Proyecciones Journal of Mathematics
Vol. 35, N^o 3, pp. 235-244, September 2016.
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An alternative proof of a Tauberian theorem for Abel summability method

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Received : December 2014. Accepted : July 2016

Abstract

Using a corollary to Karamata's main theorem [Math. Z. 32 (1930), 319–320], we prove that if a slowly decreasing sequence of real numbers is Abel summable, then it is convergent in the ordinary sense.

Subjclass [2010] : *40A05; 40E05; 40G10.*

Keywords : *Abel summability, slowly decreasing sequences, Tauberian conditions and theorems.*

1. Introduction

A number of authors such as Schmidt [9], Maddox [6], Móricz [8], and Talo and Başar [11] have proved several Tauberian theorems for some summability methods for which slowly decreasing condition for sequences is a Tauberian condition. Schmidt [9] obtained that the slowly decreasing condition for sequences of real numbers is a Tauberian condition for Abel summability. Maddox [6] introduced the slowly decreasing sequence in an ordered linear space and proved that a Cesàro summable sequence is convergent if it is slowly decreasing in an ordered linear space. Móricz [8] established a Tauberian theorem which states that ordinary convergence of a sequence follows from its statistical Cesàro summability if it is slowly decreasing. Talo and Başar [11] introduced the concept of slowly decreasing sequences for fuzzy numbers and they proved that the slowly decreasing condition for sequences is a Tauberian condition for the statistical convergence and Cesàro summability for sequences of fuzzy numbers.

Littlewood [5] proved that $n(u_n - u_{n-1}) = O(1)$ is a Tauberian condition for Abel summability of (u_n). But his proof was complicated and based on the repeated differentiation. A first clever and surprisingly simple proof based on Weierstrass approximation theorem of Littlewood's theorem was given by Karamata [2].

The main purpose of this study is to give an alternative simpler proof of the following Tauberian theorem which is more general than Littlewood's theorem [5] for Abel summability method.

Theorem 1.1. *If (u_n) is Abel summable to s and slowly decreasing, then* $\lim_n u_n = s$.

To prove Theorem 1.1, we first obtain Cesàro convergence of the generator sequence of a given sequence (u_n) by means of a corollary to Karamata's main Theorem, and then recover convergence of (u_n) by Tauber's second theorem [12].

Our proof is much easier than the existing one and uses the well known results in Tauberian theory. For a different proof of Theorem 1.1, see [1].

2. Preliminaries

For a sequence $u = (u_n)$ of real numbers, we write (u_n) in terms of (v_n) as

$$u_n = v_n + \sum_{k=1}^{n} \frac{v_k}{k} + u_0, \quad (n = 1, 2, ...) \tag{2.1}$$

where $v_n = \frac{1}{n+1} \sum_{k=1}^{n} k(u_k - u_{k-1})$. The sequence (v_n) is called a generator sequence of (u_n). We note that $\sigma_n^{(1)}(u) = \frac{1}{n+1} \sum_{k=0}^{n} u_k = u_0 + \sum_{k=1}^{n} \frac{v_k}{k}$.

Let $u = (u_n)$ be a sequence of real numbers. For each nonnegative integer m, we define $\sigma_n^{(m)}(u)$ by

$$\sigma_n^{(m)}(u) = \begin{cases} \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} \sigma_k^{(m-1)}(u) & , m \geq 1 \\ u_n & , m = 0 \end{cases}$$

A sequence (u_n) is said to be Abel summable to s if $u_0 + \sum_{n=1}^{\infty} (u_n - u_{n-1}) x^n$ converges for $0 < x < 1$, and tends to s as $x \to 1^-$.

A sequence (u_n) is called (A, m) summable to s if $(\sigma_n^{(m)}(u))$ is Abel summable to s. If $m = 0$, then (A, m) summability reduces to Abel summability. It is clear that Abel summability of (u_n) implies (A, m) summability of (u_n).

