

A generalization of Drygas functional equation

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Abstract

We obtain the solutions of the following Drygas functional equation

$$\sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S,$$

where S is an abelian semigroup, G is an abelian group, $f \in G^S$, Φ is a finite automorphism group of S with order κ , and $a_\lambda \in S$, $\lambda \in \Phi$.

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1. Introduction

Characterizing quasi-inner product spaces, Drygas considers in [9] the functional equation $f(x)+f(y)=f(x-y)+\left\{f\left(\frac{x+y}{2}\right)-f\left(\frac{x-y}{2}\right)\right\}$ which can be reduced to the following equation (see [21], Remark 9.2, p. 131)

$$(1.1) \quad f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x, y \in \mathbf{R}$$

where \mathbf{R} denotes the set of real numbers.

This equation is known in the literature as *Drygas equation* and is a generalization of the quadratic functional equation

$$(1.2) \quad f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in \mathbf{R}.$$

The general solution of Drygas equation was given by Ebanks, Kannappan and Sahoo in [10]. It has the form

$$f(x) = A(x) + Q(x),$$

where $A : \mathbf{R} \rightarrow \mathbf{R}$ is an additive function and $Q : \mathbf{R} \rightarrow \mathbf{R}$ is a quadratic function, see also [17]. A set-valued version of Drygas equation was considered by Smajdor in [23]. The Drygas functional equation on an arbitrary group G takes the form

$$(1.3) \quad f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1}).$$

The solutions of Drygas equation in abelian group are obtained by Stetkær in [24]. Various authors studied the Drygas equation, for example Ebanks et al. [10], Faiziev and Sahoo [11], Jung and Sahoo [17], Łukasik [18], Szabo [26], Yang [27].

There are several functional equations reduced to those of the Drygas functional equation (1.1), i.e. the mixed type additive, quadratic, Jensen and Pexidered equations, we refer, for example, to [1],[2],[4]-[8], [11]-[16], [19].

The present paper is actually a natural extension and complement to the work of Bouikhalene et al [4], Łukasik [18], Sinopoulos [22], Stetkær [25] and many others [2],[3], [10],[20].

We wish through this work to bring and share answers about two aspects: the characterization for solutions of the following Drygas equation

$$(1.4) \quad \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S,$$

where $f : S \rightarrow G$ is a mapping, S is an abelian semigroup, G is an abelian group, Φ is a finite automorphism group of S , $a_\lambda \in S$, $\lambda \in \Phi$, then to give illustrative examples of such situations.

This equation is an extension form of several equations, for examples,

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y), \quad x, y \in S,$$

$$f(x + y + a) + f(x + y + b) = 2f(x) + 2f(y), \quad x, y \in S,$$

$$f(x + y + a) + f(x - y + b) = 2f(x) + f(y) + f(-y), \quad x, y \in S,$$

$$f(x + y + a) + f(x + \sigma y + b) = 2f(x) + 2f(y), \quad x, y \in S,$$

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in S,$$

$$f(x + y) + f(x + \sigma(y)) = 2f(x), \quad x, y \in S,$$

$$\sum_{k=0}^{m-1} f(x + e^{\frac{2i\pi k}{m}} y) = mf(x) + \sum_{k=0}^{m-1} f(e^{\frac{2i\pi k}{m}} y), \quad x, y \in S = G = \mathbf{C}, \quad m \in \mathbf{N}^*, m \geq 2,$$

$$\text{and } \sum_{k=0}^{m-1} f(x + e^{\frac{2i\pi k}{m}} y + a_i) = mf(x), \quad x, y \in S = G = \mathbf{C}, \quad m \in \mathbf{N}^*, m \geq 2,$$

for $a, b, a_1, \dots, a_{m-1} \in \mathbf{C}$ where \mathbf{N}^* is the set of nonnegative numbers and where \mathbf{C} is the set of all complex numbers.

2. Background results

Let \mathbf{Z} designate the set of integers numbers and G^S the \mathbf{Z} -module consisting of all maps from an abelian semigroup S into an abelian group G . Let $n \in \mathbf{N}^*$ and $\mathcal{A}_n \in G^{S^n}$ be a function, then we say that \mathcal{A}_n is *n-additive* provided that

$$\mathcal{A}_n(x_1 + y_1, \dots, x_j + y_j, \dots, x_n + y_n) = \mathcal{A}_n(x_1, \dots, x_j, \dots, x_n) + \mathcal{A}_n(y_1, \dots, y_j, \dots, y_n),$$

for all $x_1, \dots, x_j, \dots, x_n, y_1, \dots, y_j, \dots, y_n \in E$; we say that \mathcal{A}_n is *symmetric* provided that

$$\mathcal{A}_n(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = \mathcal{A}_n(x_1, x_2, \dots, x_n)$$

whenever $x_1, x_2, \dots, x_n \in S$ and σ is a permutation of $\{1, 2, \dots, n\}$.

