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# The $t$-pebbling number of Jahangir graph $J_{3,m}$

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**Abstract**

*The $t$-pebbling number, $f_t(G)$, of a connected graph $G$, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, $t$ pebbles can be moved to any specified target vertex by a sequence of pebbling moves, each move removes two pebbles of a vertex and placing one on an adjacent vertex. In this paper, we determine the $t$-pebbling number for Jahangir graph $J_{3,m}$ and finally we give a conjecture for the $t$-pebbling number of the graph $J_{n,m}$.*

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**Keywords :** *Pebbling number, Jahangir graph.*

## 1. Introduction

We begin by introducing relevant terminology and background on the subject. Here, the term graph refers to a simple graph. A function $\phi : V(G) \rightarrow N \cup \{0\}$ is called a *pebbling*. Let $\phi(v)$ denote the number of pebbles on the vertex $v$ and $\phi(V(A))$ denote the number of pebbles on the vertices of the subgraph $A$ of $G$. The quantity $\sum_{x \in V(G)} \phi(x)$ is called the *size* of $\phi$; the size of $\phi$ is just the total number of pebbles assigned to vertices. A *pebbling move (step)* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. For a graph $G$, if $\phi$ is a distribution or configuration of pebbles onto the vertices of $G$ and it is possible to move a pebble to the target vertex $v$, then we say that $\phi$ is *$v$-solvable*; otherwise, $\phi$ is *$v$-unsolvable*. Then $\phi$ is *solvable* if it is $v$-solvable for all $v \in V(G)$, and *unsolvable* otherwise. If $\phi(v) = 1$ or $\phi(u) \geq 2$ where $uv \in E(G)$, then we can easily move one pebble to $v$. So, we always assume that $\phi(v) = 0$ and $\phi(u) \leq 1$ for all $uv \in E(G)$ when $v$ is the target vertex.

The *$t$-pebbling number of a vertex $v$* in a graph $G$ [2], $f_t(v, G)$, is the smallest positive integer $m$ such that however $m$ pebbles are placed on the vertices of the graph, $t$ pebbles can be moved to $v$ in finite number of pebbling moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. The *$t$-pebbling number* of $G$, $f_t(G)$, is defined to be the maximum of the pebbling numbers of its vertices. Thus the *$t$-pebbling number of a graph $G$*, $f_t(G)$, is the least $m$ such that, for any configuration of $m$ pebbles to the vertices of $G$, we can move $t$ pebbles to any vertex by a sequence of moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the *pebbling number* of $G$.

**Fact 1.** *([12]) For any vertex $v$ of a graph $G$, $f(v, G) \geq n$ where $n = |V(G)|$.*

**Fact 2.** *([12]) The pebbling number of a graph $G$ satisfies*

$$f(G) \geq max\{2^{diam(G)}, |V(G)|\}.$$

*Jahangir graph* $J_{n,m}$ [11], for $m \geq 3$, is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle $C_{nm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{nm}$ at distance $n$ to each other on $C_{nm}$.

For labeling $J_{3,m}$ ($m \geq 3$), let $v_{3m+1}$ be the label of the center vertex and $v_1, v_2, \cdots, v_{3m}$ be the label of the vertices that are incident clockwise on cycle $C_{3m}$ so that $deg(v_1) = 3$.

With regard to $t$-pebbling number of graphs, we have found the following theorems:

**Theorem 1.** *([9]). Let $K_n$ be the complete graph on $n$ vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.*

**Theorem 2.** *([2]). Let $K_1 = \{v\}$. Let $C_{n-1} = (u_1, u_2, \cdots, u_{n-1})$ be a cycle of length $n - 1$. Then the $t$-pebbling number of the wheel graph $W_n$ is $f_t(W_n) = 4t + n - 4$ for $n \geq 5$.*

**Theorem 3.**

$$f_t(G) = \begin{cases} 2t + n - 2, \text{ if } 2t \leq n - s_1 \\ 4t + s_1 - 2, \text{ if } 2t \geq n - s_1 \end{cases}$$

