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Approximate Drygas mappings on a set of measure zero

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Abstract

Let $\mathbf{R}$ *be the set of real numbers,* Y *be a Banach space and* $f : \mathbf{R} \to Y$*. We prove the Hyers-Ulam stability for the Drygas functional equation*

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y)$$

for all $(x, y) \in \Omega$*, where* $\Omega \subset \mathbf{R}^2$ *is of Lebesgue measure 0.*

Keywords*: Drygas functional equation; stability; Baire category theorem; First category; Lebesgue measure.*

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1. Introduction

To obtain a Jordan and von Neumann type characterization theorem for the quasi-inner-product spaces, Drygas [12] considered the functional equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) \tag{1.1}$$

for all $x, y \in \mathbf{R}$. However, the general solution of this functional equation was given by Ebanks, Kannappan and Sahoo [13] as

$$f(x) = A(x) + Q(x),$$

where $A : \mathbf{R} \longrightarrow \mathbf{R}$ is an additive function and $Q : \mathbf{R} \longrightarrow \mathbf{R}$ is a quadratic function.

In 2002, S. M. Jung and P. K. Sahoo [18] considered the stability problem of the following functional equation:

$$f(x+y) + f(x-y) = 2f(x) + f(y) + g(2y), \tag{1.2}$$

and as a consequence they obtained the stability theorem of functional equation of Drygas (1.1) where f and g are functions from a real vector space X to a Banach space Y.

Here we state a slightly modified version of the results in [18].

Theorem 1.1. *Let $\varepsilon \geq 0$ be fixed and let X be a real vector space and Y a Banach space. If a function $f : X \longrightarrow Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq \varepsilon, \tag{1.3}$$

for all $x, y \in X$, then there exists a unique additive mapping $A : X \longrightarrow Y$ and a unique quadratic mapping $Q : X \longrightarrow Y$ such that $S = A + Q$ is a solution of (1.1) such that
$\|f(x) - S(x)\| \leq \frac{25}{3}\varepsilon$ for all $x \in X$.

This result was improved first by Yang in [27] and later by Sikorska in [26]. In this paper we use the Sikorska's result as a basic tool in the main result. So, we need to present the following theorem.

Theorem 1.2. *[26] Let $(X, +)$ be a group and Y be a Banach space. Given an $\varepsilon > 0$, assume that $f : X \to Y$ satisfies the condition*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq \varepsilon, \qquad x, y \in X.$$

Then there exists a uniquely determined function $g : X \to Y$ such that

$$g(x) = \frac{2}{9}g(3x) - \frac{1}{9}g(-3x), \qquad x \in X,$$

and

$$\|f(x) - g(x)\| \leq \varepsilon \qquad x \in X.$$

Moreover, if X is Abelian, then g satisfies

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \qquad x, y \in X.$$

The stability and solution of the Drygas equation under some additional conditions was also studied by Forti and Sikorska in [15] in the case when X and Y are amenable groups.

It is a very natural subject to consider functional equations or inequalities satisfied on restricted domains or satisfied under restricted conditions [1]-[8], [11], [14]-[17], [19], [20], [23]-[25]. Among the results, S. M. Jung and J. M. Rassias proved the Hyers-Ulam stability of the quadratic functional equations in a restricted domain [17], [22].
It is very natural to ask if the restricted domain $D := \{(x, y) \in X^2 : \|x\| + \|y\| \geq d\}$ can be replaced by a much smaller subset $\Omega \subset D$, i.e., a subset of measure 0 in a measure space X. In 2013, J. Chung considered the stability of the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.4}$$

in a set $\Omega \subset \{(x, y) \in \mathbf{R}^2 : |x| + |y| \geq d\}$ of measure $m(\Omega) = 0$ when $f : \mathbf{R} \longrightarrow \mathbf{R}$. In 2014, J. Chung and J. M. Rassias proved the stability of the quadratic functional equation in a set of measure zero.
In this paper, we prove the Hyers-Ulam stability theorem for the Drygas functional equation (1.1) in $\Omega \subset X^2$ of Lebesgue measure 0.

