Proyecciones Journal of Mathematics
Vol. 35, N^o 1, pp. 85-98, March 2016.
Universidad Católica del Norte
Antofagasta - Chile

Odd harmonious labeling of some cycle related graphs

P. Jeyanthi
Govindammal Aditanar College for Women, India
and
S. Philo
PSN College of Engineering and Technology, India
Received : May 2015. Accepted : December 2015

Abstract

A graph $G(p,q)$ is said to be odd harmonious if there exists an injection $f : V(G) \rightarrow \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f^* : E(G) \rightarrow \{1, 3, \cdots, 2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. A graph that admits odd harmonious labeling is called odd harmonious graph. In this paper we prove that any two even cycles sharing a common vertex and a common edge are odd harmonious graphs.

Keywords : *Harmonious labeling; odd harmonious labeling; odd harmonious graph; strongly odd harmonious labeling; strongly odd harmonious graph.*

AMS Subject Classification (2010) : *05C78.*

1. Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one. For standard terminology and notation we follow Harary [6]. A graph $G = (V, E)$ with p vertices and q edges is called a (p, q) − graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. An extensive survey of various graph labeling problems is available in [3]. Labeled graphs serves as useful mathematical models for many applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. Graham and Sloane [4] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \to Z_q$ such that the induced function $f^* : E(G) \to Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod q$ is a bijection and f is called harmonious labeling of G. The concept of odd harmonious labeling was due to Liang and Bai [7]. A graph G is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, 2q - 1\}$ such that the induced function $f^* : E(G) \to \{1, 3, \cdots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. A graph that admits odd harmonious labeling is called odd harmonious graph. The odd harmoniousness of graph is useful for the solution of undetermined equations. Several results have been published on odd harmonious labeling see [1, 2, 5, 9, 10, 11]. Motivated by these results, in [8] we proved that the shadow and splitting of the graphs $K_{2,n}$, C_n for $n \equiv 0(mod\ 4)$, the graph $H_{n,n}$ and double quadrilateral snakes $DQ(n)$, $n \geq 2$ are odd harmonious. In this paper we prove that any two even cycles sharing a common vertex and a common edge are odd harmonious graphs. We use the following results and definitions in the subsequent section.

Lemma 1.1. *[2] If G is an odd harmonious Eulerian graph with q edges, then $q \equiv 0(mod\ 4)$.*

Lemma 1.2. *[2] Two copies of even cycles sharing a common edge is an odd harmonious graph and two copies of even cycles sharing a common vertex is also an odd harmonious graph, when $n \equiv 0(mod\ 4)$.*

Lemma 1.3. *[7] If G is an odd harmonious graph, then G is a bipartite graph. Hence any graph that contains an odd cycle is not an odd harmonious.*

Lemma 1.4. *[7] If a (p, q)-graph G is odd harmonious, then $2\sqrt{q} \leq p \leq 2q-1$.*

Lemma 1.5. *The graph C_n is strongly odd harmonious if and only if $n \equiv 0(mod\ 4)$.*

Definition 1. *A function f is said to be a strongly odd harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to q such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2q-1$ is a bijection.*

Definition 2. *Let C_m and C_n be two even cycles where m and n are even integers. Then the graph $C(m \circ n)$ is a bicyclic graph that share a common vertex of C_m and C_n.*

Definition 3. *Let C_m and C_n be two even cycles with m and n are even integers. Then the graph $C(m@n)$ is a graph obtained by sharing a common edge of C_m and C_n.*

2. Main Results

Theorem 2.1. *Let $G_1(p_1, q_1)$ be a strongly odd harmonious graph and $G_2(p_2, q_2)$ be any odd harmonious graph. Let $e = xy$ be an edge of G_1 with q_1 and $q_1 - 1$ are the vertex labels of x and y respectively and $e^1 = uv$ be an edge of G_2 with 0 and 1 are labels of u and v respectively. Then the graph G obtained by identifying the edges e and e^1 is a strongly odd harmonious graph.*

Proof. Add the number $q_1 - 1$ to all the vertex labels of G_2 (except for u and v) and keep the vertex labels of G_1 fixed. Then the edge labels of G_1 are remain fixed and the edge labels of G_2 are increased by $2q_1 - 2$. Hence the edge labels of G_2 are $\{2q_1 - 1, 2q_1 + 1, 2q_1 + 3, \cdots, 2q_1 + 2q_2 - 3\}$. Thus the induced edge labels of a new graph G is
$\{1, 3, 5, \cdots, 2q_1 - 1, 2q_1 + 1, 2q_1 + 3, \cdots, 2q_1 + 2q_2 - 3\}$. Therefore G is a strongly odd harmonious graph. □

In [2], it was proved that two copies of an even cycle C_n sharing a common edge is an odd harmonious graph. Now we prove that any two even cycles sharing a common edge is also an odd harmonious graph.