**Theorem 4.** *([9]). Let $K_{1,n}$ be an $n$-star where $n > 1$. Then $f_t(K_{1,n}) = 4t + n - 2$.*

**Theorem 5.** *([9]). Let $C_n$ denote a simple cycle with $n$ vertices, where $n \geq 3$. Then $f_t(C_{2k}) = t2^k$ and $f_t(C_{2k+1}) = \frac{2^{k+2}-(-1)^{k+2}}{3} + (t-1)2^k$.*

**Theorem 6.** *([9]). Let $P_n$ be a path on $n$ vertices. Then $f_t(P_n) = t(2^{n-1})$.*

**Theorem 7.** *([9]). Let $Q_n$ be the $n$-cube. Then $f_t(Q_n) = t(2^n)$.*

Lourdusamy et. al proved the $t$-pebbling number of Jahngir graph $J_{2,m}$ for $m \geq 3$ and $t \geq 1$ in [3, 6, 7, 8]. In the next section, we are going to present the pebbling number of Jahangir graph $J_{3,m}$ for $m \geq 3$. In section three, we find the $t$-pebbling number for $J_{3,m}$. In the last section, we give a conjecture for the $t$-pebbling number of Jahangir graph $J_{n,m}$, where $2 < n < m$.

## 2. The pebbling number for Jahangir graph $J_{3,m}$

Consider Jahangir graph $J_{3,m}$ ($m \geq 3$), and we can see that $J_{3,m}$ has $m$ cycles of length five, 5-cycles, that is,

$$S_1 : v_1v_2v_3v_4v_{3m+1}v_1,$$

$$S_2 : v_4v_5v_6v_7v_{3m+1}v_4,$$

$$\vdots$$

$$S_m : v_{3m-2}v_{3m-1}v_{3m}v_1v_{3m+1}v_{3m-2}.$$

For $i, j = 1, 2, , m$, $i \neq j$, two cycles $S_i$ and $S_j$ are adjacent if there exists a *common edge* $(v_k v_{3m+1})$ between them. The vertex $v_k$ is called a *common vertex* for $S_i$ and $S_j$.

**Theorem 1.** *For Jahangir graph $J_{3,3}$, $f(J_{3,3}) = 17$.*

**Proof.** Clearly, by Fact 2, $f(J_{3,3}) \geq max\{16, 10\} = 16$. Let $\phi(v_6) = 15$, $\phi(v_9) = 1$ and for all $i \neq 6, 9$, $\phi(v_i) = 0$. If $v_2$ is a target vertex, then pebbling $\phi$ is $v_2$-unsolvable. Thus $f(J_{3,3}) \geq 17$.

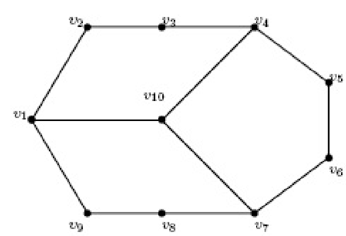


**Figure 1.** Jahangir graph $J_{3,3}$

Consider the distribution of 17 pebbles on the vertices of $J_{3,3}$.

**Case 1:** Let $v_{10}$ be the target vertex.
Without loss of generality, let $\phi(V(S_1)) \geq 5$, since $J_{3,3}$ has 17 pebbles and three 5-cycles. Hence, we can move one pebble to $v_{10}$ by Theorem 5.

**Case 2:** Let $v_1$ be the target vertex.
If $\phi(V(S_1)) \geq 5$ or $\phi(V(S_3)) \geq 5$, then we can easily move one pebble to $v_1$. Assume $\phi(v_2) + \phi(v_3) \leq 3$ and $\phi(v_8) + \phi(v_9) \leq 3$ (otherwise one pebble can be moved to $v_1$). Obviously, $\phi(V(S_2)) \geq 11$, and by Theorem 5, $f_2(C_5) = 9$. Therefore by moving two pebbles to $v_{10}$, we can move one pebble to $v_1$.

**Case 3:** Let $v_2$ be the target vertex.
If $\phi(V(S_1)) \geq 5$, then we can easily move one pebble to $v_2$. Since, $\phi(V(J_{3,3})-$

$\{v_2, v_3\}) \geq 16$, either $\phi(V(S_2)) \geq 8$ or $\phi(V(S_3)) \geq 8$.

