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# Strongly$(V^\lambda, A, \Delta^n_{(vm)}, p, q)$-summable sequence spaces defined by modulus function and statistical convergence

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**Abstract**

*In this paper we introduce strongly $(V^\lambda, A, \Delta^n_{(vm)}, p, q)$-summable sequences and give the relation between the spaces of strongly $(V^\lambda, A, \Delta^n_{(vm)}, p, q)$-summable sequences and strongly $(V^\lambda, A, \Delta^n_{(vm)}, p, q)$-summable sequences with respect to a modulus function when $A = (a_{ik})$ is an infinite matrix of complex number, $(\Delta^n_{(mv)})$ is generalized difference operator, $p = (p_i)$ is a sequence of positive real numbers and $q$ is a seminorm. Also we give the relationship between strongly $(V^\lambda, A, \Delta^n_{(vm)}, p, q)$ - convergence with respect to a modulus function and strongly $S^\lambda(A, \Delta^n_{(vm)})$- statistical convergence.*

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## 1. Introduction and Preliminaries

The idea of difference sequence spaces was introduced by Kizmaz [9]. In 1981, Kizmaz [9] defined the sequence spaces:

$$Z(\Delta) = \Big\{x = (x_k) : \Delta x \in Z\Big\},$$

for $Z = \ell_\infty, c$ and $c_0$, where $\Delta x = \Big(x_k - x_{k+1}\Big)$.

The notion was further generalized by Et and Ç$olak$ [5] by introducing the space $\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of difference sequence spaces is due to Tripathy and Esi[23]. Who studied the space $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$.Tripathy et al.[24] generalized the above notion and define these spaces as follow:
Let $m, n$ be non negative integers, then for $Z$ a given sequence space we have.

$$Z(\Delta_m^n) = \Big\{x = (x_k) : (\Delta_m^n x_k) \in Z\Big\}$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+1})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbf{N}$
Which is equivalent to the following binomial representation.

$$\Delta_m^n x_k = \sum_{i=0}^{n} (-1)^i \, (\, n \,)\, i x_{k+mi}$$

Let $m, n$ be non-negative integers and $v = (v_k)$ be a sequence of non -zero scalars. Then for Z, a given sequence space, recently Dutta [4] introduced the following sequence spaces:

$$Z(\Delta_{(vm)}^n) = \Big\{x = (x_k) : (\Delta_{(vm)}^n x_k) \in Z\Big\}, \text{ for } Z = \ell_\infty \text{ , } c \text{ and } c_0.$$

Where $(\Delta_{(vm)}^n x_k) = (\Delta_{vm}^{n-1} x_k - \Delta_{vm}^{n-1} x_{k-m})$ and $\Delta_{vm}^0 x_k = v_k x_k$ for all $k \in \mathbf{N}$ which is equivalent to the following binomial representation:

$$\Delta_{(vm)}^n x_k = \sum_{i=0}^{n} (-1)^i \, (\, n \,)\, i v_{k-mi} x_{k-mi}.$$

We take $v_{k-mi} = 0$ and $x_{k-mi} = 0$ for non-positive value of $k - mi$. Dutta [4] showed that these spaces can be made $BK$ spaces under the norm

$$\|x\| = \sup_k \mid \Delta^{n}_{(vm)} x_k \mid .$$

For $n = 1$ and $v_k = 1$ for all $k \in \mathbf{N}$. We get the spaces $\ell_{\infty}(\Delta_m), c(\Delta_m)$ and $c_o(\Delta_m)$. For $m = 1$ and $v_k = 1$ for all $k \in \mathbf{N}$, we get the spaces $\ell_{\infty}(\Delta^n), c(\Delta^n)$ and $c_o(\Delta^n)$. For $m = n = 1$ and $v_k = 1$ for all $k \in \mathbf{N}$,, we get the spaces $\ell_{\infty}(\Delta), c(\Delta)$ and $c_o(\Delta)$.