If $k \in \mathbf{N}^*$ and $\mathcal{A}_k \in G^{S^k}$ is symmetric and k -additive, let $\mathcal{A}_k^*(x) = \mathcal{A}_k(\underbrace{x, \dots, x}_k)$ for all $x \in S$. And note that $\mathcal{A}_k^*(rx) = r^k \mathcal{A}_k^*(x)$ whenever $x \in S$ and $r \in \mathbf{N}$ also we note that

$$\mathcal{A}_k(x + h, \dots, x + h) = \sum_{i=0}^k C_k^i \mathcal{A}_k(\underbrace{x, \dots, x}_i, \underbrace{h, \dots, h}_{k-i}), \quad x, h \in S.$$

The function \mathcal{A}_k^* is called a *monomial* function of degree k associated to \mathcal{A}_k .

A function $p \in G^S$ is called a *generalized polynomial function* of degree n provided there exist $\mathcal{A}_0 \in S$ and monomial functions \mathcal{A}_k^* (for $1 \leq k \leq n$) such that

$$p(x) = \mathcal{A}_0 + \sum_{i=1}^n \mathcal{A}_k^*(x),$$

for all $x \in S$. We also need to recall the definition of the *linear difference operator* Δ_h , $h \in S$ on G^S by

$$\Delta_h f(x) = f(x + h) - f(x), \quad h, x \in S.$$

Notice that these difference operators have important properties such as the linearity property

$$\Delta_h(\alpha f + \beta g) = \alpha \Delta_h(f) + \beta \Delta_h(g), \quad f, g \in G^S, \quad \alpha, \beta \in \mathbf{Z},$$

and the commutativity property

$$\Delta_{h_1} \Delta_{h_2} \dots \Delta_{h_s} = \Delta_{h_1 h_2 \dots h_s} = \Delta_{h_{\sigma(1)} h_{\sigma(2)} \dots h_{\sigma(s)}},$$

where σ is a permutation of $\{1, 2, \dots, n\}$. There are also other properties such as

$$\Delta_h^n f(x) = \sum_{i=0}^n (-1)^{n-i} C_n^i f(x + ih)$$

and if $\mathcal{A}_m : S^m \rightarrow G$ is a symmetric and m -additive mapping, then we have

$$\Delta_{h_1 \dots h_k} \mathcal{A}_m^*(x) = \begin{cases} m! \mathcal{A}_m(h_1, \dots, h_m), & \text{if } k = m \\ 0, & \text{if } k > m \end{cases} .$$

We will finish this section with some results which we will need in the sequel. Before that, we need to know that every abelian group G is said to be *$n!$ -divisible group* when it is divisible uniquely by $n!$ where $n \in \mathbf{N}^*$.

Theorem 1. [3],[7],[12],[14],[18],[19]

Let G be an abelian group $n!$ -divisible, $n \in \mathbf{N}^*$ and $f \in G^S$, then the following assertions are equivalent.

1. $\Delta_h^n f(x) = 0, x, h \in S.$
2. $\Delta_{h_1 \dots h_n} f(x) = 0, x, h_1, \dots, h_n \in S.$
3. f is a generalized polynomial function of degree at most $n - 1.$

Lemma 1. [18] Let G be an abelian group $n!$ -divisible, $n \in \mathbf{N}^*$, $x_1, x_2, \dots, x_n \in G$, then the following properties are fulfilled.

1.

$$(2.1) \quad \sum_{k=1}^n (-1)^{n-k} C_n^k k^i = 0, i \in \{1, 2, \dots, n - 1\}, n \neq 1$$

and

$$(2.2) \quad \sum_{k=1}^n (-1)^{n-k} C_n^k k^n = n!.$$

2.

$$(2.3) \quad \text{If } \sum_{i=1}^n k^i x_i = 0, k \in \{1, \dots, n\}, \text{ then } x_1 = x_2 = \dots = x_n = 0.$$

3. Main results

Using the difference operator, we adopt the operatorial approach to characterize the solutions of Drygas equation (1.4) which is not a Jensen equation or a quadratic equation.

In the remainder of this paper, we denote by S an abelian semigroup and by G an abelian $(\kappa + 1)!$ -divisible group. However, a solution f of Drygas equation (1.4) in the semigroup S can be extended to the monoid $S \cup \{0\}$ (i.e. by adding the zero element to S) by setting the value of f to zero. We will then, $f(0) = \frac{1}{2\kappa} \sum_{\lambda \in \Phi} f(a_\lambda)$. Without alter the generality of the problem studied and if necessary, we will assume that S admit a zero element.

Lemma 2. Let Φ be a finite automorphism group of S , $\kappa = \text{card}\Phi$, $a_\lambda \in S$ ($\lambda \in \Phi$), $\mathcal{A}_0 \in G$ and $\mathcal{A}_i \in G^{S^i}$ ($1 \leq i \leq \kappa$) be symmetric and i -additive mappings such that

$$(3.1) \quad p(x) = \mathcal{A}_0 + \sum_{i=1}^{\kappa} \mathcal{A}_i^*(x), \quad x \in S$$

and

$$(3.2) \quad I_p(x, y) = \sum_{\lambda \in \Phi} p(x + \lambda y + a_\lambda) - \kappa p(x) - \sum_{\lambda \in \Phi} p(\lambda y), \quad x, y \in S.$$

Then we have the following

$$(a) \quad I_p(0, 0) = \sum_{\lambda \in \Phi} \sum_{i=1}^{\kappa} \mathcal{A}_i(a_\lambda) - \kappa \mathcal{A}_0$$

(3.3)

and

$$(b) \quad I_p(x, y) = I_p(0, 0) + \sum_{\lambda \in \Phi} \sum_{j=0}^{\kappa-1} \sum_{k=0}^{\kappa-1} \sum_{2 \leq i = \max}^{\kappa} C_i^j C_{i-j}^k \mathcal{A}_i(\underbrace{x, \dots, x}_k, a_\lambda, \dots, a_\lambda, \underbrace{\lambda y, \dots, \lambda y}_j),$$

(3.4)

for all $x, y \in S$, where $\max = \max\{j+1, k+1, j+k\}$.