**Case 3.1:** Let $\phi(V(S_3)) \geq 8$.
If $\phi(V(S_3)) \geq 9$, then by Theorem 5 we can move one pebble to $v_2$. Let $\phi(V(S_3)) = 8$. Clearly, $\phi(V(S_2)) \geq 5$. Thus we can move one pebble to $v_7$ or $v_{10}$ from the pebbles on $V(S_2)$. Now, we have at least 9 pebbles on $V(S_3)$ and hence we can move one pebble to $v_2$ through $v_1$ by Theorem 5.

**Case 3.2:** Let $\phi(V(S_2)) \geq 8$.
Let $\phi(V(S_2)) \geq 9$. Clearly we can move one pebble to $v_2$ if $\phi(v_1) = 1$ or $\phi(v_3) = 1$ or $\phi(v_8) + \phi(v_9) \geq 4$. So, we assume $\phi(v_1) = 0$, $\phi(v_3) = 0$ and $\phi(v_8) + \phi(v_9) \leq 3$ such that we cannot move one pebble to $v_1$ from the pebbles on the vertices $v_8$ and $v_9$. Now, we have $\phi(V(S_2)) \geq 14$. We assume $\phi(v_4) = 0$ and $\phi(v_{10}) = 0$ (otherwise one pebble could be moved to $v_2$ by Theorem 5).

**Case 3.2.1:** Let $\phi(v_8) = 2$ or $3$ and $\phi(v_9) = 0$.
If $\phi(v_7) \geq 1$, then we move one pebble to $v_7$ from $v_8$ and then we move one pebble to $v_{10}$ from $v_7$. Now $V(S_2) - \{v_{10}\}$ contains at least 13 pebbles and hence we can easily move one pebble to $v_2$ through $v_{10}$ by Theorem 5. Assume $\phi(v_7) = 0$. Let $\phi(v_8) = 2$. If $\phi(v_6) \geq 2$, then we move one pebble $v_{10}$ through $v_7$ from $v_6$ and $v_8$ and hence we can easily move one pebble to $v_2$ through $v_{10}$ by Theorem 5. If $\phi(v_6) \leq 1$, then we can easily move one pebble to $v_2$ since $\phi(v_5) \geq 8$ and $d(v_2, v_5) = 3$. Let $\phi(v_8) = 3$. If $\phi(v_6) \geq 4$, then we move one pebble $v_8$ from $v_6$ and then we move one pebble to $v_1$. Now $V(S_2)$ contains at least 10 pebbles and hence we can move one more pebble to $v_1$ through $v_{10}$ (by Theorem 5) and then one pebble can be easily moved to $v_2$. If $\phi(v_6) \leq 3$, then we can easily move one pebble to $v_2$ since $\phi(v_5) \geq 8$ and $d(v_2, v_5) = 3$.

**Case 3.2.2:** Let $\phi(v_8) = \phi(v_9) = 1$.
If $\phi(v_6) + \phi(v_7) \geq 4$, then we can move one pebble to $v_1$ using the pebbles on $v_8$ and $v_9$. Then $\phi(V(S_2)) - 4 \geq 10$ and hence we can move another one pebble to $v_1$ through $v_{10}$ and then one pebble can be easily moved to $v_2$. Suppose, $\phi(v_6) + \phi(v_7) \leq 3$, then $\phi(v_5) \geq 8$. Thus we can easily move one pebble to $v_2$ since $d(v_2, v_5) = 3$.

**Case 3.2.3:** Let $\phi(v_8) + \phi(v_9) \leq 1$.
Since $\phi(v_8)+\phi(v_9) \leq 1$, we have $\phi(V(S_2)) \geq 16$. If $\phi(v_5) \geq 2$, then we move

one pebble to $v_4$. Then the number of remaining pebbles on the vertices of $S_2$ is at least 14 and hence we can move three more pebbles to $v_4$ by Theorem 5. Thus we can easily move one pebble to $v_2$ from $v_4$. Assume $\phi(v_5) \leq 1$. In a similar way, we may assume $\phi(v_7) \leq 1$. This implies that $\phi(v_6) \geq 14$ and clearly we can move one pebble to $v_2$ by moving seven pebbles to $v_5$ if $\phi(v_5) = 1$ (or $v_7$ if $\phi(v_7) = 1$), since, $d(v_2, v_5) = 3$ and $d(v_2, v_7) = 3$. Assume $\phi(v_5) = 0$ and $\phi(v_7) = 0$. Then $\phi(v_6) = 16$ and hence one pebble can be moved to $v_2$ since $d(v_2, v_6) = 4$.
Therefore 17 pebbles are sufficient to pebble the vertices of $J_{3,3}$ and hence $f(J_{3,3}) = 17$. □

