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# On Jensen's and the quadratic functional equations with involutions

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**Abstract**

*We determine the solutions $f : S \to H$ of the generalized Jensen's functional equation*

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x), \quad x, y \in S,$$

*and the solutions $f : S \to H$ of the generalized quadratic functional equation*

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x) + 2f(y), \quad x, y \in S,$$

*where $S$ is a commutative semigroup, $H$ is an abelian group (2-torsion free in the first equation and uniquely 2-divisible in the second) and $\sigma, \tau$ are two involutions of $S$.*

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## 1. Set up, notation and terminology

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning. $S$ is a commutative semigroup [a set equipped with an associative composition rule $(x, y) \mapsto x + y$], $\sigma, \tau : S \to S$ are two homomorphisms satisfying $\sigma \circ \sigma = \tau \circ \tau = id$, and $(H, +)$ denotes an abelian group with neutral element 0. We say that $H$ is 2-torsion free if $[h \in H \text{ and } 2h = 0] \Rightarrow h = 0$.
$H$ is said to be uniquely 2-divisible if for any $h \in H$ the equation $2x = h$ has exactly one solution $x \in H$.

A function $A : S \to H$ is said to be additive if $A(x + y) = A(x) + A(y)$ for all $x, y \in S$.

We recall that the Cauchy difference $Cf$ of a function $f : S \to H$ is defined by

$$Cf(x, y) := f(x + y) - f(x) - f(y), \quad x, y \in S.$$

## 2. Introduction

In [16], Sinopoulos determined the general solution $f : S \to H$, where $H$ is 2-torsion free, of Jensen's functional equation

$$f(x + y) + f(x + \tau(y)) = 2f(x), \quad x, y \in S, \tag{2.1}$$

and the general solution $f : S \to H$, where $H$ is uniquely 2-divisible, of the quadratic functional equation

$$f(x + y) + f(x + \tau(y)) = 2f(x) + 2f(y), \quad x, y \in S. \tag{2.2}$$

Some information, applications and numerous references concerning (2.1) and (2.2) and their further generalizations can be found, e.g., in [3-8, 10-14, 17, 18]. For more details, we refer to the monographs [9, 15, 19].

The purpose of the present paper is to solve the following functional equations

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x), \quad x, y \in S, \tag{2.3}$$

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x) + 2f(y), \quad x, y \in S. \tag{2.4}$$

Thus the contribution by our paper of new knowledge consists in introducing an involution $\sigma$ and in solving the corresponding extensions (2.3) and

(2.4) of the functional equations (2.1) and (2.2). Our solution formulas contain the previous ones as special cases.

A similar functional equation that has been studied is

$$f(x+\sigma(y))+f(x+\tau(y))=2f(x)f(y), \quad x,y\in S, \tag{2.5}$$

where $f : S \to \mathbf{C}$ is the function to determine. Eq. (2.5) was solved in a more general framework (see [2]).

## 3. On Jensen's functional equation

In this section, we solve the functional equation (2.3) by expressing its solutions in terms of additive functions.

**Lemma 3.1.** *Let $f : S \to H$ be a solution of the functional equation (2.3). Then*

$$f(x+(\tau\circ\sigma)^2(y))=f(x+y) \quad \textit{for all } x,y\in S. \tag{3.1}$$

**Proof.** Making the substitutions $(x,\sigma(y))$ and $(x,\tau(y))$ in (2.3), we get respectively

$$f(x+y)+f(x+\tau(\sigma(y)))=2f(x),$$
$$f(x+\sigma(\tau(y)))+f(x+y)=2f(x).$$

So

$$f(x+\tau(\sigma(y)))=f(x+\sigma(\tau(y))) \quad \text{for all } x,y\in S.$$

Replacing here $y$ by $\tau(\sigma(y))$, we obtain (3.1). □

**Theorem 3.2.** *Suppose that $H$ is 2-torsion free. The general solution $f : S \to H$ of the functional equation (2.3) is $f = A + c$, where $A : S \to H$ is an additive map such that $A\circ\tau = -A\circ\sigma$, and where $c \in H$ is a constant.*

**Proof.** The method used here is closely related to and inspired by the one in [16, Proof of Theorem 2]. Assume that $f : S \to H$ is a solution of (2.3). Then

$$f(x+y)+f(x+\tau(\sigma(y)))=2f(x), \quad x,y\in S. \tag{3.2}$$

Making the substitutions $(x, y + \tau(\sigma(y)))$ and $(x + z, y)$ in (3.2) and using Lemma 3.1, we get respectively

$$f(x + y + \tau(\sigma(y))) = f(x), \tag{3.3}$$
$$f(x + z + y) + f(x + z + \tau(\sigma(y))) = 2f(x + z).$$