Proof. By direct calculation, we show that

$$I_p(0, 0) = \sum_{\lambda \in \Phi} p(a_\lambda) - 2\kappa p(0).$$

Thus, by replacing p by its expression of $\mathcal{A}_i, 0 \leq i \leq \kappa$ we obtain (a). For every $x, y \in S$, we have

$$\begin{aligned} & I_p(x, y) \\ &= \kappa \mathcal{A}_0 + \sum_{\lambda \in \Phi} \left(\sum_{i=1}^{\kappa} \left(\mathcal{A}_i^*(x + \lambda y + a_\lambda) \right) \right) - \sum_{\lambda \in \Phi} \left(p(x) + p(\lambda y) \right) \\ &= \sum_{\lambda \in \Phi} \left(\sum_{i=1}^{\kappa} \left(\sum_{j=0}^i C_i^j \mathcal{A}_i(x + a_\lambda, \dots, x + a_\lambda, \underbrace{\lambda y, \dots, \lambda y}_j) \right) \right) - \sum_{i=1}^{\kappa} \kappa \mathcal{A}_i^*(x) \\ &\quad - \sum_{\lambda \in \Phi} \mathcal{A}_i^*(\lambda y) - \kappa \mathcal{A}_0 \\ &= \sum_{\lambda \in \Phi} \left(\sum_{i=1}^{\kappa} \left(\sum_{j=0}^i C_i^j \sum_{k=0}^{i-j} C_{i-j}^k \mathcal{A}_i(\underbrace{x, \dots, x}_k, a_\lambda, \dots, a_\lambda, \underbrace{\lambda y, \dots, \lambda y}_j) \right) \right) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=1}^{\kappa} \kappa \mathcal{A}_i^*(x) \\
 & - \sum_{\lambda \in \Phi} \mathcal{A}_i^*(\lambda y) - \kappa \mathcal{A}_0 \\
 & = I_p(0, 0) + \sum_{\lambda \in \Phi} \sum_{j=0}^{\kappa-1} \sum_{k=0}^{\kappa-1} \sum_{2 \leq i = \max\{j+1, k+1, j+k\}}^{\kappa} C_i^j C_{i-j}^k \\
 & \quad \mathcal{A}_i(\underbrace{x, \dots, x}_k, a_\lambda, \dots, a_\lambda, \underbrace{\lambda y, \dots, \lambda y}_j),
 \end{aligned}$$

from where (b) follows. \square

Lemma 3. Let Φ be a finite automorphism group of S , $\kappa = \text{card}\Phi$, $a_\lambda \in S$ ($\lambda \in \Phi$), $\mathcal{A}_0 \in G$ and $\mathcal{A}_i \in G^{S^i}$ ($1 \leq i \leq \kappa$) be symmetric and i -additive mappings such that

$$(3.5) \quad p(x) = \mathcal{A}_0 + \sum_{i=1}^{\kappa} \mathcal{A}_i^*(x), \quad x \in S$$

and

$$(3.6) \quad I_p(x, y) = \sum_{\lambda \in \Phi} p(x + \lambda y + a_\lambda) - \kappa p(x) - \sum_{\lambda \in \Phi} p(\lambda y), \quad x, y \in S.$$

Then the following are equivalent.

1.

$$(3.7) \quad I_p(x, y) = 0, \quad x, y \in S.$$

2. p is a solution of Eq. (1.4).

3. $\mathcal{A}_0 \in G$ and the mappings \mathcal{A}_i , $1 \leq i \leq \kappa$, satisfy the following two equalities,

a)

$$(3.8) \quad \sum_{\lambda \in \Phi} \sum_{i=1}^{\kappa} \mathcal{A}_i^*(a_\lambda) = \kappa \mathcal{A}_0$$

and

b)

$$\sum_{i=\max}^{\kappa} C_i^j C_{i-j}^k \sum_{\lambda \in \Phi} \mathcal{A}_i(\underbrace{x, x, \dots, x}_k, a_\lambda, \dots, a_\lambda, \underbrace{\lambda y, \lambda y, \dots, \lambda y}_j) = 0, \quad x, y \in S, \quad (3.9)$$

$$0 \leq k \leq \kappa - 1, \quad 0 \leq j \leq \kappa - 1, \quad 2 \leq i = \max = \max\{k + 1, j + 1, j + k\} \leq \kappa.$$