Lemma 2.2. *The graph $C(m@n)$ is odd harmonious if $m, n \equiv 0(mod\ 4)$.*

Proof. *By Lemma 1.5 and by Theorem 2.1, the graph $C(m@n)$ is odd harmonious.* $\square$

Illustration 1. *The odd harmonious labeling of the graph $C(8@12)$ is given in Figure 1.*

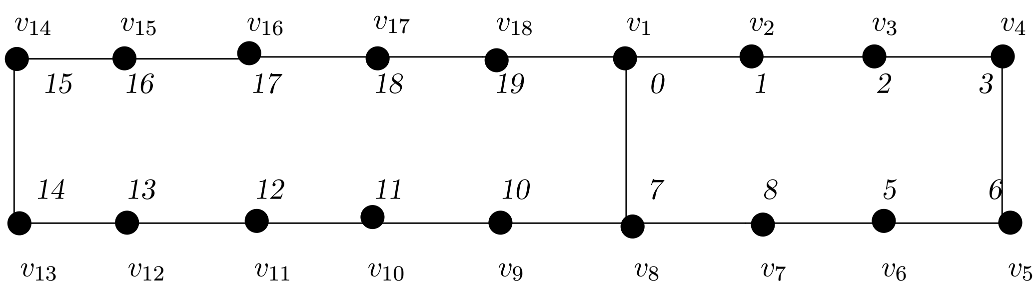

Figure 1: The odd harmonious labeling of $C(8@12)$

Lemma 2.3. *The graph $C(m@n)$ is odd harmonious if $m, n \equiv 2(mod\ 4)$.*

Proof. *Consider the cycles C_m and C_n with $m, n \equiv 2(mod\ 4)$, and take $m = 4l + 2$ and $n = 4k + 2$. Hence, the graph $C(m@n)$ has $4l + 4k + 2$ vertices and $4l+4k+3$ edges. Without loss of generality we assume that $m < n$.*

Define a labeling $f : V(G) \rightarrow \{0, 1, 2, \cdots, 2(4l + 4k + 3) - 1\}$ as follows:

$f(v_i) = i - 1$, $1 \leq i \leq (2k + 2)$.

For $2k + 3 \leq i \leq 2l + 2k$, $f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even,} \end{cases}$ possible only if $l \neq 1$.

For $2l + 2k + 1 \leq i \leq 4l + 4k + 2$, $f(v_i) = i + 1$.

The induced edge labels are
$f^*(v_i v_{i+1}) = 2i - 1,\ \ 1 \leq i \leq (2k + 1)$,

$f^*(v_1v_{4k+2}) = 4k+3,$
$f^*(v_iv_{i+1}) = 2i+1, \;\; 2k+2 \leq i \leq 2l+2k,$
$f^*(v_1v_{4l+4k+2}) = 4k+4l+3,$
$f^*(v_iv_{i+1}) = 2i+3, \;\; 2l+2k+1 \leq i \leq 4l+4k+1.$

The induced edge labels are $\{1, 3, 5, \cdots, 4k+1, 4k+3, \cdots, 8k+8l+5\}$. *Hence the graph* $C(m@n)$ *is odd harmonious if* $m, n \equiv 2(mod\ 4)$. □

Illustration 2. *The odd harmonious labeling of the graph* $C(6@10)$ *is given in Figure 2.*

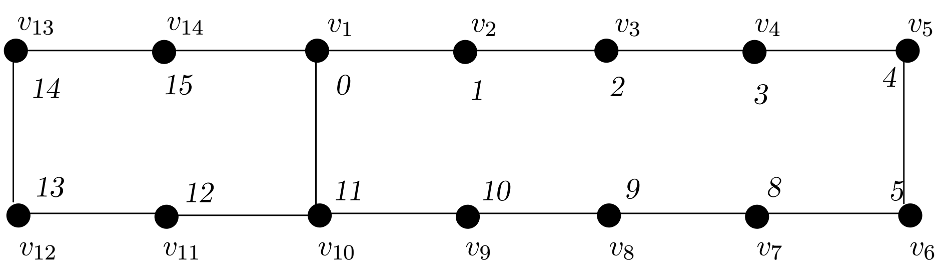

Figure 2: The odd harmonious labeling of $C(6@10)$

Lemma 2.4. *The graph* $C(m@n)$ *is odd harmonious if* $m \equiv 0(mod\ 4)$, *and* $n \equiv 2(mod\ 4)$.