Throughout this work, the symbol $[\lambda n]$ denotes the integral part of the product λn.

A sequence (u_n) is said to be slowly decreasing [9] if

$$\lim_{\lambda \to 1^+} \liminf_{n \to \infty} \min_{n+1 \leq k \leq [\lambda n]} (u_k - u_n) \geq 0 \tag{2.2}$$

or equivalently [8],

$$\lim_{\lambda \to 1^-} \liminf_{n \to \infty} \min_{[\lambda n]+1 \leq k \leq n} (u_n - u_k) \geq 0. \tag{2.3}$$

Notice that (u_n) is slowly decreasing if the classical one-sided Tauberian condition of Landau [1] is satisfied, that is, there exists a positive constant $C > 0$ such that

$$n(u_n - u_{n-1}) \geq -C \tag{2.4}$$

for all nonnegative n. Indeed, for any $k > n$, we have

$$u_k - u_n = \sum_{j=n+1}^{k} (u_j - u_{j-1}) \geq -C \sum_{j=n+1}^{k} \frac{1}{j} \geq -C \log\left(\frac{k}{n}\right)$$

whence we conclude that

$$\liminf_{n\to\infty} \min_{n+1\le k\le[\lambda n]} (u_k - u_n) \ge -C\log\lambda, \ \ \lambda > 1.$$

Taking $\lambda \to 1^+$, we have the inequality (2.2).

Note that we used C to denote a constant, possibly different at each occurrence.

A sequence (u_n) is slowly increasing if and only if $(-u_n)$ is slowly decreasing, and an equivalent definition of a slowly increasing sequence as follows:

A sequence (u_n) is said to be slowly increasing if

$$\lim_{\lambda\to 1^+} \limsup_{n\to\infty} \max_{n+1\le k\le[\lambda n]} (u_k - u_n) \le 0. \tag{2.5}$$

The condition (2.5) is reformulated as follows (see [8]):

$$\lim_{\lambda\to 1^-} \limsup_{n\to\infty} \max_{[\lambda n]+1\le k\le n} (u_n - u_k) \le 0. \tag{2.6}$$

It is plain that a sequence (u_n) is said to be slowly oscillating if and only if (u_n) is both slowly increasing and slowly decreasing. Notice that each of the conditions (2.2) and (2.5) is necessary for convergence (see [4]).

If (u_n) converges to s, then (u_n) is Abel summable to s. However, the converse of this statement is not always true. Note that Abel summability of (u_n) implies convergence of (u_n) under certain additional hypotheses called Tauberian conditions. Any theorem which states that convergence of sequence (u_n) follows from Abel summability of (u_n) and some Tauberian condition(s) is called a Tauberian theorem for Abel summability method.

3. Corollary to Karamata's Main Theorem and Lemmas

Our proof is based on the following corollary to Karamata's main theorem and three Lemmas.

Corollary to Karamata's Main Theorem. ([2]) If $u = (u_n)$ is Abel summable to s and $u_n \ge -C$ for some nonnegative C, then $\lim_n \sigma_n^{(1)}(u) = s$.

Lemma 3.1. *([3]) If, for $x \to 1^-$, a function $f(x)$, which is integrable in $[0,1]$, satisfies the limiting relation*

$$(1-x)^2 f(x) \to s, \tag{3.1}$$

then, for $x \to 1^-$, we also have

$$(1-x)\int_0^x f(t)\,dt \to s. \tag{3.2}$$

The next lemma gives a necessary condition for a slowly decreasing sequence in terms of the generator sequence (v_n).

Lemma 3.2. *([7]) If (u_n) is slowly decreasing, then $v_n \geq -C$ for some C, where $v_n = \frac{1}{n+1}\sum_{k=1}^{n} k(u_k - u_{k-1})$.*

Next, we represent the difference $u_n - \sigma_n^{(1)}(u)$ in two different ways.