2. General approach

Through this paper, we denote by X and Y a real normed space and a real Banach space. For given $x, y, a \in X$, we define

$$P_{x,y,a} := \Big\{(x+y, a), (x-y, a), (x, y+a), (x, y-a), (y, a), (-y, -a)\Big\}$$

Let $\Omega \subset X^2$. Throughout this section, we assume that Ω satisfies the condition: For given $x, y \in X$, there exists $a \in X$ such that

$$(C) \qquad P_{x,y,a} \subset \Omega.$$

In the following, we prove the Hyers-Ulam stability theorem for the Drygas functional equation (1.1) in Ω.

Theorem 2.1. *Let $\varepsilon \geq 0$ be fixed. Suppose that $f : X \longrightarrow Y$ satisfies the functional inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq \varepsilon \tag{2.1}$$

for all $(x, y) \in \Omega$. Then there exists a unique mapping $g : X \longrightarrow Y$ such that g is a solution of (1.1) and

$$\|f(x) - g(x)\| \leq 3\varepsilon \tag{2.2}$$

for all $x \in X$.

Proof. Let $D(x, y) = f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)$. Since Ω satisfies (C), for given $x, y \in X$, there exists $a \in X$ such that

$$\|D(x+y, a)\| \leq \varepsilon, \qquad \|D(x-y, a)\| \leq \varepsilon, \qquad \|D(x, y+a)\| \leq \varepsilon,$$

$$\|D(x, y-a)\| \leq \varepsilon, \qquad \|D(y, a)\| \leq \varepsilon, \qquad \|D(-y, -a)\| \leq \varepsilon.$$

Thus, using the triangle inequality we have

$$\Big\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\Big\| = \Big\|-\frac{1}{2}D(x+y, a)-\frac{1}{2}D(x-y, a)$$

$$+\frac{1}{2}D(x, y+a) + \frac{1}{2}D(x, y-a) + \frac{1}{2}D(y, a) + \frac{1}{2}D(-y, -a)\Big\| \leq 3\varepsilon$$

for all $x, y \in X$. Next, according Theorem 1.1, there exists a unique mapping $g : X \longrightarrow Y$ such that

$$\|f(x) - g(x)\| \leq 3\varepsilon$$

for all $x \in X$. This completes the proof. $\square$ The following corollary is a particular case of Theorem 2.1, where $\varepsilon = 0$.

Corollary 2.2. *Suppose that $f : X \longrightarrow Y$ satisfies the functional equation*

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) \tag{2.3}$$

for all $(x, y) \in \Omega$. Then, (2.3) holds for all $x, y \in X$.

3. Construction of a set Ω of Lebesgue measure zero

In this section we construct a set Ω of measure zero satisfying the condition (C) when $X = \mathbf{R}$. From now on,we identify $\mathbf{R}^2$ with $\mathbf{C}$. The following lemma is a crucial key of our construction [[22], Theorem 1.6].

Lemma 3.1. *The set* $\mathbf{R}$ *of real numbers can be partitioned as* $\mathbf{R} = F \cup K$ *where* F *is of first Baire category, i.e.,* F *is a countable union of nowhere dense subsets of* $\mathbf{R}$*, and* K *is of Lebesgue measure 0.*

The following lemma was proved by J. Chung and J. M. Rassias in [9] and [10].

Lemma 3.2. *Let* K *be a subset of* $\mathbf{R}$ *of measure 0 such that* $K^c := \mathbf{R}\backslash K$ *is of first Baire category. Then, for any countable subsets* $U \subset \mathbf{R}$, $V \subset \mathbf{R}\backslash\{0\}$ *and* $M > 0$*, there exists* $a \geq M$ *such that*

$$U + aV = \{u + av : u \in U, v \in V\} \subset K. \tag{3.1}$$

In the following theorem, we give the construction of a set Ω of Lebesgue measure zero.