**Theorem 2.** *For Jahangir graph $J_{3,4}$, $f(J_{3,4}) = 21$.*

**Proof.** Let $\phi(v_8) = 15$, $\phi(v_5) = 3$, $\phi(v_9) = \phi(v_{12}) = 1$ and $\phi(v_i) = 0$ for all $i \neq 5, 8, 9, 12$. Then we cannot move one pebble to $v_2$. Thus $f(J_{3,4}) \geq 21$.

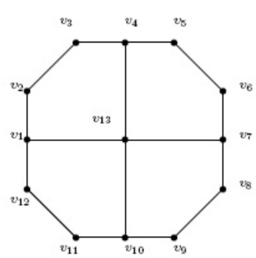


**Figure 2.** Jahangir graph $J_{3,4}$

Consider the distribution of 21 pebbles on the vertices of $J_{3,4}$.

**Case 1:** Let $v_{13}$ be the target vertex.
Without loss of generality, let $\phi(V(S_1)) \geq 5$, since $J_{3,4}$ has 21 pebbles and four 5-cycles. Hence, we can move one pebble to $v_{13}$ by Theorem 5.

**Case 2:** Let $v_1$ be the target vertex.
If $\phi(V(S_1)) \geq 5$ or $\phi(V(S_4)) \geq 5$, then we can easily move one pebble to $v_1$. Assume $\phi(v_2) + \phi(v_3) \leq 3$ and $\phi(v_{11}) + \phi(v_{12}) \leq 3$ (otherwise one pebble can be moved to $v_1$). Thus, $\phi(V(S_2)) + \phi(V(S_3)) \geq 15$. If both

$\phi(V(S_2)) \geq 5$ and $\phi(V(S_3)) \geq 5$, then we can easily move one pebble to $v_1$. Without loss of generality, let $\phi(V(S_3)) \leq 4$. So, $\phi(V(S_2)) \geq 11$ and hence we can move one pebble to $v_1$ by moving two pebbles to $v_{13}$, since $f_2(C_5) = 9$ by Theorem 5.

**Case 3:** Let $v_2$ be the target vertex.
If $\phi(V(S_1)) \geq 5$, then clearly we can move one pebble to $v_2$. So, we have $\phi(V(S_2)) + \phi(V(S_3)) + \phi(V(S_4)) \geq 20$ and hence $\phi(V(S_i)) \geq 7$ for some $i = 2, 3, 4$.

**Case 3.1:**Let $\phi(V(S_2)) \geq 7$.
Let $\phi(V(S_2)) \geq 9$. If $\phi(v_1) = 1$ or $\phi(v_3) = 1$ or $\phi(v_{11}) + \phi(v_{12}) \geq 4$, then clearly we can move one pebble to $v_2$. Assume $\phi(v_1) = 0$, $\phi(v_3) = 0$ and $\phi(v_{11}) + \phi(v_{12}) \leq 3$ such that we cannot move one pebble to $v_1$. We have $\phi(V(S_2)) + \phi(V(S_3)) \geq 18$.