Let $\lambda = (\lambda_r)$ be a non- decreasing sequence of positive numbers tending to $\infty$ such that

$$\lambda_{r+1} \leq \lambda_r + 1,\ \lambda_1 = 1.$$

The generalized de la Vall*éé*-Pousin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{i \in I_r} x_i,$$

where $I_r = [r - \lambda_r + 1, r]$ for $r = 1, 2, ....$

A sequence $x = (x_i)$ is said to be $(V, \lambda)$- summable to a number $s$, if $t_r(x) \to s$ as $r \to \infty$[11].

If $\lambda_r = r$, then $(V, \lambda)$- summability is reduced to $(C, 1)$- summability. We write

$$[V, \lambda] = \left\{ x = (x_i) : \lim_{r \to \infty} \lambda_r^{-1} \sum |x_i - s| = 0 \text{ for some s} \right\}$$

the set of sequences $x = (x_i)$ which are strongly $(V, \lambda)$-summable to s that is $x_i \to s[V, \lambda]$. The strongly $(V, \lambda)$-summable as well as generalized this kind of summable sequence spaces have been studied by various authors( Bilgin[2],Gunor et al[8], Savas[19] and others). The idea of modulus function was introduced by Nakano[15].

We recall that a modulus $f$ is a function from $[0, \infty) \to [0, \infty)$ such that

$(i)$ $f(x) = 0$ if and only if $x = 0$,

$(ii)$ $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, \quad y \geq 0$,

$(iii)$ $f$ is increasing,

$(iv)$ $f$ is continuous from right at 0.

It follows that $f$ must be a continuous everywhere on $[0, \infty)$. The Belgin [2], Kolack [10] Maddox[12,13],Öztürk and Bilgin [2], Ruckle [17] and others used a modulus function for defining some new sequence spaces.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write $Ax = (A_i(x))$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik}x_k$ converges for each $i$.

Recently,the concept of strong $(V, \lambda)$- summability was generalized by Bilgin [1] as follow:

$$V^{\lambda}[A, f] = \left\{ x = (x_i) : \lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} f(\mid A_i(x) - s \mid) = 0 \text{ for some s} \right\}.$$

In this paper we introduce the strongly $(V^{\lambda}, A, \Delta^n_{(vm)}, p, q)$-summable sequences and give the relation between the spaces of strongly $(V^{\lambda}, A, \Delta^n_{(vm)}, p, q)$-summable sequences and strongly and strongly $(V^{\lambda}, A, \Delta^n_{(vm)}, p, q)$-summable sequences with respect to a modulus function when $A = (a_{ik})$ be an infinite matrix of real or complex number, $(\Delta^n_{(mv)})$ is generalized difference operator, $p = (p_i)$ is a sequence of positive real numbers and $q$ is a seminorm. Also we give the natural relationship between strongly $(V^{\lambda}, A, \Delta^n_{(vm)}, p, q)$ -convergence with respect to a modulus function and strongly $S^{\lambda}(A, \Delta^n_{(vm)})$-statistical convergence.
The following inequality will be used throughout the paper:

$$\mid a_i + b_i \mid^{p_i} \leq T\Big( \mid a_i \mid^{p_i} + \mid b_i \mid^{p_i} \Big) \qquad (1).$$

where $a_i$ and $b_i$ are complex numbers, $T = \max\Big(1, 2^{H-1}\Big)$ and $H = \sup p_i < \infty$.

## 2. Main Results

### 2. Strongly $(V^{\lambda}, A, \Delta^n_{(vm)}, p, q)$-summable sequences

Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $p = (p_i)$ be bounded sequence of positive real numbers $(0 < h = \inf p_i \leq p_i \leq \sup p_i = H < \infty)$, and $F = (f_k)$ be a sequence of modulus function. We define

$$V^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$$

$$= \Big\{x : \lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \Big[f_k\Big(q\Big( \mid \Delta^n_{(vm)} A_i(x) - s \mid \Big)\Big)\Big]^{p_i} = 0 \text{ for some s}\Big\}.$$

$$V_0^{\lambda}[A, \Delta^n_{(vm)}, F, p, q] = \Big\{x : \lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \Big[f_k\Big(q\Big( \mid \Delta^n_{(vm)} A_i(x) \mid \Big)\Big)\Big]^{p_i} = 0\Big\}.$$

$$V_\infty^{\lambda}[A, \Delta^n_{(vm)}, F, p, q] = \Big\{x : \sup_r \lambda_r^{-1} \sum_{i\in I_r} \Big[f_k\Big(q\Big( \mid \Delta^n_{(vm)} A_i(x) \mid \Big)\Big)\Big]^{p_i} < \infty\Big\}.$$