Proof. Note first that by Lemma 2 the condition (2) is satisfied if and only if the condition (1) is satisfied. Suppose that (1) is satisfied, then by Lemma 2 we obtain (3)(a) and we have:

$$I_p(x, y) = \sum_{\lambda \in \Phi} \sum_{j=0}^{\kappa-1} \sum_{k=0}^{\kappa-1} \sum_{2 \leq i = \max(j+1, k+1, j+k)}^{\kappa} C_i^j C_{i-j}^k \mathcal{A}_i(\underbrace{x, \dots, x}_k, a_\lambda, \dots, a_\lambda, \underbrace{\lambda y, \dots, \lambda y}_j) = 0, \quad (3.10)$$

for all $x, y \in S$. To prove (3) we define, for every $0 \leq j \leq \kappa - 1$, $0 \leq k \leq \kappa - 1$ the mappings $g_j, h_{(k,j)} : S \times S \rightarrow G$ by

$$g_j(x, y) = \sum_{\lambda \in \Phi} \sum_{k=0}^{\kappa-1} \sum_{i=j+k}^{\kappa} C_i^j C_{i-j}^k \mathcal{A}_i(\underbrace{x, \dots, x}_k, a_\lambda, \dots, a_\lambda, \underbrace{\lambda y, \dots, \lambda y}_j), \quad x, y \in S,$$

$$h_{(k,j)}(x, y) = \sum_{\lambda \in \Phi} \sum_{1 \leq i=j+k}^{\kappa} C_i^j C_{i-j}^k \mathcal{A}_i(\underbrace{x, \dots, x}_k, a_\lambda, \dots, a_\lambda, \underbrace{\lambda y, \dots, \lambda y}_j), \quad x, y \in S.$$

Note that,

$$I_p(x, y) = \sum_{j=0}^{\kappa-1} g_j(x, y), \quad \sum_{k=0}^{\kappa-1} h_{(k,j)}(x, y) = g_j(x, y)$$

and

$$g_0(x, y) = h_{(0,j)}(x, y) = h_{(k,0)}(x, y) = 0, \quad \text{for all } x, y \in S. \text{ However, as}$$

$$g_j(x, ny) = n^j g_j(x, y), \quad n \in \mathbf{N}^*, \quad x, y \in S, \quad 0 \leq j \leq \kappa - 1,$$

we have

$$\sum_{j=0}^{\kappa-1} n^j g_j(x, y) = \sum_{j=0}^{\kappa-1} g_j(x, ny) = 0, \quad n \in \mathbf{N}^*, \quad x, y \in S.$$

By Lemma 1, we get

$$g_j(x, y) = 0, \quad x, y \in S, \quad 0 \leq j \leq \kappa - 1.$$

We deduced from the definition of $h_{(k,j)}$ that

$$h_{(k,j)}(nx, y) = n^k h_{(k,j)}(x, y), \quad n \in \mathbf{N}^*, \quad x, y \in S, \quad 0 \leq k \leq \kappa - j, \quad 0 \leq j \leq \kappa - 1,$$

and we have

$$\sum_{k=0}^{\kappa-1} n^k h_{(k,j)}(x, y) = \sum_{k=0}^{\kappa-1} h_{(k,j)}(nx, y) = g_j(nx, y) = 0, \quad n \in \mathbf{N}^*, \quad x, y \in S, \\ 0 \leq j \leq \kappa - 1.$$

By the same manner as above we obtain

$$h_{(k,j)}(x, y) = 0, \quad j \in \{0, \dots, \kappa - 1\}, \quad k \in \{0, \dots, \kappa - 1\}.$$

Thus, Lemma 1 gives the expected result, (3)(b). The converse of this implication is immediate. This completes the proof. \square

Lemma 4. Let Φ be a finite automorphism group of S , $\kappa = \text{card}\Phi$, $a_\lambda \in S$ ($\lambda \in \Phi$), and $f \in G^S$ such that

$$(3.11) \quad \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S.$$

Then, for every $x, y \in S$, $\Delta_y^\kappa f(x)$ is independent of x and we have

$$(3.12) \quad \Delta_y^{\kappa+1} f(x) = 0, \quad x, y \in S.$$

Proof. The proof used here goes along the same lines as the one in [18]. We will denote by $\Phi_{i,j} \subset \Phi$, $i \in \{0, \dots, \kappa\}$, $j \in \{1, \dots, C_\kappa^i\}$ the C_κ^i pairwise different sets such that $\text{card}\Phi_{i,j} = \kappa - i$ and by $g \in G^S$, the application defined by

$$g(y) = - \sum_{i=0}^{\kappa} (-1)^{\kappa-i} \sum_{j=1}^{C_\kappa^i} f \left(\sum_{\lambda \in \Phi_{i,j}} \lambda y \right), \quad y \in S.$$

Let $\lambda \in \Phi$, $i \in \{0, \dots, \kappa\}$ and $j \in \{1, \dots, C_\kappa^i\}$, then the set $\lambda\Phi_{i,j} \subset \Phi$ has $\kappa - i$ elements. So, there is $k \in \{1, \dots, C_\kappa^i\}$ satisfies the following two equalities

$$\lambda\Phi_{ij} = \Phi_{i,k} \text{ and } \lambda^{-1}\Phi_{i,k} = \Phi_{i,j}.$$

