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Vertex equitable labeling of union of cyclic snake related graphs

P. Jeyanthi
Govindammal Aditanar College for Women, India
A. Maheswari
Kamaraj College of Engineering and Technology, India
and
M. Vijayalakshmi
Dr. G. U. Pope College of Engineering, India
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Abstract

Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in $A, |v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$. In this paper, we prove that key graph $KY(m, n), P(2.QS_n), P(m.QS_n)$, $C(n.QS_m)$, $NQ(m)$ and $K_{1,n} \times P_2$ are vertex equitable graphs.

Keywords : *Vertex equitable labeling, vertex equitable graph, comb graph, key graph, path union graph, quadrilateral snake graph.*

AMS Subject Classification : *05C78.*

1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a detailed survey of graph labeling can be found in [2]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan in [3] and further studied in [4]-[10]. Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there exists a vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. The vertex labeling f is known as vertex equitable labeling. A graph G is said to be a vertex equitable if it admits vertex equitable labeling. In this paper, we extend our study on vertex equitable labeling and prove that key graph $KY(m,n), P(2.QS_n), P(m.QS_n)$, $C(n.QS_m)$, $NQ(m)$ and $K_{1,n} \times P_2$ are vertex equitable graphs. In [3], it is proved that the comb graph $P_n \odot K_1$ is a vertex equitable graph. In the following theorem we give an another vertex equitable labeling for the same graph $P_n \odot K_1$.

Theorem 1.1. *The comb graph $P_n \odot K_1$ is a vertex equitable graph.*

Proof. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$. Here $|V(P_n \odot K_1)| = 2n$ and $E(P_n \odot K_1)| = 2n - 1$. Let $A = \{0, 1, 2, \ldots, \lceil \frac{2n-1}{2} \rceil\}$.

Define a vertex labeling $f : V(P_n \odot K_1) \to A$ as follows:

Case (i). When n is even.
$f(u_{2i-1}) = 2(i-1), f(u_{2i}) = 2i, f(v_{2i-1}) = f(v_{2i}) = 2i - 1$ if $1 \leq i \leq \frac{n}{2}$.

Case (ii). When n is odd.
$f(u_{2i-1}) = 2i - 1, f(v_{2i-1}) = 2(i-1)$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil, f(v_{2i}) = 2i, f(u_{2i}) = 2i - 1$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. It can be verified that the induced edge labels of $P_n \odot K_1$ are $1, 2, \ldots, 2n - 1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $P_n \odot K_1$ is a vertex equitable graph. $\square$

We use the following theorem and definitions in the subsequent section.

Theorem 1.2. *[3] The cycle C_n is a vertex equitable graph if and only if $n \equiv 0$ or $3(mod\ 4)$.*

Theorem 1.3. *[7] The kC_4-snake is a vertex equitable graph.*

Theorem 1.4. *[4] Let $G_1(p_1, 2n+1)$ and $G_2(p_2, q_2)$ be any two vertex equitable graphs with equitable labeling f and g respectively. Let u and v be the vertices of G_1 and G_2 respectively such that $f(u) = n+1$ and $g(v) = 0$. Then the graph G obtained by joining u and v by an edge is a vertex equitable graph.*

Theorem 1.5. *[9] Let $G_1(p_1, q), G_2(p_2, q), \ldots, G_m(p_m, q)$ be the vertex equitable graphs with q is odd and u_i, v_i be the vertices of $G_i (1 \le i \le m)$ labeled by 0 and $\lceil \frac{q}{2} \rceil$. Then the graph G obtained by joining v_1 with u_2 and v_2 with u_3 and v_3 with u_4 and so on until we join v_{m-1} with u_m by an edge is also a vertex equitable graph.*

Definition 1.6. *Let $NQ(m)$ be the n^{th} quadrilateral snake obtained from the path $u_1, u_2, \ldots, u_m$ by joining u_i, u_{i+1} with $2n$ new vertices v_j^i and $w_j^i, 1 \le i \le m-1, 1 \le j \le n$.*

Definition 1.7. *The key graph is a graph obtained from K_2 by appending one vertex of C_m to one end point and comb graph $P_n \odot K_1$ to the other end of K_2. It is denoted as $KY(m, n)$.*