Interchanging $y$ and $z$ in the last equation we have

$$f(x + y + z) + f(x + y + \tau(\sigma(z))) = 2f(x + y).$$

Adding the last two equations we obtain

$$\begin{aligned} 2f(x + y + z) + f(x + y + \tau(\sigma(z))) + f(x + z + \tau(\sigma(y))) \\ = 2f(x + z) + 2f(x + y). \end{aligned} \tag{3.4}$$

Using Lemma 3.1, we get that

$$f(x + z + \tau(\sigma(y))) = f(x + (\tau \circ \sigma)[y + (\tau \circ \sigma)(z)]).$$

So, using (3.2), we can reformulate (3.4) to

$$2f(x + y + z) + 2f(x) = 2f(x + z) + 2f(x + y).$$

Setting here $z = \tau(\sigma(x))$ and using (3.3) and the fact that $H$ is 2-torsion free, we get

$$f(y) + f(x) = f(x + \tau(\sigma(x))) + f(x + y). \tag{3.5}$$

Interchanging $x$ and $y$ in (3.5), we get that

$$f(x + \tau(\sigma(x))) = f(y + \tau(\sigma(y)))$$

for all $x, y \in S$. So $f(x + \tau(\sigma(x)))$ is a constant, say $c$. By using (3.5), we infer that the function $A(x) := f(x) - c$ is additive. Substituting $f$ into (2.3) we see that $A \circ \tau = -A \circ \sigma$.
The other direction of the proof is trivial to verify. □

As a immediate consequence of Theorem 3.2, we have the following result.

**Corollary 3.3.** *[16, Theorem 2] Suppose that $H$ is 2-torsion free. The general solution $f : S \to H$ of the functional equation (2.1) is $f = A + c$, where $A : S \to H$ is an additive map such that $A \circ \tau = -A$, and where $c \in H$ is a constant.*

## 4. On the quadratic functional equation

In this section, we generalize Sinopoulos's result [16, Theorem 3] on semigroups by solving the functional equation (2.4). The following lemma lists pertinent basic properties of any solution $f : S \to H$ of (2.4).

**Lemma 4.1.** *Suppose that $H$ is 2-torsion free and let $f : S \to H$ be a solution of the functional equation (2.4).*

a) $f \circ \sigma + f \circ \tau = 2f$.

b) *Let $A : S \to H$ be $A := f \circ \sigma - f \circ \tau$. Then $A$ is additive and $A \circ \sigma = A \circ \tau = -A$.*

c) *For all $x, y, z \in S$, we have*

$$f(x + y + z) = f(x + y) + f(x + z) + f(y + z) - f(x) - f(y) - f(z). \tag{4.1}$$

d) *$Cf : S \times S \to H$ is a symmetric, bi-additive map satisfying $Cf(x, \tau(y)) = -Cf(x, \sigma(y))$ for all $x, y \in S$.*

e) *Let $\varphi : S \to H$ be $\varphi(x) := A(x) + 2f(x + \tau(x))$, $x \in S$. Then $\varphi \circ \sigma + \varphi \circ \tau = 2\varphi$ and $\varphi$ satisfies that*

$$\varphi(x + y) = \varphi(x) + \varphi(y) + 4\{Cf(x, y) - Cf(x, \sigma(y))\}, \quad x, y \in S.$$

f) *$\varphi$ is a solution of (2.4).*

g) *$4f(x) = 2Cf(x, \sigma(x)) + \varphi(x)$ for all $x \in S$.*

**Proof.** (*a*) Let us first observe that $f \circ \sigma + f \circ \tau$ is a solution of (2.4). We next replace $x$, first by $\sigma(x)$ and then by $\tau(x)$, in (2.4) we find that

$$f(\sigma(x) + \sigma(y)) + f(\sigma(x) + \tau(y)) = 2f(\sigma(x)) + 2f(y), \tag{4.2}$$

$$f(\tau(x) + \sigma(y)) + f(\tau(x) + \tau(y)) = 2f(\tau(x)) + 2f(y). \tag{4.3}$$

Summing these two equations and using (2.4) and the fact that $H$ is 2-torsion free, we obtain

$$[f(x) + f(\sigma(y))] + [f(x) + f(\tau(y))] = f(\sigma(x)) + f(\tau(x)) + 2f(y),$$

i.e.