It follows,

$$(3.13) \quad \sum_{j=1}^{C_\kappa^i} f \left(\sum_{\mu \in \Phi_{i,j}} \lambda\mu y \right) = \sum_{j=1}^{C_\kappa^i} f \left(\sum_{\mu \in \Phi_{i,j}} \mu y \right), \quad x \in S.$$

For given x, y , we set

$$u_i = x + iy, \quad v_{ij} = \sum_{\mu \in \Phi_{i,j}} \mu y, \quad i \in \{0, \dots, \kappa\}, \quad j \in \{1, \dots, C_\kappa^i\}.$$

Otherwise, let $\lambda \in \Phi$, $i \in \{0, \dots, \kappa\}$ and $j \in \{1, \dots, C_\kappa^i\}$, then we have the following two cases:

Case 1. Let $\lambda^{-1} \in \Phi_{i,j}$, then $i \neq \kappa$ and, $\Phi_{i,j} = \Phi_{i+1,j} \cup \{\lambda^{-1}\}$.

It follows that

$$\begin{aligned} u_i + \lambda v_{ij} &= x + iy + \sum_{\mu \in \Phi_{i,j}} \lambda\mu y \\ &= x + (i+1)y + \sum_{\mu \in \Phi_{i+1,k}} \lambda\mu y \\ &= u_{i+1} + \lambda v_{i+1,k}, \end{aligned}$$

for a suitable k in $\{1, \dots, C_\kappa^{i+1}\}$.

Case 2. Let $\lambda^{-1} \in \Phi_{i,j}$, then $i \neq 0$ and, $\Phi_{i-1,j} = \Phi_{i,j} \cup \{\lambda^{-1}\}$. We can write,

$$\begin{aligned} u_i + \lambda v_{i,j} &= x + iy + \sum_{\mu \in \Phi_{i,j}} \lambda\mu y \\ &= x + (i-1)y + \sum_{\mu \in \Phi_{i-1,k}} \lambda\mu y \\ &= u_{i-1} + \lambda v_{i-1,k}, \end{aligned}$$

for a suitable k in $\{1, \dots, C_\kappa^{i+1}\}$. Taking into account (3.13) and the calculation results of the previous two cases, we have:

$$\begin{aligned}
 & \kappa \Delta_y^\kappa f(x) - \kappa g(y) \\
 = & \kappa \sum_{i=0}^{\kappa} (-1)^{\kappa-i} C_\kappa^i f(x + iy) + \kappa \sum_{i=0}^{\kappa-1} (-1)^{\kappa-i} \sum_{j=1}^{C_\kappa^i} f\left(\sum_{\mu \in \Phi_{i,j}} \mu y\right) \\
 = & \kappa \sum_{i=0}^{\kappa} (-1)^{\kappa-i} C_\kappa^i f(x + iy) + \sum_{i=0}^{\kappa-1} (-1)^{\kappa-i} \sum_{j=1}^{C_\kappa^i} \sum_{\lambda \in \Phi} f\left(\sum_{\mu \in \Phi_{i,j}} \lambda \mu y\right) \\
 = & \sum_{i=0}^{\kappa} (-1)^{\kappa-i} \sum_{j=1}^{C_\kappa^i} \left(\kappa f(u_i) + \sum_{\lambda \in \Phi} f(\lambda v_{ij}) \right) \\
 = & \sum_{i=0}^{\kappa} (-1)^{\kappa-i} \sum_{j=1}^{C_\kappa^i} \sum_{\lambda \in \Phi} f(u_i + \lambda v_{ij} + a_\lambda) \\
 = & 0, \quad x, y \in S.
 \end{aligned}$$

This shows that for every $x, y \in S$, $\Delta_y^\kappa f(x)$ is independent of x and

$$\Delta_y^\kappa f(x + y) - \Delta_y^\kappa f(x) = 0, \quad x, y \in S,$$

and more accurately

$$\Delta_y^{\kappa+1} f(x) = 0, \quad x, y \in S,$$

from which the desired result follows. \square

Remark 1. Under the assumptions of Lemma 4, if in addition we assume that

$$\sum_{\lambda \in \Phi} f(\lambda y) = 0, \quad y \in S,$$

then

$$\Delta_y^\kappa f(x) = 0, \quad x, y \in S.$$

Theorem 2. Let $f \in G^S$, Φ a finite automorphism group of S , $\kappa = \text{card}\Phi$ and $a_\lambda \in S$ ($\lambda \in \Phi$). Then the function $f : S \rightarrow G$ is a solution of equation

$$(3.14) \quad \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S,$$

if and only if f has the following form

$$(3.15) \quad f(x) = \mathcal{A}_0 + \sum_{i=1}^{\kappa} \mathcal{A}_i^*(x), \quad x \in S,$$

where $\mathcal{A}_0 \in G$ and $\mathcal{A}_k : S^k \rightarrow G$, $k \in \{1, 2, \dots, \kappa\}$ are symmetric and k -additive functions satisfying the two conditions:

- i) $\sum_{i=\max}^{\kappa} C_i^j C_{i-j}^k \sum_{\lambda \in \Phi} \mathcal{A}_i(\underbrace{x, x, \dots, x}_k, a_\lambda, \dots, a_\lambda, \underbrace{\lambda y, \lambda y, \dots, \lambda y}_j) = 0$, $x, y \in S$,
 $0 \leq k \leq \kappa - 1$, $0 \leq j \leq \kappa - 1$, $2 \leq \max = \max\{j + 1, k + 1, k + j\} \leq i \leq \kappa$ and
- ii) $\sum_{\lambda \in \Phi} \sum_{i=1}^{\kappa} \mathcal{A}_i^*(a_\lambda) = \kappa \mathcal{A}_0$.