Definition 1.8. *[11] Let $G_1, G_2, \ldots, G_n, n \ge 2$ be n graphs and u_i be a vertex of G_i for $1 \le i \le n$. The graph obtained by adding an edge between u_i and u_{i+1} for $1 \le i \le n-1$ is called a path union of $G_1, G_2, \ldots, G_n$ and is denoted by $P(G_1, G_2, \ldots, G_n)$. When all the n graphs are isomorphic to a graph G, it is denoted by $P(n.G)$.*

Definition 1.9. *Let $G_1, G_2, \ldots, G_n$, be n graphs and u_i be a vertex of G_i for $1 \le i \le n$. The graph obtained by adding an edge between u_i and $u_{i+1} (1 \le i \le n-1), u_n$ and u_1 is called a cycle union of $G_1, G_2, \ldots, G_n$ and is denoted by $C(G_1, G_2, \ldots, G_n)$. When all the n graphs are isomorphic to a graph G, it is denoted by $C(n.G)$.*

2. Main Results

Theorem 2.1. *The key graph $KY(m, n)$ is a vertex equitable graph if $m \equiv 0$ or $3(mod\ 4)$.*

Proof.
Case(i). $m \equiv 3(mod\ 4)$.

Let $G_1 = C_m, G_2 = P_n \odot K_1$. Since G_1 has m edges and G_2 has $2n-1$ edges, By Theorem 1.1 and Theorem 1.2, $P_n \odot K_1, C_m$ are vertex equitable graphs. Hence, by Theorem 1.4, $KY(m,n)$ is a vertex equitable graph.

Case(ii). $m \equiv 0(mod\ 4)$.

Let $G_1 = P_n \odot K_1, G_2 = C_m$. Since G_1 has $2n-1$ edges and G_2 has m edges, By Theorem 1.1 and Theorem 1.2 $P_n \odot K_1$ and C_n are vertex equitable graphs. Hence by Theorem 1.4, $KY(m,n)$ is a vertex equitable graph. □

An example for the vertex equitable labeling of $KY(7,5)$ is shown in Figure 1.

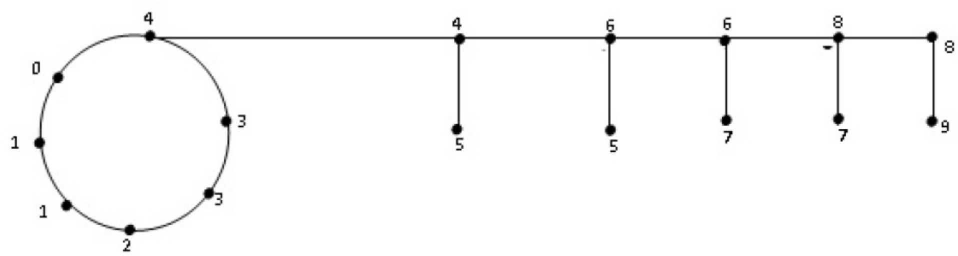

Figure 1

Theorem 2.2. *The path union graph $P(2.QS_n)$ is a vertex equitable graph.*

Proof. Let $V(P(2.QS_n)) = \{u_i, v_{ij}, w_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$ and $E(P(2.QS_n)) = \{u_1u_2, u_iv_{i1}, u_iw_{i1} : 1 \leq i \leq 2\} \cup \{u_{ij}v_{ij}, u_{ij}w_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{u_{ij}v_{ij+1}, u_{ij}w_{ij+1} : 1 \leq i \leq 2, 1 \leq j \leq n-1\}$. Here $|V(P(2.QS_n))| = 6n+2$ and $|E(P(2.QS_n))| = 8n+1$. Let $A = \{0, 1, 2, \ldots, \lceil \frac{8n+1}{2} \rceil\}$.

Define a vertex labeling $f : V((P(2.QS_n))) \to A$ as follows:

$f(u_1) = 0, f(u_2) = \lceil \frac{8n+1}{2} \rceil$.