**Case 3.1.1:** Let $\phi(v_{11}) = 0$ and $\phi(v_{12}) = 2$ or $\phi(v_{11}) = 0$ and $\phi(v_{12}) = 3$.
If $\phi(v_{10}) = 1$ or $\phi(v_9) \geq 2$, then we move one pebble to $v_{13}$. If $\phi(V(S_3)) - 2 \geq 5$, then we can move one more pebble to $v_{13}$ and hence we are done since $\phi(V(S_2)) \geq 9$. Assume that $\phi(V(S_3)) - 2 \leq 4$. Thus $\phi(V(S_2)) \geq 12$. If $\phi(v_8) \geq 2$, then we move one pebble to $v_7$. Now, $V(S_2)$ contains at least 13 pebbles and hence we can move three pebbles to $v_{13}$ and so we can move one pebble to $v_2$. Let $\phi(v_8) \leq 1$. Clearly, $\phi(V(S_2)) \geq 13$ and hence we can move one pebble to $v_2$ through $v_{13}$. Let $\phi(v_{10}) = 0$, $\phi(v_9) \leq 1$ and $\phi(v_8) \leq 3$. Thus, $\phi(V(S_2)) \geq 15$. If $\phi(v_{13}) = 1$ or $\phi(v_4) = 1$ or $\phi(v_5) \geq 2$ or $\phi(v_7) \geq 2$, then we can move three additonal pebbles to either $v_4$ or $v_{13}$, since $\phi(V(S_2)) - 2 \geq 13$ and hence we can easily move one pebble to $v_2$. So, we assume that $\phi(v_{13}) = 0$, $\phi(v_4) = 0$, $\phi(v_5) \leq 1$ and $\phi(v_7) \leq 1$ and thus $\phi(v_6) \geq 13$. Let $\phi(v_7) = 1$. If $\phi(V(S_3)) = 4$, then we can move one pebble to $v_7$ and then we move six pebbles to $v_7$ from $v_6$ and hence we can move one pebble to $v_2$, since $d(v_2, v_7) = 3$. If not, then we can move seven pebbles to $v_7$ from $v_6$ and hence we are done. Thus we assume $\phi(v_7) = 0$. For the same reason we assume $\phi(v_5) = 0$ and thus $\phi(v_6) \geq 15$. If $\phi(V(S_3)) \geq 3$, then we can move one pebble to $v_7$ and then we move seven pebbles to $v_7$ from $v_6$ and hence we can move one pebble to $v_2$, since $d(v_2, v_7) = 3$. If not, then we can move eight pebbles to $v_7$ from $v_6$ and hence we are done.

**Case 3.1.2:** Let $\phi(v_{11}) = 1$ and $\phi(v_{12}) = 1$.
Clearly, we can move one pebble to $v_1$ using the pebbles on $v_{11}$ and $v_{12}$

if $\phi(V(S_3)) \geq 7$ and then we move one more pebble to $v_1$ from $\phi(V(S_2))$ and hence we are done. Assume $\phi(V(S_3)) \leq 6$ and so $\phi(V(S_2)) \geq 13$. If $\phi(V(S_3)) \geq 5$, then also we can easily move one pebble to $v_2$. Assume $\phi(V(S_3)) \leq 4$ and thus $\phi(V(S_2)) \geq 15$. We do the same thing as we did above when $\phi(V(S_2)) \geq 15$ to put one pebble at $v_2$. Next, we assume $\phi(V(S_2)) = 7$ or 8. If $\phi(V(S_3)) \geq 5$ and $\phi(V(S_4)) \geq 5$, then we can easily move one pebble to $v_2$. Assume that $\phi(V(S_3)) \leq 4$ and so $\phi(V(S_4)) \geq 8$. Either we can move one more pebble to $v_{10}$ from $\phi(V(S_3))$ or we should have $\phi(V(S_4)) \geq 9$ and hence we can easily move one pebble to $v_2$. Assume that $\phi(V(S_4)) \leq 4$ and so $\phi(V(S_3)) \geq 8$. If $\phi(v_{11}) + \phi(v_{12}) \geq 4$ or $\phi(v_3) = 1$, then we can move one pebble to $v_2$ either through $v_1$ or $v_3$. So we assume that $\phi(v_{11}) + \phi(v_{12}) \leq 3$ and $\phi(v_3) = 0$ and hence we can move 'four pebbles to $v_{13}$' or 'one pebble to $v_1$ and two pebbles to $v_{13}$', like we did above. Thus we can easily move one pebble to $v_2$.

**Case 3.2:** Let $\phi(V(S_4)) \geq 7$.
Either $\phi(V(S_4)) \geq 9$ or we can add one or two pebbles (if $\phi(V(S_4)) = 8$ or 7) to $V(S_4)$ from $V(S_2)$ and $V(S_3)$. Thus we can easily move one pebble to $v_2$.