Proof. The necessary condition is obtained by Lemma 4, Theorem 1 and Lemma 3. By Lemma 3 we get the sufficient condition which completes the proof of Theorem. \square

Remark 2. Under the assumptions of Theorem 2, if in addition we assume that

$$\sum_{\lambda \in \Phi} f(\lambda y) = 0, \quad y \in S,$$

then the result (with some modifications on the control of indices i, j and k) can be obtained by requiring the assumption "G is $\kappa!$ -divisible" instead of "G is $(\kappa + 1)!$ -divisible".

4. Consequences

The following corollaries are immediate consequences of Theorem 2. On this occasion, we obtain the following three corollaries 1, 2 and 3 which have been proved by Sinopoulos [22], Stetkær [25], Łukasik [18], Bouikhalene and Elqorachi [4] respectively.

Corollary 1. [22][25] Let $\sigma : S \rightarrow S$ be an involution of S and G be an abelian group divisible by 2. Then the function $f : S \rightarrow G$ is a solution of equation

$$(4.1) \quad f(x + y) + f(x + \sigma(y)) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in S$$

if and only if f has the following form

$$(4.2) \quad f(x) = \mathcal{A}_1(x) + \mathcal{A}_2^*(x), \quad x \in S,$$

where $\mathcal{A}_1 : S \rightarrow G$ is an arbitrary additive function and $\mathcal{A}_2 : S \times S \rightarrow G$ is an arbitrary symmetric biadditive function with $\mathcal{A}_2(x, y) + \mathcal{A}_2(x, \sigma(y)) = 0$, $x, y \in S$.

Corollary 2. [18] Let S be an abelian semigroup, G be an abelian group divisible by $\kappa!$, Φ be a finite automorphism group of S with order κ . Then the function $f : S \rightarrow G$ is a solution of equation

$$(4.3) \quad \sum_{\lambda \in \Phi} f(x + \lambda y) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S,$$

if and only if f has the following form

$$(4.4) \quad f(x) = \sum_{i=1}^{\kappa} \mathcal{A}_i^*(x), \quad x \in S,$$

where $\mathcal{A}_k : S^k \rightarrow G$, $k \in \{1, 2, \dots, \kappa\}$ are arbitrary symmetric and k -additive functions which satisfy the following condition:

$$\sum_{\lambda \in \Phi} \mathcal{A}_i(x, x, \dots, x, \underbrace{\lambda y, \lambda y, \dots, \lambda y}_j) = 0, \quad x, y \in S, \quad 1 \leq j \leq i - 1, \quad 2 \leq i \leq \kappa.$$

Proof. In this case, with the notations of Theorem 2, as $\{a_\lambda, \lambda \in \Phi\} = \{0\}$, $k + j = i$.

Furthermore, we can write that

$$\begin{aligned} 0 &= \sum_{i=\max(k+j, k+1)}^{\kappa} C_i^k C_j^{i-j} \sum_{\lambda \in \Phi} \mathcal{A}_i k(x, \dots, x, a_\lambda, \dots, a_\lambda, \underbrace{j \lambda y, \dots, \lambda y}) \\ &= \sum_{i=k+j}^{\kappa} C_i^k \sum_{\lambda \in \Phi} \mathcal{A}_i k(x, \dots, x, \underbrace{j \lambda y, \dots, \lambda y}) \\ &= \sum_{j=1}^{i-1} C_i^k \sum_{\lambda \in \Phi} \mathcal{A}_i(x, \dots, x, \underbrace{j \lambda y, \dots, \lambda y}), \quad x, y \in S, \quad 2 \leq i \leq \kappa. \end{aligned}$$

For $1 \leq j \leq i - 1$, $2 \leq i \leq \kappa$, we define the mappings $q_{(j,i)} : S \times S \rightarrow G$ by

$$q_{(j,i)}(x, y) = C_i^j \sum_{\lambda \in \Phi} \mathcal{A}_i(x, \dots, x, \underbrace{j \lambda y, \lambda y, \dots, \lambda y}) \quad x, y \in S.$$

So, we have

$$q_{(j,i)}(x, ny) = n^j q_{(j,i)}(x, y), \quad x, y \in S, \quad n \in \mathbf{N}^*, \quad 0 \leq j \leq i - 1, \quad 2 \leq i \leq \kappa$$

and

$$\sum_{j=1}^{\kappa} n^j q_{(j,i)}(x, y) = \sum_{j=1}^{\kappa} q_{(j,i)}(x, ny) = 0, \quad x, y \in S, \quad 2 \leq i \leq \kappa.$$

According to Lemma1 we get the sought result. \square

Corollary 3. [4] Let S be an abelian group, G be a Banach space and $a \in S$. Then, the general solution $f : S \rightarrow G$ of the functional equation

$$(4.5) \quad f(x + y + a) = f(x) + f(y), \quad x, y \in S,$$

is

$$(4.6) \quad f(x) = \mathcal{A}_1(a) + \mathcal{A}_1(x), \quad x \in S.$$

where $\mathcal{A}_1 : S \rightarrow G$ is an arbitrary additive function.