For $1 \leq j \leq n$, $f(u_{1j}) = f(w_{1j}) = 2j$, $f(v_{1j}) = 2j-1$, $f(v_{2j}) = f(v_{2j}) = f(u_{2j}) = \lceil \frac{8n+1}{2} \rceil - 2j$, $f(w_{2j}) = \lceil \frac{8n+1}{2} \rceil - 2j + 1$.

It can be verifed that the induced edge labels of $P(2.QS_n)$ are $1, 2, \ldots, 8n+1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $P(2.QS_n)$ is a vertex equitable graph. □

Theorem 2.3. *The path union graph $P(m.QS_n)$ is a vertex equitable graph if $m > 2$.*

Proof. Here $V|P(m.QS_n)| = m(3n+1)$ and $E|P(m.QS_n)| = 4mn + m - 1$.

Case(i). m is even.

Let $G_i = P(2.QS_n)$ for $1 \leq i \leq \frac{m}{2}$. By Theorem 2.2, $P(2.QS_n)$ is a vertex equitable graph. Since each G_i has $8n+1$ edges, by Theorem 1.5, $P(m.QS_n)$ admits vertex equitable labeling if m is even.

Case(ii). m is odd and take $m = 2k+1$.

By Case (i) $P(2k.QS_n)$ is a vertex equitable graph. By Theorem 1.3, nC_4 snake is a vertex equitable graph. Let $G_1 = P(2k.QS_n)$ and $G_2 = nC4$. Since G_1 has $8mn + 2m - 1$ edges, by Theorem 1.4, $P(2m+1.QS_n)$ admits vertex equitable labeling. □

An example for the vertex equitable labeling of the graph obtained by the path union of 4 copies of $3C_4$-snake is shown in Figure 2.

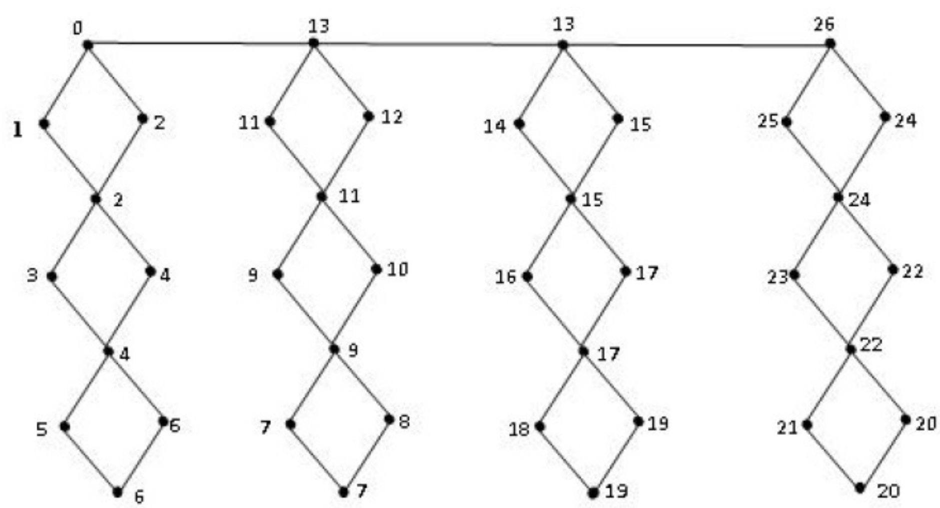

Figure 2

Theorem 2.4. *The graph obtained by the cycle union of n copies of mC_4-snake, $C(n.QS_m)$ is a vertex equitable graph if $n \equiv 0, 3(mod\ 4)$.*

Proof. Let $V(C(n.QS_m)) = \{u_i, u_{ij}, v_{ij}, w_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(C(n.QS_m)) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\} \cup \{u_iv_{i1}, uw_{i1} : 1 \leq i \leq n\} \cup \{u_{ij}v_{ij}, u_{ij}w_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_{ij}v_{ij+1}, u_{ij}w_{ij+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$. Here $|V(C(n.QS_m))| = 3mn + n$ and $|E(C(n.QS_m))| = 4mn + n$. Let $A = \{0, 1, 2, \ldots, \lceil \frac{4mn+n}{2} \rceil\}$.