A sequence $x = (x_i)$ is said to be strongly $(V^{\lambda}, A, \Delta^n_{(vm)}, p, q)$-convergent to a number $s$ with respect to a modulus if there is a complex number $s$ such that $x \in (V^{\lambda}, A, \Delta^n_{(vm)}, p, q)$. If $x$ is strongly $(V^{\lambda}, A, \Delta^n_{(vm)}, p, q)$-convergent to s with respect to a modulus $F = (f_k)$, then we write $x_i \to s(V^{\lambda}, \mid A, \Delta^n_{(vm)}, F, p, q \mid)$.
Throughout this paper $\varphi$ will denote one of the notation $0, 1$ or $\infty$.

When $F(x) = x$ then we write the spaces $V_\varphi^{\lambda}[A, \Delta^n_{(vm)}, p, q]$ in place of $V_\varphi^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$, If $p_i = 1$ for all $i$, then $V_\varphi^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ reduces to $V_\varphi^{\lambda}[A, \Delta^n_{(vm)}, F, q]$ if $q = x$ then $V_\varphi^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ reduces to $V_\varphi^{\lambda}[A, \Delta^n_{(vm)}, F, p]$.

In this section we examine some topological properties of $V^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ spaces and investigate some inclusion relations between these spaces.

**Theorem 2.1.** Let $F = (f_k)$ be a sequence of moduli, q be a seminorm, $p = (p_i)$ be a sequence of positive real numbers and $X$ denotes the anyone of the spaces $V^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$,$V_0^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ or $V_\infty^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$. Then $X$ is linear space over the complex field **C**.

**Proof.** Since the proof is analogus for the space $V^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$, and $V_\infty^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$. So we give the proof of $V_0^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$. Let

$x, y \in V_0^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ and $a, b \in \mathbf{C}$. Then there exist integers $T_a$ and $T_b$ such that $|a| \leq T_a$ and $|b| \leq |T_b$. We have

$$\lambda_r^{-1} \sum_{i \in I_r} \Big[ f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i \Big( ax + by \Big) \Big) \mid \Big) \Big) \Big]^{p_i}$$

$$\leq \lambda_r^{-1} \sum_{i \in I_r} \Big[ f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i ax + \Delta^n_{(vm)} A_i by \mid \Big) \Big) \Big]^{p_i}$$

$$\leq T \Big\{ \lambda_r^{-1} \sum_{k \in I_r} \Big[ T_a f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i x \mid \underline{-}ig) \Big]^{p_i}$$

$$+ \lambda_r^{-1} \sum_{i \in I_r} \Big[ T_b f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i y \mid \Big) \Big) \Big]^{p_i} \Big\}$$

$$\leq T \Big\{ [T_a]^H \lambda_r^{-1} \sum_{i \in I_r} f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i x \mid \Big)^{p_i}$$

$$+ [Tb]^H \lambda_r^{-1} \sum_{i \in I_r} f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i y \mid \Big) \Big) \Big]^{p_i} \Big\}$$

as $r \to \infty$.This proves that $V_0^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ is linear.

**Theorem 2.2.** Let $F = (f_k)$ be a sequence of moduli, q be a seminorm and $p = (p_i)$ be a sequence of positive real numbers, then the inclusions $V_0^{\lambda}[A, \Delta^n_{(vm)}, F, p, q] \subset V^{\lambda}[A, \Delta^n_{(vm)}, F, p, q] \subset V_{\infty}^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ hold.