**Case 3.3:** Let $\phi(V(S_3)) \geq 7$.
Let $\phi(V(S_3)) \geq 9$. Clearly, we can move one pebble to $v_2$ if $\phi(V(S_4)) \geq 5$ or $\phi(v_1) = 1$ or $\phi(v_{11}) + \phi(v_{12}) \geq 4$. So we assume that $\phi(v_1) = 0$ and $\phi(v_{11}) + \phi(v_{12}) \leq 3$.

**Case 3.3.1:** Let $\phi(V(S_2)) \geq 7$.
Clearly, we can move one pebble to $v_2$ by Case 3.1.

**Case 3.3.2:** Let $\phi(V(S_2)) = 5$ or 6.
Clearly we can move one pebble to $v_2$ if $\phi(v_3) = 1$. So, we assume $\phi(v_3) = 0$. If $\phi(v_{11}) \geq 2$, then we move one pebble to $v_{10}$ and hence $V(S_3)$ contains 13 pebbles. Thus we can move four pebbles to $v_{13}$ from $V(S_2)$ and $V(S_3)$ and hence we can move one pebble to $v_2$. Let $\phi(v_{11}) \leq 1$. Thus, $\phi(S_3) \geq 13$ and hence we can move one pebble to $v_2$ through $v_{13}$.

**Case 3.3.3:** Let $\phi(V(S_2)) \leq 4$.
We have $\phi(V(S_3)) \geq 13$. Hence, we can move 'four pebbles to $v_{13}$' or 'one pebble to $v_1$ and two pebbles to $v_{13}$', by considering the cases $\phi(v_{11}) = 2$ or 3 or $\phi(v_{11}) = 1$ and $\phi(v_{12}) = 1$ or $\phi(v_{12}) + \phi(v_{12}) \leq 1$. Otherwise, $\phi(V(S_3)) \geq 17$ and hence we can easily move one pebble to $v_2$ by Theorem

5.

For the case $\phi(V(S_3)) = 7$ or 8, one could see that we can easily always move a pebble to $v_2$.

Thus we can always move one pebble to $v_2$ using 21 pebbles on the vertices of $J_{3,4}$. So, $f(J_{3,4}) = 21$. □

**Theorem 3.** *For Jahangir graph $J_{3,m}$ ($m \geq 5$), $f(J_{3,m}) = 3m + 10$.*

**Proof.** Consider the following distribution: $\phi(v_8) = 15$, $\phi(v_9) = \phi(v_{3m}) = \phi(v_{3m-1}) = 1$, $\phi(v_4) = 3$ and $\phi(v_i) = 3$ for all $i \in \{12+3k\}$ $(0 \leq k \leq m-5)$. Then we cannot move one pebble to $v_2$. Since this distribution contains $15 + 1 + 1 + 1 + 3 + (m - 4)3 = 3m + 9$ pebbles, $f(J_{3,m}) \geq 3m + 10$ for $m \geq 5$.

Now, we consider the distribution of $3m + 10$ pebbles on the vertices of $J_{3,m}$ where $m \geq 5$.

**Case 1:** Let $v_{3m+1}$ be the target vertex.
If any one of the 5-cycle contains five or more pebbles, then we can easily move one pebble to $v_{3m+1}$. Consider every 5-cycle contains at most four pebbles only. Since we have placed $3m + 10$ pebbles on the vertices of $J_{3,m}$, ten 5-cycles must contain exactly four pebbles. Without loss of generality, let $\phi(V(S_1)) = 4$. If one of adjacect cycle also has four pebbles or the adjacent vertex of a common vertex from the adjacent cycle contains more than two pebbles, then we can move one pebble to $v_{3m+1}$ through the common vertex $v_1$ or $v_4$. If both the adjacent vertices of a common vertex have more than one pebble each, then also we can move one pebble to $v_{3m+1}$. Otherwise, the graph $J_{3,m}$ must contain at most $3m + 1$ pebbles - which is a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$.