Define a vertex labeling $f : V(C(n.QS_m)) \to A$ as follows:

Case (i). $n \equiv 0(mod\ 4)$.

f(u_{2i}) $= (4m+1)i$ if $1 \leq i \leq \frac{n}{2}$,

$$f(u_{2i-1}) = \begin{cases} (4m+1)(i-1) & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)(i-1)+1 & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2}, \end{cases}$$

For $1 \leq j \leq m$,

$$\text{f}(\text{v}_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2j & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)(i-1)+(2j-1) & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2}, \end{cases}$$

$f(v_{2i)j}) = (4m+1)i - 1 - 2(j-1)$ if $1 \leq i \leq \frac{n}{2}$,

$$f(w_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2j-1 & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)(i-1)+2j & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2} \end{cases}$$

$$f(w_{(2i)j}) = \begin{cases} (4m+1)i-2j & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)i-2(j-1) & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2}, \end{cases}$$

$$f(u_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2j & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)(i-1)+1+2j & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2}, \end{cases}$$

$f(u_{(2i)j}) = (4m+1)i - 2j$ if $1 \leq i \leq \frac{n}{2}$.

Case (ii). $n \equiv 3(mod\ 4)$.

f(u_{2i}) $= (4m+1)(i-1) + (2m+1)$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$f(u_{2i-1}) = \begin{matrix} (4m+1)(i-1)+2m & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+(2m+1) & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{matrix}$$

$For 1 \leq j \leq m$,

$f(u_{(2i)j}) = (4m+1)(i-1) + (2m+1) + 2j$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$\text{f}(\text{u}_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2m-2j & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m-2j+1 & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases}$$

$$f(v_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2m-2(j-1) & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m+1-2j & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases}$$

$$f(v_{(2i)j}) = \begin{cases} (4m+1)(i-1)+2m+2j-1 & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m+2j & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \end{cases}$$

$$f(w_{2i-1)j}) = \begin{cases} (4m+1)(i-1)+2m-2j+1 & \text{if } 1 \le i \le \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m-2(j-1) & \text{if } \lceil \frac{n}{4} \rceil + 1 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$f(w_{(2i)j}) = \begin{cases} (4m+1)(i-1)+2m+2j & \text{if } 1 \le i \le \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m+2j+1 & \text{if } \lceil \frac{n}{4} \rceil + 1 \le i \le \lfloor \frac{n}{2} \rfloor. \end{cases}$$

It can be verified tha the induced edge labels of $C(n.QS_m)$ are $1, 2, \ldots, 4mn+n$ and $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$. Hence $C(n.QS_m)$ is a vertex equitable graph. □

An example for the vertex equitable labeling of the graph obtained by the cycle union of 7 copies of $2C_4$-snake is shown in Figure 3.

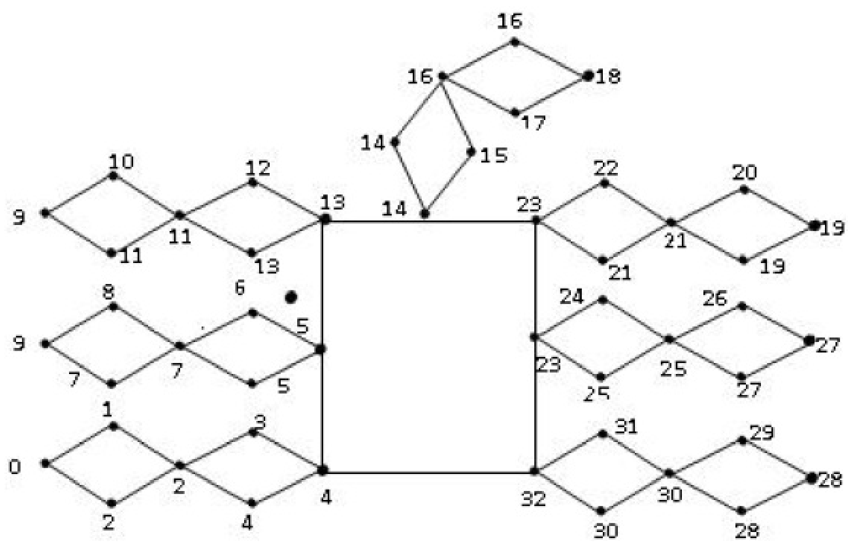

Figure 3

Theorem 2.5. *The n^{th} quadrilateral snake $NQ(m)$ is a vertex equitable graph if $n \ge 2$ is even.*