In the following corollaries we prove new others special cases of the equation 1.4 that is, according to our knowledge, not in the literature.

Corollary 4. Let S be an abelian semigroup, G be an abelian group divisible by 2 and $a, b \in S$. Then, the general solution $f : S \rightarrow G$ of the functional equation

$$(4.7) \quad f(x + y + a) + f(x - y + b) = 2f(x) + f(y) + f(-y), \quad x, y \in S,$$

is

$$(4.8) \quad f(x) = \frac{1}{2}(\mathcal{A}_1(a + b)) + \mathcal{A}_1(x) + \mathcal{A}_2^*(x), \quad x \in S$$

where $\mathcal{A}_1 : S \rightarrow G$ is an arbitrary additive function and $\mathcal{A}_2 : S \times S \rightarrow G$ is an arbitrary symmetric biadditive function with $\mathcal{A}_2(x, a) = \mathcal{A}_2(x, b) = 0$, $x \in S$.

Corollary 5. Let S be an abelian semigroup, σ be an involution of S , G be an abelian group divisible by 2 and $a, b \in S$. Then, the general solution $f : S \rightarrow G$ of the functional equation

$$(4.9) \quad f(x + y + a) + f(x + \sigma(y) + b) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in S,$$

is

$$(4.10) \quad f(x) = \frac{1}{2}(\mathcal{A}_1(a + b)) + \mathcal{A}_1(x) + \mathcal{A}_2^*(x), \quad x \in S,$$

where $\mathcal{A}_1 : S \rightarrow G$ is an arbitrary additive function and $\mathcal{A}_2 : S \times S \rightarrow G$ is an arbitrary symmetric biadditive function with

$$\mathcal{A}_2(x, a) = \mathcal{A}_2(x, b) = 0, \quad x \in S \quad \text{and} \quad \mathcal{A}_2(x, y) + \mathcal{A}_2(x, \sigma(y)) = 0, \quad x, y \in S.$$

Proof. Keeping in mind the notation of Theorem 2, we apply it where $\kappa = 2$. Then there are an element $\mathcal{A}_0 \in G$ and symmetric i -additives mappings $\mathcal{A}_i \in G^{S^i}$, $i \in \{1, 2\}$ satisfy

1. $f(x) = \mathcal{A}_0 + \mathcal{A}_1(x) + \mathcal{A}_2^*(x)$, $x \in S$
 on the other side, they satisfy the following conditions of Theorem 2
 :
 - i) $k = 0, j = 1, \mathcal{A}_2(a, y) + \mathcal{A}_2(b, \sigma(y)) = 0, y \in S$,
 - ii) $k = 1, j = 0, \mathcal{A}_2(y, a) + \mathcal{A}_2(y, b) = 0, y \in S$,
 - iii) $k = 1, j = 1, \mathcal{A}_2(x, y) + \mathcal{A}_2(x, \sigma(y)) = 0, x, y \in S$.

Thus, $\mathcal{A}_2(y, a) = \mathcal{A}_2(y, b) = 0, y \in S; \mathcal{A}_2(x, y) + \mathcal{A}_2(x, \sigma(y)) = 0, x, y \in S$ and $2\mathcal{A}_0 = \mathcal{A}_1(a + b)$. \square

Corollary 6. Let j be a primitive cube root of unity and a be complex number. Then, the general continuous solution $f : \mathbf{C} \rightarrow \mathbf{C}$ of the functional equation

$$f(x+y+ja)+f(x+jy+j^2a)+f(x+j^2y+a) = 3f(x)+f(y)+f(jy)+f(j^2y), \quad x, y \in \mathbf{C}, \tag{4.11}$$

is of the form

$$f(x) = \alpha_1x + \beta_1\bar{x} + \alpha_2x^2 + \beta_2\bar{x}^2, \tag{4.12}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbf{C}$.

Proof. According the Theorem 2, there are $\alpha_0 \in \mathbf{C}$, and symmetric i -additive mappings $\mathcal{A}_i : \mathbf{C}^i \rightarrow \mathbf{C}$, $i \in \{1, 2, 3\}$ such that

$$f(z) = \alpha_0 + \mathcal{A}_1(z) + \mathcal{A}_2^*(z) + \mathcal{A}_3^*(z), \quad z \in \mathbf{C}.$$

Taking into account that j is a primitive cube root of unity, we have $1 + j + j^2 = 0$. In addition, the continuity of f show that $\mathcal{A}_1, \mathcal{A}_2$ et \mathcal{A}_3 can be written as the following

$$\begin{aligned} \mathcal{A}_1(z) &= \alpha_1z + \beta_1\bar{z}, \quad \alpha_1, \beta_1 \in \mathbf{C}, \\ \mathcal{A}_2^*(z) &= \alpha_2z^2 + \beta_2\bar{z}^2 + \beta_3|z|^2, \quad \alpha_2, \beta_2, \beta_3 \in \mathbf{C}, \\ \mathcal{A}_3^*(z) &= \gamma_1z^3 + \gamma_2\bar{z}^3, \quad \gamma_1, \gamma_2 \in \mathbf{C}. \end{aligned}$$