Theorem 3.3. *Let* $\Omega = e^{-\frac{\pi}{6}i}(K \times K)$ *be the rotation of* $K \times K$ *by* $-\frac{\pi}{6}$*, i.e.,*

$$\Omega = \left\{(p, q) \in \mathbf{R}^2 : \frac{\sqrt{3}}{2}p - \frac{1}{2}q \in K, \frac{1}{2}p + \frac{\sqrt{3}}{2}q \in K\right\}. \tag{3.2}$$

Then Ω *satisfies the condition* (C) *which has two-dimensional Lebesgue measure 0.*

Proof. By the construction of Ω, the condition (C) is equivalent to the condition that for every $x, y \in \mathbf{R}$, there exists $a \in \mathbf{R}$ such that

$$e^{-\frac{\pi}{6}i}P_{x,y,a} \subset K \times K. \tag{3.3}$$

The inclusion (3.3) is equivalent to

$$S_{x,y,a} := \left\{\frac{\sqrt{3}}{2}u - \frac{1}{2}v, \frac{1}{2}u + \frac{\sqrt{3}}{2}v : (u, v) \in P_{x,y,a}\right\} \subset K. \tag{3.4}$$

It is easy to check that the set $S_{x,y,a}$ is contained in a set of form $U + aV$, where

$$U = \left\{\frac{\sqrt{3}}{2}(x + y), \frac{\sqrt{3}}{2}(x - y), \frac{1}{2}(x + y), \frac{1}{2}(x - y), (\frac{\sqrt{3}}{2}x - \frac{1}{2}y), (\frac{1}{2}x + \frac{\sqrt{3}}{2}y),\right.$$

$$\left.\frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}y, \frac{1}{2}y, -\frac{1}{2}y\right\},$$

$$V = \left\{\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\}.$$

By Lemma 3.2, for given $x, y \in \mathbf{R}$ and $M > 0$ there exists $a \geq M$ such that

$$S_{x,y,a} \subset U + aV \subset K. \tag{3.5}$$

Thus, Ω satisfies (C). This completes the proof. □

Corollary 3.4. *Suppose that* $f : \mathbf{R} \longrightarrow \mathbf{R}$ *satisfies*

$$|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)| \longrightarrow 0 \tag{3.6}$$

as $(x, y) \in \Omega$, $|x| + |y| \longrightarrow \infty$. *Then* f *is a Drygas mapping.*

Proof. The condition (3.6) implies that for each $n \in \mathbf{N}$, there exists $d_n > 0$ such that

$$|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)| \leq \frac{1}{n} \tag{3.7}$$

for all $(x, y) \in \Omega_{d_n} := \{(x, y) \in \Omega : |x| + |y| \geq d_n\}$. In view of the proof of Theorem 2.1, the inclusion (3.5) implies that for every $x, y \in \mathbf{R}$ and $M > 0$ there exists $a \geq M$ such that

$$P_{x,y,a} \subset \Omega. \tag{3.8}$$

For given $x, y \in \mathbf{R}$ if we take $M = d_n + |x| + |y|$ and if $a \geq M$, then we have

$$P_{x,y,a} \subset \{(p, q) : |x| + |y| \geq d_n\}. \tag{3.9}$$

It follows from (3.8) and (3.9) that for every $x, y \in \mathbf{R}$ there exists $a \in \mathbf{R}$ such that

$$P_{x,y,a} \subset \Omega_{d_n}. \tag{3.10}$$

So, Ω_{d_n} satisfies the condition (C). Thus, by Theorem 2.1, there exists a unique additive mapping $A : \mathbf{R} \longrightarrow \mathbf{R}$ and a unique quadratic mapping $Q : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$\|f(x) - A_n(x) - Q_n(x)\| \leq \frac{25}{n} \tag{3.11}$$

for all $x \in \mathbf{R}$. Replacing $n \in \mathbf{N}$ by $m \in \mathbf{N}$ in (3.11) and using the triangle inequality we have

—A$_n(x) - A_m(x) + Q_n(x) - Q_m(x)| \leq |A_n(x) + Q_n(x) - f(x)| + |f(x) - A_m(x) - Q_m(x)|$

$\leq \frac{25}{n} + \frac{25}{m} \leq 50$

for all $m, n \in \mathbf{N}$ and $x \in \mathbf{R}$. Hence, $A_n + Q_n - A_m - Q_m$ is bounded. So, we get that

$$A_n + Q_n(x) = A_m + Q_m(x)$$

for all $m, n \in \mathbf{N}$. Then, $A_n = A_m$ and $Q_n = Q_m$ for all $m, n \in \mathbf{N}$. Now, letting $n \longrightarrow \infty$ in (3.11) we get the result. $\square$

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