Proof. Let $V(NQ(m)) = \{u_i/1 \leq i \leq m\} \cup \{v_j^i/1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{w_j^i/2 \leq i \leq m, 1 \leq j \leq n\}, E(NQ(m)) = \{u_i u_{i+1}/1 \leq i \leq n-1\} \cup \{u_i v_j^i/1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{u_i w_j^i/2 \leq i \leq m, 1 \leq j \leq n\}$. Clearly $NQ(m)$ has $2(m-1)n+m$ vertices and $3(m-1)n+m-1$ edges. Let $A = \{0, 1, 2, \ldots, \lceil \frac{3n(m-1)+m-1}{2} \rceil\}$.

Define a vertex labeling $f : V(NQ(m)) \to A$ as follows:

For $1 \leq i \leq m$, $f(u_i) = \lceil \frac{(3n+1)(i-1)}{2} \rceil$,

For $1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j \leq n$, $f(v_j^{2i-1}) = (3n+1)(i-1)+j$,

$f(v_j^{2i}) = (3n+1)(i-1) + \lceil \frac{(3n+1)}{2} \rceil + (j-1)$.

For $1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j \leq \frac{n}{2}$, $f(w_j^{2i-1}) = (3n+1)(i-1) + \lceil \frac{(3n+1)}{2} \rceil - 2j$,

$f(w_{\frac{n}{2}+j}^{2i-1}) = (3n+1)(i-1) + \lceil \frac{(3n+1)}{2} \rceil - (2j-1)$.

For $1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j \leq \frac{n}{2}$, $f(w_j^{2i}) = (3n+1)i - (2j-1)$,

$f(w_{\frac{n}{2}+j}^{2i}) = (3n+1)i - (2j-2)$.

It can be verified that the induced edge labels of $NQ(m)$ are $1, 2, \ldots, 3(m-1)n+m-1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $NQ(m)$ is a vertex equitable graph. □

An example for the vertex equitable labeling of $4Q(4)$ is shown in Figure 4.

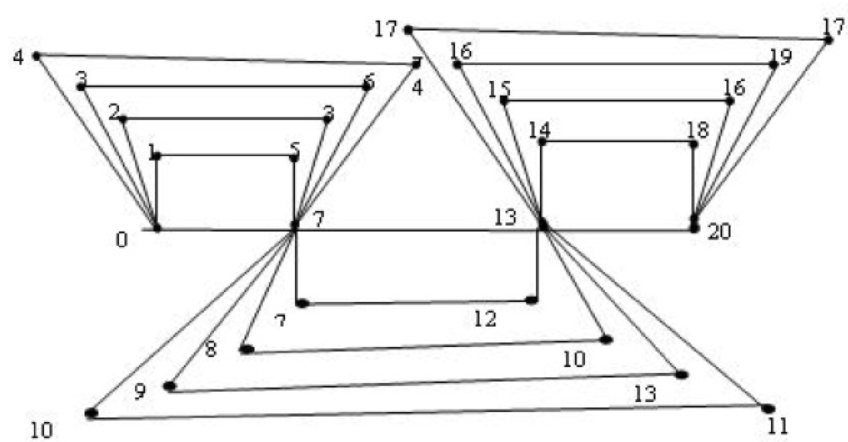

Figure 4

Corollary 2.6. *The book graph $K_{1,n} \times P_2$ is a vertex equitable graph.*

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P. Jeyanthi
Research Centre
Department of Mathematics
Govindammal Aditanar College for Women
Tiruchendur-628 215, Tamilnadu,
India
e-mail : jeyajeyanthi@rediffmail.com

A. Maheswari
Department of Mathematics
Kamaraj College of Engineering and Technology
Virudhunagar, Tamilnadu,
India
e-mail : bala_nithin@yahoo.co.in

and

M. Vijayalakshmi
Department of Mathematics
Dr. G. U. Pope College of Engineering
Sawyerpuram, Tamilnadu,
India
e-mail : viji_mac@rediffmail.com