**Proof.** The inclusion $V_0^{\lambda}[\Delta^n_{(vm)}, F, p, q] \subset V^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ is obvious. Now let $x \in V^{\lambda}[\Delta^n_{(vm)}, A, F, p, q]$ such that $x_i \to s\Big( V^{\lambda}[\Delta^n_{(vm)}, A, F, p, q]$. By using (1), we have

$$\sup_r \lambda_r^{-1} \sum_{i \in I_r} f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i(x) \mid \Big)^{p_i}$$

$$= \sup_r \lambda_r^{-1} \sum_{i \in I_r} f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i(x) - s + s \mid \Big)^{p_i}$$

$$\leq T \Big\{ \sup_r \lambda_r^{-1} \sum_{i \in I_r} f_k \Big( q \Big( \mid \Delta^n_{(vm)} A_i(x) - s \mid \Big)^{p_i}$$

$$+\sup_r \lambda_r^{-1} \sum_{i\in I_r} f_k\Big(q\Big( \mid s \mid \Big)^{p_i}\Big\}$$

$$\leq T\Big\{ \sup_r \lambda_r^{-1} \sum_{i\in I_r} f_k\Big(q\Big( \mid \Delta^{n}_{(vm)} A_i(x) - s \mid \Big)^{p_i}$$

$$+\max\Big\{ f_k\Big(q\Big( \mid s \mid \Big)^{h}, f_k q\Big( \mid s \mid \Big)^{H}\Big\} < \infty.$$

Hence $x \in V^{\lambda}[A, \Delta^{n}_{(vm)}, F, p, q]$. This proves that inclusion $V^{\lambda}[A, \Delta^{n}_{(vm)}, F, p, q] \subset V^{\lambda}_{\infty}[A, \Delta^{n}_{(vm)}, F, p, q]$ holds, which completes the proof.

**Corollary 1.** $V^{\lambda}_{0}[A, \Delta^{n}_{(vm)}, F, p, q]$ and $V^{\lambda}[A, \Delta^{n}_{(vm)}, F, p, q]$ are nowhere dense subsets of $V^{\lambda}_{\infty}[A, \Delta^{n}_{(vm)}, F, p, q]$. Let $X$ be a sequence space.
$(i)$ $X$ is called solid(or normal) if $(\alpha_i x_i) \in X$ , whenever $(x_i) \in X$ for all sequences $(\alpha_i)$ of scalars with$\mid \alpha_i \mid \leq 1$, for all $i \in \mathbf{N}$.
$(ii)$ Monotone provided $X$ contains the canonical pre - images of all its step spaces. If $X$ is normal, then $X$ is monotone.

**Theorem 2.3.** The sequence spaces $V^{\lambda}_{0}[A, \Delta^{n}_{(vm)}, F, p, q]$ and $V^{\lambda}_{\infty}[A, \Delta^{n}_{(vm)}, F, p, q]$ are solid and hence monotone.

**Proof.** Let $\alpha = (\alpha_i)$ be a sequence of scalars such that $\mid \alpha_i \mid \leq 1$, for all $i \in \mathbf{N}$. Since $F = (f_k)$ is monotone, we get

$$\lambda_r^{-1} \sum_{i\in I_r} f_k\Big(q\Big( \mid \Delta^{n}_{(vm)} A_i(\alpha x) \mid \Big)^{p_i}$$

$$\leq \lambda_r^{-1} \sum_{i\in I_r} f_k\Big(q\Big( \sup \mid \alpha_i \mid\mid \Delta^{n}_{(vm)} A_i(x) \mid \Big)^{p_i} \leq \lambda_r^{-1} \sum_{i\in I_r} f_k\Big(q\Big( \mid \Delta^{n}_{(vm)} A_i(x) \mid \Big)^{p_i}$$

Which leads to the proof.

**Theorem 2.4.** Let $F = (f_k)$ be any modulus. Then $V^{\lambda}_{\varphi}[A, \Delta^{n}_{(vm)}, p, q] \subset V^{\lambda}_{\varphi}[A, \Delta^{n}_{(vm)}, F, p, q]$.