Proof. *Consider the graphs* C_m *and* C_n *with* $m \equiv 0(mod\ 4)$, *and* $n \equiv 2(mod\ 4)$, *and take* $m = 4l$ *and* $n = 4k+2$. *Here, the graph* $C(m@n)$ *has* $4l+4k$ *vertices and* $4l+4k+1$ *edges.*
We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(4l+4k+1)-1\}$ *by considering the following two cases:*

Case (i): $m < n$.
$f(v_i) = i-1\ ,\ 1 \leq i \leq (2k+3).$

$$\text{For } 2k+4 \leq i \leq 2l+2k+1,\; f(v_i) = \begin{cases} i-1 & \text{if } i \text{ is odd} \\ i+1 & \text{if } i \text{ is even,} \end{cases} \quad \text{possible only if } l \neq 1.$$

For $2l+2k+2 \leq i \leq 4l+4k$. $f(v_i) = \begin{cases} i-1 & \text{if } i \text{ is odd} \\ i+3 & \text{if } i \text{ is even.} \end{cases}$

The induced edge labels are
$f^*(v_iv_{i+1}) = 2i-1$, $1 \leq i \leq (2k+2)$,
$f^*(v_1v_{4k+2}) = 4k+5$,
$f^*(v_iv_{i+1}) = 2i+1$, $2k+3 \leq i \leq 2l+2k$,
$f^*(v_1v_{4l+4k}) = 4k+4l+3$,
$f^*(v_iv_{i+1}) = 2i+3$, $2l+2k+1 \leq i \leq 4l+4k-1$.

Case (ii): $m > n$.

$f(v_1) = 0$ and $f(v_i) = 4k+4l+3-i$, $2 \leq i \leq (2l+2k+2)$.

For $2l+2k+3 \leq i \leq 2l+4k$

$$f(v_i) = \begin{cases} 4k+4l+1-i & \text{if } i \text{ is odd} \\ 4k+4l+3-i & \text{if } i \text{ is even,} \end{cases} \quad \text{possible only if } k \neq 1.$$

For $2l+4k+1 \leq i \leq 4l+4k.$, $f(v_i) = 4l+4k+1-i$.

The induced edge labels are
$f^*(v_1v_{4k+4l}) = 1$,
$f^*(v_{i-1}v_i) = 2(4k+4l+1-i)+1$, $4k+2l+2 \leq i \leq 4k+4l-1$,
$f^*(v_1v_{4k+2}) = 4l+1$,
$f^*(v_iv_{i-1}) = 2(4k+4l+1-i)+3$, $4k+2l+1 \leq i \leq 2l+2k+3$,
$f^*(v_1v_2) = 4k+4l+1$,
$f^*(v_iv_{i-1}) = 2(4k+4l+3-i)+1$, $2l+2k+2 \leq i \leq 2$.

Hence the graph $C(m@n)$ *is odd harmonious if* $m \equiv 0(mod\ 4)$ *and* $n \equiv 2(mod\ 4)$. □

Illustration 3. *The odd harmonious labeling of the graphs* $C(8@10)$ *and* $C(12@6)$ *are given in Figures 3 and 4.*