**Case 2:** Let $v_1$ be the target vertex.
If $\phi(V(S_1)) \geq 5$ or $\phi(V(S_m)) \geq 5$, then we can easily move one pebble to $v_1$. Also if $\phi(v_2 + v_3) \geq 4$ or $\phi(v_{3m} + v_{3m-1}) \geq 4$, then we can move one pebble to $v_1$. So, we assume $\phi(v_2 + v_3) \leq 3$ and $\phi(v_{3m-1} + v_{3m}) \leq 3$. If $\phi(V(S_i)) \geq 5$ and $\phi(V(S_j))) \geq 5$ for some $i \neq 1$, $m$, $j \neq 1$, $m$, then we can move two pebbles to $v_{3m+1}$ and hence one pebble is moved to $v_1$. Assume $\phi(V(S_i)) \geq 5$ and all other cycles contain at most four pebbles each except $S_1$ and $S_m$. Suppose we cannot move one more pebble to $v_{3m+1}$ or we cannot move one pebble to $v_1$, then the graph $J_{3,m}$ contains at most $3m + 2$ pebbles - a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$. Next, we assume that every cycle contains at most

four pebbles only. If we have two adjacent cycles with four pebbles each on them, then we can move two pebbles to $v_{3m+1}$ and hence we move one pebble to $v_1$. Thus we assume only one adjacent copy has four pebbles each on them. We move one pebble to $v_{3m+1}$ from that adjacent cycle. Suppose if we cannot move one more pebble to $v_{3m+1}$ or if we cannot move one pebble to $v_1$, then the grpah has at most $3m + 2$ pebbles - a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$. Suppose if there is no such adjacent cycles, then we can move two pebbles to $v_{3m+1}$, since we have $3m + 10$ pebbles and ten cycles have exactly four pebbles. If we cannot move one pebble to $v_1$, then the graph $J_{3,m}$ has at most $3m$ pebbles - a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$.

**Case 3:** Let $v_2$ be the target vertex.
If $\phi(V(S_1)) \geq 5$, then clearly we can move one pebble to $v_2$. Suppose four cycles have five pebbles each on them. Then we can move four pebbles to $v_{3m+1}$ pebbles and hence one pebbles is moved to $v_2$. Let three cycles only have more than four pebbles. So we can move three pebbles to $v_{3m+1}$. If we cannot move one more pebble to $v_{3m+1}$ or if we cannot move one pebble to $v_2$, then the graph $J_{3,m}$ has at most $3m + 8$ pebbles - a contradiction. Let two cycles have more than four pebbles. Suppose if we cannot move one pebble to $v_2$, then the graph has at most $3m+6$ pebbles - a contradiction. Let only one cycle has more than four pebbles. Suppose if we cannot move one pebble to $v_2$, then the graph has at most $3m + 9$ pebbles - a contradiction. Assume every cycle has at most four pebbles only. Suppose if we cannot move one pebble to $v_2$, then the graph has at most $3m + 5$ pebbles - a contradiction.

Thus we can always move one pebble to $v_2$ using $3m + 10$ pebbles on the vertices of $J_{3,m}$ . So, $f(J_{3,m}) = 3m + 10$. □

## 3. The $t$-pebbling number of Jahangir graph $J_{3,m}$

**Theorem 1.** *For Jahangir graph $J_{3,3}$, $f_t(J_{3,3}) = 16t + 1$.*

**Proof.** Let $\phi(v_6) = 16(t - 1) + 15$, $\phi(v_9) = 1$ and $\phi(v_i) = 0$ for all $i \neq 6, 9$. Then we cannot move $t$ pebbles to $v_2$. Thus $f_t(J_{3,3}) > 16t$.

Now, consider the distribution of the $16t + 1$ pebbles on the vertices of $J_{3,3}$. Clearly the result is true for $t = 1$. Assume the result is true for $2 \leq t' < t$. Clearly, the graph $J_{3,3}$ has at least 33 pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles.

Then the remaining number of pebbles on the vertices of $J_{3,3}$ is at least $16(t-1)+1$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,3}) \leq 16t+1$. $\square$

**Theorem 2.** *For Jahangir graph $J_{3,4}$, $f_t(J_{3,4}) = 16t+5$.*

**Proof.** Let $\phi(v_8) = 16(t-1)+15$, $\phi(v_5) = 3$, $\phi(v_9) = \phi(v_{12}) = 1$ and $\phi(v_i) = 0$ for all $i \neq 5, 8, 9, 12$. Then we cannot move one pebble to $v_2$. Thus $f(J_{3,4}) > 16t+4$.