Lemma 3.3. *([10]) Let $u = (u_n)$ be a sequence of real numbers.*
(i) For $\lambda > 1$ and sufficiently large n,

$$u_n - \sigma_n^{(1)}(u) = \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)\right) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}(u_k - u_n). \tag{3.3}$$

(ii) For $0 < \lambda < 1$ and sufficiently large n,

$$u_n - \sigma_n^{(1)}(u) = \frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n^{(1)}(u) - \sigma_{[\lambda n]}^{(1)}(u)\right) + \frac{1}{n-[\lambda n]}\sum_{k=[\lambda n]+1}^{n}(u_n - u_k). \tag{3.4}$$

4. Proof of Theorem 1.1

Proof. Since (u_n) is Abel summable to s, then $(\sigma_n^{(1)}(u))$ is also Abel summable to s. Hence, we conclude by (2.1) that $(v_n) = \left(\frac{1}{n+1}\sum_{k=0}^{n} k(u_k - u_{k-1})\right)$ is Abel summable to zero by Lemma 3.1. It follows by Lemma 3.2 that there exists a nonnegative C such that

$$v_n \geq -C. \tag{4.1}$$

Taking (4.1) and the fact that (v_n) is Abel summable to zero into account, we obtain by Corollary to Karamata's Main Theorem that $\sigma_n^{(1)}(v) =$

$o(1)$ as $n \to \infty$. Since $(\sigma_n^{(1)}(u))$ is Abel summable to s and $\sigma_n^{(1)}(v) = o(1)$ as $n \to \infty$, we have that $(\sigma_n^{(1)}(u))$ converges to s by Tauber's second theorem [12].

By the fact that every convergent sequence is slowly increasing, we have

$(\sigma_n^{(1)}(u))$ is slowly increasing. Thus, $(-\sigma_n^{(1)}(u))$ is slowly decreasing. Since (s_n) is slowly decreasing, (v_n) is slowly decreasing.

By Lemma 3.3 (i), we have

$$v_n - \sigma_n^{(1)}(v) = \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(v) - \sigma_n^{(1)}(v)\right) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}(v_k - v_n). \tag{4.2}$$

It is easy to verify that for $\lambda > 1$ and sufficiently large n,

$$\frac{\lambda}{2(\lambda-1)} \le \frac{[\lambda n]+1}{[\lambda n]-n} \le \frac{3\lambda}{2(\lambda-1)}. \tag{4.3}$$

By $\sigma_n^{(1)}(v) = o(1)$ as $n \to \infty$ and (4.10), for all $\lambda > 1$,

$$\lim_{n\to\infty} \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(v) - \sigma_n^{(1)}(v)\right) = 0. \tag{4.4}$$

By (4.2) and (4.3), we have

$$v_n - \sigma_n^{(1)}(v) \le \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(v) - \sigma_n^{(1)}(v)\right) - \min_{n+1\le k\le[\lambda n]}(v_k - v_n). \tag{4.5}$$

Taking lim sup of both sides of (4.5), we have

$$\begin{aligned}\limsup_n (v_n - \sigma_n^{(1)}(v)) &\le \limsup_{n\to\infty} \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(v) - \sigma_n^{(1)}(v)\right) \\ &- \liminf_n \min_{n+1\le k\le[\lambda n]}(v_k - v_n).\end{aligned} \tag{4.6}$$

The inequality (4.6) becomes

$$\limsup_n (v_n - \sigma_n^{(1)}(v)) \leq -\liminf_n \min_{n+1 \leq k \leq [\lambda n]} (v_k - v_n) \tag{4.7}$$

by (4.4). Taking $\lambda \to 1^+$ in (4.7), we have

$$\limsup_n (v_n - \sigma_n^{(1)}(v)) \leq 0 \tag{4.8}$$

by (2.2).