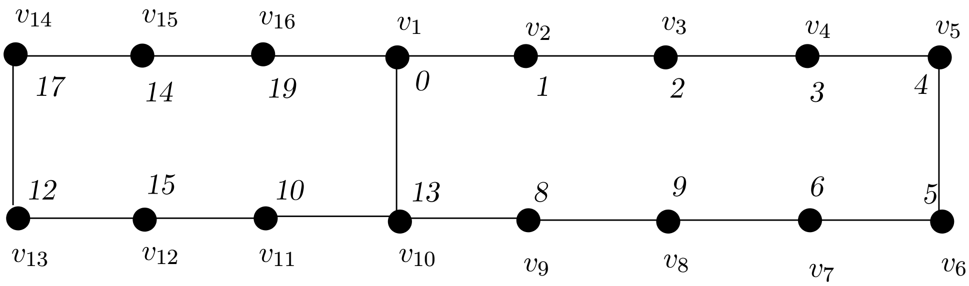

Figure 3: The odd harmonious labeling of $C(8@10)$

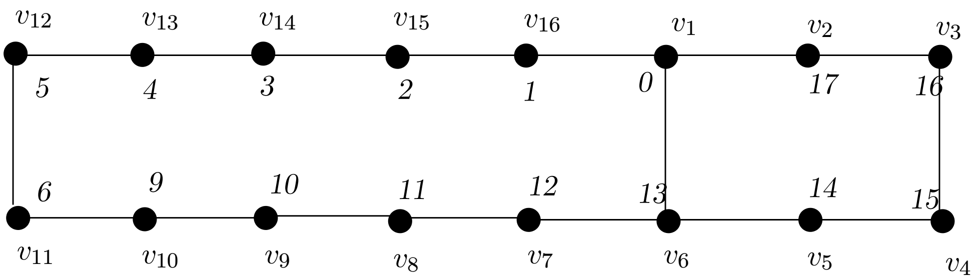

Figure 4: The odd harmonious labeling of $C(12@6)$

Theorem 2.5. *The graph $C(m@n)$ is an odd harmonious graph if and only if both m and n are even integers.*

Proof. *By Lemmas 2.2, 2.3 and 2.4 , the graph $C(m@n)$ is an odd harmonious graph if both m and n are even integers.*
Conversely, if either m or n is an odd integer, then the graph $C(m@n)$ has an odd cycle. By Lemma 1.3, a graph which has an odd cycle is not odd harmonious. $\square$

In [2], it was proved that two copies of an even cycle C_n sharing a common vertex is an odd harmonious graph. Now we prove that any two even cycles sharing a common vertex is also an odd harmonious graph.

Lemma 2.6. *The graph $C(m \circ n)$ is odd harmonious if $m, n \equiv 2(mod\ 4)$.*

Proof. *Consider the graphs C_m and C_n with $m, n \equiv 2(mod\ 4)$ and take $m = 4k+2$ and $n = 4l+2$. Here the graph $C(m \circ n)$ has $4l + 4k + 3$ vertices and $4l + 4k + 4$ edges. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(4l+4k+4)-1\}$ by considering the following two cases:*

Case (i): $k \leq l \leq 2k-2$.

$f(v_i) = i$, $1 \leq i \leq 2l+1$.

For $2l+2 \leq i \leq 2l+2k$, $f(v_i) = \begin{cases} i+2 & \text{if } i \text{ is even,} \\ i & \text{if } i \text{ is odd.} \end{cases}$

Also $f(v_i) = i+2$, $2l+2k+1 \leq i \leq 4l+1$.

$f(v_i) = 0$, if $i = 4l+2$ and $f(v_i) = 8l+7$, if $i = 4l+3$.

$f(v_i) = i$, $4l+4 \leq i \leq 6l+5$.

$f(v_i) = \begin{cases} i+2 & \text{if } i \text{ is even,} \\ i & \text{if } i \text{ is odd.} \end{cases}$, $6l+6 \leq i \leq 4l+4k+3$.

The induced edge labelings are

$f^*(v_{4l+2}v_1) = 1$,
$f^*(v_iv_{i+1}) = 2i+1$, $1 \leq i \leq 2l$,
$f^*(v_nv_{n-1}) = 4l+3$,
$f^*(v_iv_{i+1}) = 2i+3$, $2l+1 \leq i \leq 2l+2k-3$,
$f^*(v_nv_{4k+4l+3}) = 4k+4l+3$,
$f^*(v_iv_{i+1}) = 2i+5$, $2l+2k \leq i \leq 4l$,
$f^*(v_nv_{n+1}) = 2n+3$,
$f^*(v_iv_{i+1}) = 2i+1$, $4l+4 \leq i \leq 3l+3k+7$,
$f^*(v_{4l+3}v_{4l+4}) = 8l+4k+15$,
$f^*(v_iv_{i+1}) = 2i+3$, $3l+3k+8 \leq i \leq 4l+4k+2$.

Case (ii):$l \geq 2k-1$.

$f(v_i) = i - 1$, $1 \leq i \leq 2l + 2$.

