connected unicyclic graphs for which $\gamma(G) = \gamma_{ctd}(G)$. To this, we define $\mathcal{C}$ to be the family of all graphs G for which exists a tree T belonging to the family $\mathcal{T}$, such that G is obtained from T by the operation:
Operation $\mathcal{B}$: Let u, v be any two support vertices of T. Let x and y be the leaves adjacent to u and v, respectively. Identify x with y.
Let us also assume that C_3 and C_4 belong to $\mathcal{C}$ and observe that C_3 is obtained from $P_4 \in \mathcal{T}$ by the above operation.

Lemma 2. *If G belong to the family $\mathcal{C}$, then $\gamma(G) = \gamma_{ctd}(G)$.*

Proof. If G is a cycle belonging to $\mathcal{C}$, then the result is immediate. Let us now assume that G is obtained from a tree $T \in \mathcal{T}$ by Operation $\mathcal{B}$. Let G be obtained from T by identifying the leaves x and y. Denote by w the vertex obtained by identifying x and y. It is easy to see that $L(G) \cup \{w\}$ is a minimum dominating set of G. Thus $\gamma(G) = |L(G)| + 1$. On the other hand, $L(G) \cup \{w\}$ is a complementary tree dominating set of G. Thus we have $|L(G)| + 1 = \gamma(G) \leq \gamma_{ctd}(G) \leq |L(G)| + 1$. Thus we have $\gamma(G) = \gamma_{ctd}(G)$. $\square$

Lemma 3. *If G is a connected unicyclic graph with $\gamma(G) = \gamma_{ctd}(G)$, then G belongs to family $\mathcal{C}$.*

Proof. Let G be a connected unicyclic graph, where $C_k = (v_1, v_2, v_3, \cdots, v_k)$ is the unique cycle of G. Assume first that each vertex of C_k is of degree 2. Then G is a cycle C_k for some $k \geq 3$. It is clear that $\gamma_{ctd}(C_k) = k - 2$ for $k \geq 3$. On the other hand, $\gamma(C_k) < k - 2$ for $k \geq 5$. Thus $\gamma(C_k) = \gamma_{ctd}(C_k)$ if $k \in \{3, 4\}$.

Assume that G is not a cycle. If $v_i \in V(C_k)$, then let $T(v_i)$ be the tree obtained from G by removing edges v_iv_{i+1} and $v_{i-1}v_i$ (where the indices are taken modulo k added 1) and containing v_i. Let v_i be the root of $T(v_i)$. Let D_{ctd} be a minimum complementary tree dominating set of G.

Assume without loss of generality, that $d_G(v_1) \geq 3$, and denote by x any element of $V(T(v_1))$ which is neither a leaf nor a support vertex. Let $x \in D_{ctd}$. Then either $V(G) \setminus D_{ctd} \subseteq V(T(x))$ or $V(G) \setminus D_{ctd} \subseteq V(G) \setminus V(T(x))$, Let $V(G) \setminus D_{ctd} \subseteq V(G) \setminus V(T(x))$. It is clear that $V(G) \setminus D_{ctd}$ contains a cycle, contradiction to the definition of D_{ctd}. Now assume $V(G) \setminus D_{ctd} \subseteq V(T(x))$. Then $D_{ctd} \setminus \{u\}$ where u is a leaf in $T(x)$ is a dominating set of G of smaller cardinality than $\gamma(G)$, a contradiction. Let $x \notin D_{ctd}$. Arguing as above, we get a contradiction. Hence, we conclude that every vertex in $T(v_1)$ is either a support vertex or a leaf.

Assume $V(C) \cap D_{ctd} = \phi$. The complement of D_{ctd} contains a cycle, a contradiction. Now assume, without loss of generality, that $v_1 \in V(C) \cap D_{ctd}$ and $d_G(v_1) \geq 3$. Then obviously $V(T(v_1)) \subseteq D_{ctd}$. It is easy to see that $D_{ctd} \setminus \{u\}$ where u is a leaf in $T(v_1)$ is a dominating set of G of smaller cardinality than $\gamma(G)$, a contradiction. Hence, $d_G(v_1) = 2$.

Now assume $d_G(v_2) \geq 3$ and $d_G(v_k) \geq 3$. Suppose v_2 and v_k is in D_{ctd}, then $D_{ctd} \setminus \{v_1\}$ is a dominating set of cardinality smaller than $\gamma(G)$, a contradiction. Without loss of generality, assume $v_2 \in D_{ctd}$. Arguing as in the previous case, we get $d_G(v_2) = 2$. Assume that v_3 and v_k not in D_{ctd}. Then since D_{ctd} is a complementary tree dominating set, exactly two vertices of $V(C_k)$ belong to D_{ctd}, namely v_1 and v_2. It is easy to see that $T(v_i) \setminus \{v_i\} \in D_{ctd}, 3 \leq i \leq k$. It is obvious that $D_{ctd} \setminus \{v_2\}$ is a dominating set of cardinality smaller than $\gamma(G)$, a contradiction. Thus v_1 is the only vertex in D_{ctd} set of G.

Denote by G_1 the graph obtained from G by removing the edge v_1v_2 and attaching the vertex x to the vertex v_2. It is obvious that $\gamma(G) \leq \gamma(G_1)$. Suppose D_{ctd} is a $\gamma_{ctd}(G_1)$-set of cardinality smaller than $\gamma_{ctd}(G)+1$. The vertex x is a leaf in G_1. By observation 1, the leaf $x \in D_{ctd}$. Then $D'_{ctd} = D_{ctd} \setminus \{x\}$ is a complementary tree dominating set of G. Thus $\gamma(G) = \gamma_{ctd}(G) \leq |D'_{ctd}| \leq \gamma_{ctd}(G_1)-1$. It is easy to observe that $D'_{ctd} \cup \{x\}$ is a complementary tree dominating set of G_1, so $\gamma_{ctd}(G_1) \leq \gamma_{ctd}(G) + 1$. Equivalently, $\gamma_{ctd}(G) \geq \gamma_{ctd}(G_1)-1$. This implies that $\gamma_{ctd}(G) = \gamma_{ctd}(G_1)-1$. Since G_1 is a tree, theorem 1 implies that G_1 belongs to the family $\mathcal{T}$. We conclude that G is obtained from a tree belonging to the family $\mathcal{T}$ by operation $\mathcal{B}$. Therefore, G belongs to the family $\mathcal{C}$. □

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