$$2f(x) - f(\sigma(x)) - f(\tau(x)) = 2f(y) - f(\sigma(y) - f(\tau(y)),$$

for all $x, y \in S$. From this last equation we infer that $2f - f\circ\sigma - f\circ\tau$ is a constant in $H$, say $c$. Using the fact that $2f - (f\circ\sigma + f\circ\tau)$ is a solution of (2.4) and that $H$ is 2-torsion free, we see that $c = 0$.
($b$) We subtract (4.3) from (4.2) and get that

$$(f\circ\sigma - f\circ\tau)(x+y) + [f(\sigma(x)+\tau(y)) - f(\tau(x)+\sigma(y))] = 2(f\circ\sigma - f\circ\tau)(x), \tag{4.4}$$

for all $x, y \in S$. By using (2.4) and ($a$), we have

$$\begin{aligned} & f(\sigma(x)+\tau(y)) - f(\tau(x)+\sigma(y)) \\ &\quad = [f(\sigma(x)+\tau(y)) + f(\tau(x)+\tau(y))] - 2f\circ\tau(x) - 2f(y) \\ &\quad = 2f(x) + 2f\circ\tau(y) - 2f\circ\tau(x) - 2f(y) \\ &\quad = (f\circ\sigma - f\circ\tau)(x) - (f\circ\sigma - f\circ\tau)(y), \end{aligned}$$

which turns the identity (4.4) into

$$(f\circ\sigma - f\circ\tau)(x+y) = (f\circ\sigma - f\circ\tau)(x) + (f\circ\sigma - f\circ\tau)(y),$$

for all $x, y \in S$. This show that the function $A = f\circ\sigma - f\circ\tau$ is additive. Using ($a$), we see that

$$A = 2f - 2f\circ\tau = 2f\circ\sigma - 2f.$$

So, $A\circ\sigma = A\circ\tau = -A$.
($c$) Making the substitutions $(x+y, \sigma(z))$, $(x+\tau(\sigma(z)), \sigma(y))$, and $(x, \sigma(y+z))$ in (2.4), we get respectively

$$\begin{aligned} f(x+y+z) + f(x+y+\tau(\sigma(z))) &= 2f(x+y) + 2f(\sigma(z)), \\ f(x+\tau(\sigma(z))+y) + f(x+\tau(\sigma(y+z))) &= 2f(x+\tau(\sigma(z)) + 2f(\sigma(y)) \\ &= 2\,[2f(x) + 2f(\sigma(z)) - f(x+z)] + 2f(\sigma(y)), \\ f(x+y+z) + f(x+\tau(\sigma(y+z))) &= 2f(x) + 2f(\sigma(y+z)). \end{aligned}$$

Subtracting the middle identity from the sum of the other two we get that

$$\begin{aligned} 2f(x+y+z) \quad = \quad & 2f(x+y) + 2f(x+z) + 2f(\sigma(y+z)) \\ & -2f(x) - 2f(\sigma(y)) - 2f(\sigma(z)). \end{aligned}$$

Replacing here $2f\circ\sigma$ by $2f + A$ and using the fact that $H$ is 2-torsion free, we get (4.1).

($d$) That $Cf$ is symmetric and bi-additive follows immediately from the very definition of $Cf$ and (4.1). Let $x, y \in S$ be arbitrary. By help of (4.1) and ($a$), we get that

$$\begin{aligned} Cf(x,\tau(y)) &= f(x+\tau(y)) - f(x) - f(\tau(y)) \\ &= 2f(x) + 2f(y) - f(x+\sigma(y)) - f(x) - f(\tau(y)) \\ &= f(x) + f(\sigma(y)) - f(x+\sigma(y)) \\ &= -Cf(x,\sigma(y)). \end{aligned}$$

($e$) For all $x \in S$, we have

$$\begin{aligned} &(\varphi\circ\sigma + \varphi\circ\tau)(x) \\ &\quad = A(\sigma(x)) + 2f(\sigma(x) + \tau(\sigma(x))) + A(\tau(x)) + 2f(\tau(x) + x) \\ &\quad = -2A(x) + 2f(x+\tau(x)) + 8f(\sigma(x)) - 2f(x+\sigma(x)) \\ &\quad = -2A(x) + 2f(x+\tau(x)) + 8f(x) + 4A(x) - 2f(x+\sigma(x)) \\ &\quad = 2A(x) + 4f(x+\tau(x)) \\ &\quad = 2\varphi(x). \end{aligned}$$

So, $\varphi\circ\sigma + \varphi\circ\tau = 2\varphi$.