$f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd.} \\ i-1 & \text{if } i \text{ is even.} \end{cases}$, $2l + 3 \leq i \leq 3l + 2$.

For $3l + 3 \leq i \leq 2k + 3l$.

$f(v_i) = i + 1$, *if* l *is odd and* $f(v_i) = \begin{cases} i-1 & \text{if } i \text{ is even} \\ i+3 & \text{if } i \text{ is odd,} \end{cases}$ *if* l *is even,* *possible only if* $k \neq 1$.

For $2k + 3l + 1 \leq i \leq 4k + 4l + 2$.

$f(v_i) = \begin{cases} i+3 & \text{if } i \text{ is odd} \\ i+1 & \text{if } i \text{ is even.} \end{cases}$

$f(v_{4k+4l+2}) = 2l + 2$.

The induced edge labels are

$f^*(v_i v_{i+1}) = 2i - 1$, $1 \leq i \leq 2l + 1$,

$f^*(v_1 v_{4l+2}) = 4l + 3$,

$f^*(v_i v_{i+1}) = 2i + 1$, $2l + 2 \leq i \leq 3l + 1$,

$f^*(v_{4l+2} v_{4k+4l+3}) = \begin{cases} 4l + 4k + 5 & \text{if } l \text{ is even} \\ 4l + 4k + 3 & \text{if } l \text{ is odd.} \end{cases}$

$f^*(v_i v_{i+1}) = 2i + 3$, $3l + 2k \leq i \leq 2k + 3l$,

$f^*(v_{4k+4l+2} v_{4k+4l+3}) = 4k + 6l + 5$,

$f^*(v_i v_{i+1}) = 2i + 5$, $2k + 3l + 1 \leq i \leq 4k + 4l + 1$.

Hence the graph $C(m \circ n)$ *is odd harmonious if* $m, n \equiv 2(mod\ 4)$. □

Illustration 4. *The odd harmonious labeling of the graphs* $C(14 \circ 18)$ *and* $C(10 \circ 14)$ *are given in Figures 5 and 6.*

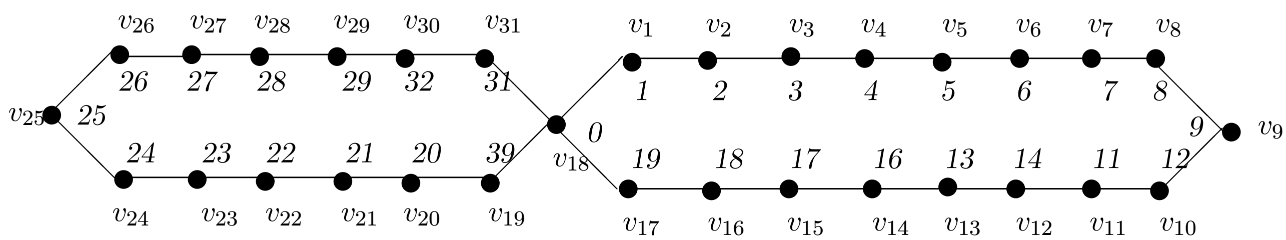

Figure 5: *The odd harmonious labeling of* $C(14@18)$

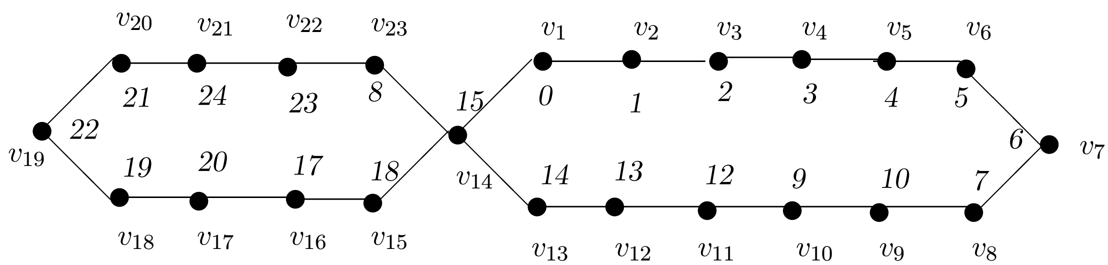

Figure 6: *The odd harmonious labeling of* $C(10 \circ 14)$

Lemma 2.7. *The graph* $C(m \circ n)$ *is odd harmonious if* $m, n \equiv 0(mod\ 4)$.