By Lemma 3.3 (ii), we have

$$v_n - \sigma_n^{(1)}(v) = \frac{[\lambda n]+1}{n-[\lambda n]} \left(\sigma_n^{(1)}(v) - \sigma_{[\lambda n]}^{(1)}(v)\right) + \frac{1}{n-[\lambda n]} \sum_{k=[\lambda n]+1}^{n} (v_n - v_k). \tag{4.9}$$

It is easy to verify that for $0 < \lambda < 1$ and sufficiently large n,

$$\frac{\lambda}{2(1-\lambda)} \leq \frac{[\lambda n]+1}{n-[\lambda n]} \leq \frac{3\lambda}{2(1-\lambda)}. \tag{4.10}$$

By $\sigma_n^{(1)}(v) = o(1)$ as $n \to \infty$ and (4.10), for all $0 < \lambda < 1$,

$$\lim_{n \to \infty} \frac{[\lambda n]+1}{n-[\lambda n]} \left(\sigma_n^{(1)}(v) - \sigma_{[\lambda n]}^{(1)}(v)\right) = 0. \tag{4.11}$$

By (4.9) and (4.10), we have

$$v_n - \sigma_n^{(1)}(v) \geq \frac{[\lambda n]+1}{n-[\lambda n]} \left(\sigma_n^{(1)}(v) - \sigma_{[\lambda n]}^{(1)}(v)\right) + \min_{[\lambda n]+1 \leq k \leq n} (v_n - v_k). \tag{4.12}$$

Taking lim inf of both sides of (4.12), we have

$$\begin{aligned} \liminf_n (v_n - \sigma_n^{(1)}(v)) \quad &\geq \quad \liminf_n \frac{[\lambda n]+1}{n-[\lambda n]} \left(\sigma_n^{(1)}(v) - \sigma_{[\lambda n]}^{(1)}(v)\right) \\ &+ \quad \liminf_n \min_{[\lambda n]+1 \leq k \leq n} (v_n - v_k). \end{aligned} \tag{4.13}$$

The inequality (4.13) becomes

$$\liminf_n (v_n - \sigma_n^{(1)}(v)) \geq \liminf_n \min_{[\lambda n]+1 \leq k \leq n} (v_n - v_k) \tag{4.14}$$

by (4.11).

Taking $\lambda \to 1^-$ in (4.14), we have

$$\liminf_n (v_n - \sigma_n^{(1)}(v)) \geq 0 \tag{4.15}$$

by (2.3).

Combining (4.8) and (4.15) yields that $v_n = o(1)$ as $n \to \infty$. Since (u_n) is Abel summable to s and $v_n = o(1)$ as $n \to \infty$, $\lim_n u_n = s$ by Tauber's second theorem [12]. This completes the proof. $\square$

Using Theorem 1.1, we show that slow decrease of (u_n) is also a Tauberian condition for (A, m) summability method.

Theorem 4.1. *If (u_n) is (A, m) summable to s and slowly decreasing, then* $\lim_n u_n = s$.

Proof. Let (u_n) be slowly decreasing. Then, we have $v_n \geq -C$ for some C by Lemma 3.2. Since $n(\sigma_n^{(1)}(u) - \sigma_{n-1}^{(1)}) = v_n$ for all nonnegative n, we conclude that $(\sigma_n^{(1)}(u))$ is slowly decreasing if we replace u_n in (2.4) by $\sigma_n^{(1)}(u)$.

It easily follows that $(\sigma_n^{(m)}(u))$ is slowly decreasing for each nonnegative m.

Since (u_n) is (A, m) summable to s, we have

$$\lim_n \sigma_n^{(m)}(u) = s \tag{4.16}$$

by Theorem 1.1. By definition, we have

$$\sigma_n^{(m)}(u) = \sigma_n^{(1)}(\sigma^{(m-1)}(u)). \tag{4.17}$$

From (4.16) and (4.17) it follows that (u_n) is $(A, m-1)$ summable to s. Since $(\sigma_n^{(m-1)}(u))$ is slowly decreasing, we have $\lim_n \sigma_n^{(m-1)}(u) = s$ by Theorem 1.1. Continuing in this way, we obtain that $\lim_n u_n = s$. $\square$

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