**Proof.** We consider the case $V^{\lambda}_{0}[A, \Delta^{n}_{(vm)}, p, q] \subset V^{\lambda}_{0}[A, \Delta^{n}_{(vm)}, F, p, q]$. Let $x \in V^{\lambda}_{0}[A, \Delta^{n}_{(vm)}, p, q]$ and $\epsilon > 0$. We choose $0 < \delta < 1$ such that $f_k(u) < \epsilon$ for every $u$ with $0 \leq u \leq \delta$.

we can write

$$\lambda_r^{-1}\sum_{i\in I_r} f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}$$
$$=\lambda_r^{-1}\sum_{1} f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}+\lambda_r^{-1}\sum_{2} f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}.$$
$$\le \max\Big(\epsilon^h,\epsilon\Big)+\max\Big(1,(2f_k(1)\delta^{-1})\Big)^H\Big)\lambda_r^{-1}\sum_{2} f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}.$$

where
$\sum_1 f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}\le\delta$ and $\sum_2 f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}>\delta.$

Hence
$$\lambda_r^{-1}\sum_{i\in I_r} f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}$$
$$\le \max\Big(\epsilon^h,\epsilon\Big)+\max\Big(1,(2f_k(1)\delta^{-1})\Big)^H\Big)\lambda_r^{-1}\sum_{i\in I_r} f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}.$$

therefore, $x\in V_0^\lambda[\Delta^n_{(vm)},A,F,p,q]$

**Theorem 2.5.** Let $F=(f_k)$ be any modulus. If $\lim_{t\to\infty}\frac{f(t)}{t}=\beta>0$, then $V_\varphi^\lambda[A,\Delta^n_{(vm)},p,q]=V_\varphi^\lambda[A,\Delta^n_{(vm)},F,p,q]$.

**Proof.** The existence of positive limit for any modulus function given with $\beta$ was introduced by Maddox[13]
Let $\beta>0$ and Let $x\in V_\varphi^\lambda[A,\Delta^n_{(vm)},,F,p,q]$. Since $\beta>0$, we have $f_k(t)\ge\beta t$ for all $t>0$ It is easy to see that $x\in V_\varphi^\lambda[A,\Delta^n_{(vm)},p,q]$, by using Theorem 2.4 the proof is completed.
we consider that $(p_i)$ and $p_i^{'}$ are any bounded sequences of positive real numbers. We can prove $V_\varphi^\lambda[A,\Delta^n_{(vm)},F,p^{'},q]\subset V_\varphi^\lambda[A,\Delta^n_{(vm)},F,p,q]$ only under addition condition

**Theorem 2.6.** Let $0<p_i\le p_i^{'}$, for all$i$ and let $\frac{p_i^{'}}{p_i}$ be bounded. Then $V_\varphi^\lambda[A,\Delta^n_{(vm)}F,p^{'},q]\subset V_\varphi^\lambda[A,\Delta^n_{(vm)},F,p,q]$

**Proof.** If we take $t_i=f_k(\mid A_i(x)\mid)^{p_i^{'}}$ for all $i$, then using the same technique in proof of Theorem 2.2 of Öztürk and Bilgin [16], it is easy to prove the theorem

**Corollary 2.**

if $0 < \inf p_i \leq 1$ for all $i$, $V_{\varphi}^{\lambda}[A, \Delta^n_{(vm)} F, q] \subset V_{\varphi}^{\lambda}[A, \Delta^n_{(vm)}, F, p, q]$ if $1 \leq p_i \leq \sup p_i = H < \infty$, then $V_{\varphi}^{\lambda}[A, \Delta^n_{(vm)}, F, p, q] \subset V_{\varphi}^{\lambda}[A, \Delta^n_{(vm)}, F, q]$

## 3. $S^{\lambda}(A, \Delta^n_{(vm)})$-Statistical Convergence

In this section,we introduce natural relationship between strongly $V^{\lambda}[A, \Delta^n_{(vm)}, p, q]$-convergence with respect to modulus function and strongly $S^{\lambda}(A, \Delta^n_{(vm)})$- statistical convergence. In [6], Fast introduce the idea of statistical convergence.These idea was later studied by Connor [3], Freedman and Sember [7], Salat[19],Savas[20], Schoenberg [21], Rath and Tripathy [18],Tripathy [22],Tripathy and Sen [25,26] and the other authors independently.