Proof. *Consider the graphs* C_m *and* C_n *with* $m, n \equiv 0(mod\ 4)$ *and take* $m = 4k$ *and* $n = 4l$. *Here the graph* $C(m \circ n)$ *has* $4l + 4k - 1$ *vertices and* $4l + 4k$ *edges.*
We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(4l + 4k) - 1\}$ *as follows.*

$f(v_i) = i - 1$, $1 \leq i \leq 2l$.
For $2l + 1 \leq i \leq 4l + 2k - 2$.

$$f(v_i) = \begin{cases} i + 1 & \text{if } i \text{ is odd.} \\ i - 1 & \text{if } i \text{ is even} \end{cases}$$

For $4l + 2k - 1 \leq i \leq 4l + 4k - 1$, $f(v_i) = i + 1$.

The induced edge labels are
$f^*(v_iv_{i+1}) = 2i-1$, $1 \le i \le 2l-1$,
$f^*(v_1v_n) = n-1$,
$f^*(v_iv_{i+1}) = 2i+1$, $2l \le i \le 4l+2k-2$,
$f^*(v_nv_{4k+4l-1}) = 4k+8l-1$,
$f^*(v_iv_{i+1}) = 2i+1$, $4l+2k-1 \le i \le 4l+4k-2$.

Hence the graph $C(m \circ n)$ *is odd harmonious if* $m, n \equiv 0(mod\ 4)$. □

Illustration 5. *The odd harmonious labeling of the graph* $C(4 \circ 8)$ *is given in Figure 7.*

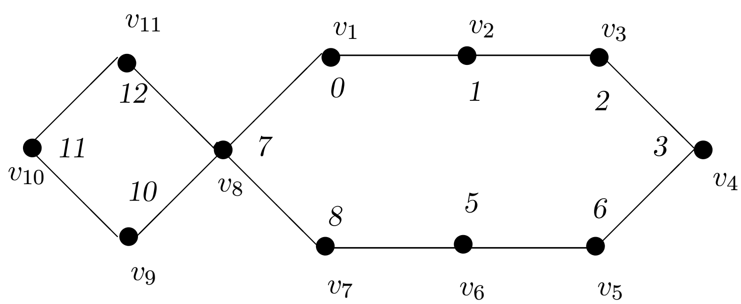

Figure 7: The odd harmonious labeling of $C(4 \circ 8)$

Lemma 2.8. *If* $m \equiv 0(mod\ 4)$ *and* $n \equiv 2(mod\ 4)$ *or vice versa, then the graph* $C(m \circ n)$ *is not an odd harmonious graph.*

Proof. *Without loss of generality we assume that* $m \equiv 0(mod\ 4)$ *and* $n \equiv 2(mod\ 4)$. *Take* $m = 4k$ *and* $n = 4l+2$. *Then the graph* $C(m \circ n)$ *has* $4k+4l+1$ *vertices and* $4k+4l+2$ *edges. Let the vertex set* $V = \{u_1, u_2, \cdots, u_{4l+1}\} \cup \{v_1, v_2, \cdots, v_{4k-1}\} \cup \{u\}$.
$2\sum_{i=1}^{4l+1} f(u_i) + 2\sum_{i=1}^{4k-1} f(v_j) + 4f(u) = q^2 = (m+n)^2 = (4l+2+4k)^2 = 4(2l+2k+1)^2$.

Hence $2\sum_{i=1}^{4l+1} f(u_i) + 2\sum_{i=1}^{4k-1} f(v_j) + 2f(u) = 2(2l+2k+1)^2$.
$2\sum_{i=1}^{4l+1} f(u_i) + 2\sum_{i=1}^{4k-1} f(v_j) = even.......(1)$.

Case (i):$f(u)$ is even.
In C_n there are $2l+1$ even labels and $2l+1$ odd labels. Hence $2\sum_{i=1}^{4l+1} f(u_i)$ is the sum of $2l$ even integers and $2l+1$ odd integers. Therefore $2\sum_{i=1}^{4l+1} f(u_i)$ is odd.
Also $2\sum_{i=1}^{4k-1} f(v_j)$ is the sum of $2k-1$ even integers and $2k$ odd integers. Therefore $2\sum_{i=1}^{4k-1} f(v_j)$ is even. Hence $2\sum_{i=1}^{4l+1} f(u_i)+2\sum_{i=1}^{4k-1} f(v_j)$ is odd, which is a contradiction to (1).