So the conditions of Theorem 2 do not satisfy where $\gamma_1 = \gamma_2 = \beta_3 = 0$ which finish the proof. \square

References

- [1] M. Ait Sibaha, B. Bouikhalene, E. Elqorachi, *Hyers-Ulam-Rassias stability of the K -quadratic functional equation*, J. Ineq. Pure and appl. Math **8**, (2007).
- [2] L. M. Arriola, W. A. Beyer, *Stability of the Cauchy functional equation over p -adic fields*, Real Analysis Exchange, **31** (1), pp. 125-132, (2005).
- [3] J. Baker, *A general functional equation and its stability*, Proceedings of the American Mathematical Society, **133**(6), pp. 1657-1664, (2005).
- [4] B. Bouikhalene and E. Elqorachi, *Hyers-Ulam-Rassias stability of the Cauchy linear functional equation*, Tamsui Oxford Journal of Mathematical Sciences **23** (4), pp. 449-459, (2007).
- [5] A. Charifi, B. Bouikhalene, E. Elqorachi, *Hyers-Ulam-Rassias stability of a generalized Pexider functional equation*, Banach J. Math. Anal, **1** (2), pp. 176-185, (2007).
- [6] A. Charifi, B. Bouikhalene, E. Elqorachi, A. Redouani, *Hyers-Ulam-Rassias stability of a generalized Jensen functional equation*, Aust. J. Math. Anal. Appl, **6** (1), pp. 1-16, (2009).
- [7] AB. Chahbi, A. Charifi, B. Bouikhalene, S. Kabbaj, *Operatorial approach to the non-Archimedean stability of a Pexider K -quadratic functional equation*, Arab Journal of Mathematical Sciences, **21** (1), pp. 67-83, (2015).
- [8] D. Ž. Djoković, *A representation theorem for $(X_1-1)(X_2-1)\dots(X_n-1)$ and its applications*, In Annales Polonici Mathematici **22** (2), pp. 189-198, (1969).
- [9] H. Drygas, *Quasi-inner products and their applications*, Springer Netherlands., pp. 13-30, (1987).
- [10] B. R. Ebanks, P. L. Kannappan, P. K. Sahoo, *A common generalization of functional equations characterizing normed and quasi-inner-product spaces*, Canad. Math. Bull, **35** (3), pp. 321-327, (1992).

- [11] V. A. Faiziev, P. K. Sahoo, *On Drygas functional equation on groups*, Int. J. Appl. Math. Stat. **7**, pp. 59-69, (2007).
- [12] M. Frechet, *Une définition fonctionnelles des polynômes*, Nouv. Ann. **9**, pp. 145-162, (1909).
- [13] A. Giányi, *A characterization of monomial functions*, Aequationes Math. **54**, pp. 343-361, (1997).
- [14] D. H. Hyers, *Transformations with bounded n -th differences*, Pacific J. Math., **11**, pp. 591-602, (1961).
- [15] S. -M. Jung, *Stability of the quadratic equation of Pexider type*, Abh. Math. Sem. Univ. Hamburg, **70**, pp. 175-190, (2000).
- [16] S. -M. Jung, P. K. Sahoo, *Hyers-Ulam stability of the quadratic equation of Pexider type*, J. Korean Math. Soc., **38** (3), pp. 645-656, (2001).
- [17] S.-M. Jung, P. K. Sahoo, *Stability of a functional equation of Drygas*, Aequationes Math., **64** (3), pp. 263-273, (2002).
- [18] R. Lukasik, *Some generalization of Cauchy's and the quadratic functional equations*, Aequat. Math., **83**, pp. 75-86, (2012).
- [19] S. Mazur, W. Orlicz, *Grundlegende Eigenschaften der Polynomischen Operationen*, Erst Mitteilung, Studia Math., **5**, pp. 50-68, (1934).
- [20] A. K. Mirmostafae, *Non-Archimedean stability of quadratic equations*, Fixed Point Theory, **11** (1), pp. 67-75, (2010).
- [21] P. K. Sahoo and Pl. Kannappan, *Introduction to Functional Equations*, CRC Press, Boca Raton, Florida, (2011).
- [22] P. Sinopoulos, *Functional equations on semigroups*, Aequationes Math. **59**, pp.255-261, (2000).
- [23] W. Smajdor, *On set-valued solutions of a functional equation of Drygas*, Aequ. Math. **77**, pp. 89-97, (2009).
- [24] H. Stetkær, *Functional equations on abelian groups with involution. II*, Aequationes Math. **55**, pp. 227-240, (1998).
- [25] H. Stetkær, *Functional equations involving means of functions on the complex plane*, Aequationes Math. **55**, pp. 47-62, (1998).

- [26] Gy. Szabo, *Some functional equations related to quadratic functions*, Glasnik Math. **38**, pp. 107-118, (1983).
- [27] D. Yang, *Remarks on the stability of Drygas equation and the Pexider-quadratic equation*, Aequationes Math. **68**, pp. 108-116, (2004).

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