Next, let $x, y \in S$ be arbitrary. Using (4.1) repeatedly and the fact that $2f\circ\tau = 2f - A$ and that $A\circ\tau = -A$ we find

$$\begin{aligned} \varphi(x+y) &= A(x+y) + 2f((x+\tau(x)) + y + \tau(y)) \\ &= A(x) + A(y) + 2f(x+\tau(x)+y) + 2f(x+\tau(x)+\tau(y)) \\ &\quad +2f(y+\tau(y)) - 2f(x+\tau(x)) - 2f(y) - 2f(\tau(y)) \\ &= A(x) + A(y) + [2f(x+\tau(x)) + 2f(x+y) + 2f(\tau(x)+y) \\ &\quad -2f(x) - 2f(\tau(x)) - 2f(y)] + [2f(x+\tau(x)) + 2f(x+\tau(y)) \\ &\quad +2f\circ\tau(x+y) - 2f(x) - 2f(\tau(x)) - 2f(\tau(y))] \\ &\quad +2f(y+\tau(y)) - 2f(x+\tau(x)) - 2f(y) - 2f(\tau(y)) \\ &= \varphi(x) + \varphi(y) + [2f(x+y) + 2f\circ\tau(x+y)] + [2f(x+\tau(y)) \\ &\quad +2f\circ\tau(x+\tau(y))] - 4f(x) - 4f\circ\tau(x) - 4f(y) - 4f\circ\tau(y) \\ &= \varphi(x) + \varphi(y) + 4f(x+y) + 4f(x+\tau(y)) - A(x+y) \\ &\quad -A(x+\tau(y)) - 8f(x) + 2A(x) - 4f(y) - 4f\circ\tau(y) \\ &= \varphi(x) + \varphi(y) + 4Cf(x,y) + 4Cf(x,\tau(y)) \\ &= \varphi(x) + \varphi(y) + 4\left\{Cf(x,y) - Cf(x,\sigma(y))\right\}. \end{aligned}$$

$(f)$ Using $(e)$ and $(d)$, we get

$$\begin{aligned}&\varphi(x+\sigma(y))+\varphi(x+\tau(y))\\&\quad=2\varphi(x)+\varphi(\sigma(y))+\varphi(\tau(y))+4\{Cf(x,\sigma(y))-Cf(x,y)\}\\&\qquad+4\{Cf(x,\tau(y))-Cf(x,\sigma(\tau(y)))\}\\&\quad=2\varphi(x)+2\varphi(y)+4\{Cf(x,\sigma(y))-Cf(x,y)\}\\&\qquad+4\{-Cf(x,\sigma(y))+Cf(x,y)\}\\&\quad=2\varphi(x)+2\varphi(y).\end{aligned}$$

So, $\varphi$ is a solution of (2.4).
$(g)$ Using the equality $A=2f\circ\sigma-2f$ and (2.4), we obtain

$$\begin{aligned}&2Cf(x,\sigma(x))+\varphi(x)\\&\quad=2f(x+\sigma(x))-2f(x)-2f(\sigma(x))+A(x)+2f(x+\tau(x))\\&\quad=[2f(x+\sigma(x))+2f(x+\tau(x))]-2f(x)-2f(\sigma(x))+2f(\sigma(x))\\&\qquad-2f(x)\\&\quad=8f(x)-4f(x)\\&\quad=4f(x)\quad\text{for all } x\in S.\end{aligned}$$

□

The second main theorem of the present paper reads as follows.
**Theorem 4.2.** Suppose that $H$ is uniquely 2-divisible. The general solution $f:S\to H$ of the functional equation (2.4) is

$$f(x)=Q(x,\sigma(x))+\psi(x),\quad x\in S,$$

where $Q:S\times S\to H$ is an arbitrary symmetric, bi-additive map such that $Q(x,\tau(y))=-Q(x,\sigma(y))$ for all $x,y\in S$, and where $\psi:S\to H$ is an arbitrary solution of

$$\psi(x+y)=\psi(x)+\psi(y)+2\{Q(x,y)-Q(x,\sigma(y))\},\quad x,y\in S,$$

such that $\psi\circ\sigma+\psi\circ\tau=2\psi$.

**Proof.** That all solutions of (2.4) have this form is a consequence of Lemma 4.1 and the fact that $H$ is uniquely 2-divisible. Conversely, simple computations based on the properties of $Q$ and $\psi$, show that the indicated functions are solutions. □

As a immediate consequence of Theorem 4.2, we have the following result.

**Corollary 4.2.** *[16, Theorem 3] Suppose that $H$ is uniquely 2-divisible. The general solution $f : S \to H$ of the functional equation (2.2) is*

$$f(x) = Q(x, x) + \psi(x),$$

*where $Q : S \times S \to H$ is an arbitrary symmetric, bi-additive map such that $Q(x, \tau(y)) = -Q(x, y)$ for all $x, y \in S$, and where $\psi : S \to H$ is an arbitrary additive map such that $\psi \circ \tau = \psi$.*

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