Case (ii):$f(u)$ is odd.
$2\sum_{i=1}^{4l+1} f(u_i)$ is the sum of $2l$ odd integers and $2l+1$ even integers. Therefore $2\sum_{i=1}^{4l+1} f(u_i)$ is even.
Also $2\sum_{i=1}^{4k-1} f(v_j)$ is the sum of $2k-1$ odd integers and $2k$ even integers. Therefore $2\sum_{i=1}^{4k-1} f(v_j)$ is odd. Hence $2\sum_{i=1}^{4l+1} f(u_i)+2\sum_{i=1}^{4k-1} f(v_j)$ is odd, which is a contradiction to (1).

Therefore the graph $C(m \circ n)$ is not an odd harmonious graph if $m \equiv 0(mod\ 4)$ and $n \equiv 2(mod\ 4)$. □

Theorem 2.9. *The graph $C(m \circ n)$ is an odd harmonious graph if and only if either both $m, n \equiv 0(mod\ 4)$ or both $m, n \equiv 2(mod\ 4)$.*

Proof. *By Lemmas 2.5 and 2.6 the graph $C(m \circ n)$ is an odd harmonious graph if and only if either both $m, n \equiv 0(mod\ 4)$ or both $m, n \equiv 2(mod\ 4)$. Conversely, by Lemma 2.7, if $m \equiv 0(mod\ 4)$ and $n \equiv 2(mod\ 4)$ or vice versa, then the graph $C(m \circ n)$ is not an odd harmonious graph. Therefore,$C(m \circ n)$ is an odd harmonious graph if and only if either both $m, n \equiv 0(mod\ 4)$ or both $m, n \equiv 2(mod\ 4)$.* □

References

[1] M. E. Abdel-Aal, Odd Harmonious Labelings of Cyclic Snakes, *International Journal on applications of Graph Theory in Wireless Adhoc networks and Sensor Networks*, 5 (3), pp. 1–13, (2013).

[2] M. E. Abdel-Aal, New Families of Odd Harmonious Graphs, *International Journal of Soft Computing, Mathematics and Control*, 3 (1), pp. 1–13, (2014).

[3] J. A. Gallian, A Dynamic Survey of Graph Labeling, The Electronics Journal of Combinatorics, (2015) #DS6.

[4] R. L. Graham and N. J. A Sloane, On Additive bases and Harmonious Graphs, *SIAM J. Algebr. Disc. Meth.*, 4, pp. 382–404, (1980).

[5] A. Gusti Saputri, K. A. Sugeng and D. Froncek, The Odd Harmonious Labeling of Dumbbell and Generalized Prism Graphs, *AKCE Int. J. Graphs Comb.*, 10 (2), pp. 221–228, (2013).

[6] F. Harary, *Graph Theory*, Addison-Wesley, Massachusetts, (1972).

[7] Z. Liang, Z. Bai, On the Odd Harmonious Graphs with Applications, *J. Appl. Math. Comput.*, 29, pp. 105–116, (2009).

[8] P. Jeyanthi , S. Philo and Kiki A. Sugeng, Odd harmonious labeling of some new families of graphs, *SUT Journal of Mathematics.*, Vol. 51, No 2, pp. 53-65, (2015).

[9] P. Selvaraju, P. Balaganesan and J.Renuka, Odd Harmonious Labeling of Some Path Related Graphs, *International J. of Math. Sci. & Engg.Appls.*, 7 (III), pp. 163–170, (2013).

[10] S. K. Vaidya and N. H. Shah, Some New Odd Harmonious Graphs, *International Journal of Mathematics and Soft Computing*, 1, pp. 9–16, (2011).

[11] S. K. Vaidya, N. H. Shah, Odd Harmonious Labeling of Some Graphs, *International J. Math. Combin.*, 3, pp. 105–112, (2012).

P. Jeyanthi
Research Centre, Department of Mathematics,
Govindammal Aditanar College for Women,
Tiruchendur - 628 215, Tamil Nadu,
India
e-mail: jeyajeyanthi@rediffmail.com

and

S. Philo
Department of Mathematics,
PSN College of Engineering and Technology,
Melathediyoor,
Tirunelveli, Tamil Nadu,
India
e-mail: lavernejudia@gmail.com