Now, consider the distribution of the $16t+5$ pebbles on the vertices of $J_{3,4}$. Clearly the result is true for $t = 1$. Assume the result is true for $2 \leq t' < t$. Clearly, the graph $J_{3,4}$ has at least 37 pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles. Then the remaining number of pebbles on the vertices of $J_{3,4}$ is at least $16(t-1)+5$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,4}) \leq 16t+5$. $\square$

**Theorem 3.** *For Jahangir graph $J_{3,m}$ $(m \geq 5)$, $f_t(J_{3,m}) = 16t+3m-6$.*

**Proof.** Consider the following distribution: $\phi(v_8) = 16(t-1)+15$, $\phi(v_9) = \phi(v_{3m}) = \phi(v_{3m-1}) = 1$, $\phi(v_4) = 3$ and $\phi(v_i) = 3$ for all $i \in \{12+3k\}$ $(0 \leq k \leq m-5)$. Then we cannot move one pebble to $v_2$. Since this distribution contains $16(t-1)+3m+9$ pebbles, $f(J_{3,m}) \geq 16t+3m-6$ for $m \geq 5$.

Now, consider the distribution of the $16t+3m-6$ pebbles on the vertices of $J_{3,m}$. Clearly the result is true for $t = 1$. Assume the result is true for $2 \leq t' < t$. Clearly, the graph $J_{3,m}$ has at least $3m+24$ pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles. Then the remaining number of pebbles on the vertices of $J_{3,m}$ is at least $16(t-1)+3m-6$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,m}) \leq 16t+3m-6$. $\square$

## 4. An upper bound for the $t$-pebbling number of Jahangir graph $J_{n,m}$

Here, we present the known results about the pebbling number of Jahangir graphs from [6, 7, 8]. The pebbling number of Jahangir graph $J_{2,m}$ $(m \geq 3)$ is as follows:

**Theorem 1.** *[6] For Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$.*

**Theorem 2.** *[6] For Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$.*

**Theorem 3.** *[6] For Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.*

**Theorem 4.** *[6] For Jahangir graph $J_{2,6}$, $f(J_{2,6}) = 21$.*

**Theorem 5.** *[6] For Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.*

**Theorem 6.** *[7] For Jahangir graph $J_{2,m}$ where $m \geq 8$, $f(J_{2,m}) = 2m+10$.*

The $t$-pebbling number of Jahangir graph $J_{2,m}$ $(m \geq 3)$ is as follows:

**Theorem 7.** *[8] For Jahangir graph $J_{2,3}$, $f_t(J_{2,3}) = 8t$.*

**Theorem 8.** *[8] For Jahangir graph $J_{2,4}$, $f_t(J_{2,4}) = 16t$.*

**Theorem 9.** *[8] For Jahangir graph $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.*

**Theorem 10.** *[8] For Jahangir graph $J_{2,m}$, $f_t(J_{2,m}) = 16(t-1) + f(J_{2,m})$ where $m \geq 6$.*

From the above results and the results from this paper, we can conclude that $f_t(J_{n,m}) \geq t(2^k)$, where $k = 2^{2\lfloor \frac{n}{2} \rfloor + 2}$ is the diameter of $J_{n,m}$ for $3 \leq n < m$ (for $n = 2$, we take $m \geq 4$).

After seeing the behaviour of Jahangir graph $J_{n,m}$, we give the following conjecture for the $t$-pebbling number of $J_{n,m}$.

**Conjecture 1.** *For Jahangir graph $J_{n,m}$ $(3 \leq n < m)$,*

$$f_t(J_{n,m}) \leq \begin{cases} t(2^k) + (m-2)(2^{\lfloor \frac{n}{2} \rfloor} - 1) & \text{if } n \text{ is even} \\ t(2^k) + (m-3)(2^{\lfloor \frac{n}{2} \rfloor + 1} - 1) + n & \text{if } n \text{ is odd,} \end{cases}$$

*where $k = 2^{2\lfloor \frac{n}{2} \rfloor + 2}$ is the diameter of $J_{n,m}$.*

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