A complex number sequence $x = (x_i)$ is said to be statistically convergent to the number $\ell$ if for every $\epsilon > 0$, $\lim_{n\to\infty} \mid \frac{K(\epsilon)}{n} \mid = 0$, where $\mid K(\epsilon) \mid$ denotes the number of elements in $K(\epsilon) = \Big\{ i \in \mathbf{N} : \mid x_i - \ell \mid \geq \epsilon \Big\}$. The set of statistically convergent sequences is denoted by $S$.

A sequence $x = (x_i)$ is said to strongly $S^{\lambda}(A, \Delta^n_{(vm)})$- statistically convergent to $s$ if any $\epsilon > 0$, $\lim_{r\to\infty} \lambda_r^{-1} \mid KA(\epsilon) \mid = 0$, where$\mid K(\epsilon) \mid$ denotes the number of elements in
$KA(\epsilon) = \Big\{ i \in I_r : \mid \Delta^n_{(vm)} A_i(x) - s \mid \geq \epsilon \Big\}$.

The set of all strongly $S^{\lambda}(A, \Delta^n_{(vm)})$- statistically convergent sequences is denoted by $S^{\lambda}(A, \Delta^n_{(vm)})$.

Now we give the relation between $S^{\lambda}(A, \Delta^n_{(vm)})$-statistically convergence and strongly $V^{\lambda}(A, \Delta^n_{(vm)}, p, q)$- convergence with respect to modulus.

**Theorem 3.1.** Let $F = (f_k)$ be any modulus. Then
$V^{\lambda}[A, \Delta^n_{(vm)}, F, p, q] \subset S^{\lambda}(A, \Delta^n_{(vm)})$.

**Proof.** Let $x \in V^{\lambda}(A, \Delta^n_{(vm)}, F, p, q)$. Then

$$\lambda_r^{-1}\sum_{i\in I_r} f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}$$

$$\geq \lambda_r^{-1}\sum_1 f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i} \geq \lambda_r^{-1}\sum_1 f_k\Big(\epsilon\Big)^{p_i}.$$

$$\geq \lambda_r^{-1}\sum_1 \min\Big(f_k(\epsilon)^h, f_k(\epsilon)\Big)^H$$

$$\geq \lambda_r^{-1}\mid \Big\{i\in I :\mid \Delta^n_{(vm)}A_i(x)-s\mid\geq\epsilon\Big\}\mid \min\Big\{f_k(\epsilon)^h,(\epsilon)^H\Big\}.$$

where the summation $\sum_1$ is over $(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)\geq\epsilon$. Hence
$S^\lambda(\mid \Delta^n_{(vm)}A_i(x)\mid\Big)$

**Theorem 3.2.** Let $F=(f_k)$ be any modulus. Then
$V^\lambda[A,\Delta^n_{(vm)},F,p,q]\subset S^\lambda(A,\Delta^n_{(vm)},q)$.

**Proof.** By Theorem 3.1. it is sufficient to show that
$S^\lambda[A,\Delta^n_{(vm)},q]\subset S^\lambda(A,\Delta^n_{(vm)},F,p,q)$.

Let $x\in S^\lambda(A,\Delta^n_{(vm)},q)$. Since $f_k$ is bounded, so there exists an integer $K>0$ such that
$f_k(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)\leq K$. Then for a given $\epsilon>0$, we have.

$$\lambda_r^{-1}\sum_{i\in I_r} f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}$$

$$=\lambda_r^{-1}\sum_1 f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}+\lambda_r^{-1}\sum_2 f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)^{p_i}.$$

$$\leq K^H\lambda_r^{-1}\mid\{i\in I :\mid \Delta^n_{(vm)}A_i(x)-s\mid\geq\epsilon\Big\}\mid+\max\Big\{f_k(\epsilon)^h,f_k(\epsilon)^H\Big\}.$$

where the summation $\sum_1 f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)\geq\epsilon$ and
$\sum_2 f_k\Big(q\Big(\mid \Delta^n_{(vm)}A_i(x)-s\mid\Big)<\epsilon$. Taking $\epsilon\to 0$ and $r\to\infty$. It follows that $x\in V^\lambda(A,\Delta^n_{(vm)},F,p,q)$. This completes the proof.

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