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On the stability and boundedness of certain third order non-autonomous differential equations of retarded type

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Abstract

In this paper, based on the Lyapunov-Krasovskii functional approach, we obtain sufficient conditions which guarantee stability, uniformly stability, boundedness and uniformly boundedness of solutions of certain third order non- autonomous differential equations of retarded type. Our results complement and improve some recent ones.

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1. Introduction

In this paper, we are interested in obtaining sufficient conditions for all solutions of the third order non-autonomous differential equation of retarded type

$$x'''(t) = a(t)\varphi(x''(t-\tau))x''(t)+b(t)\psi(x'(t))+h(x(t))+f(x(t),x'(t-\tau))+p(t), \tag{1.1}$$

to be bounded and uniformly bounded, and in case $p(t) \equiv 0$, sufficient conditions for the zero solution to be stable and uniformly stable. Our motivation comes partially from a recent paper of El-Nahhas [11] who studied the stability of the autonomous differential equation of retarded type

$$x'''(t) = ax''(t) + bx'(t) + cx(t) + f(x(t), x'(t-\tau)), \tag{1.2}$$

where a, b and c are negative constants; τ (> 0) is constant retardation; $f(0, 0) = 0$.

Differential equations of the type (1.1) and (1.2) have been shown to be useful in modeling many phenomena in various fields of science and engineering and in more recent years to problems in biomathematics (see, for example, Cronin-Scanlon [8] and Smith [19]). One special case of nonlinear differential equations of third order is what is known as the jerky dynamics equation

$$x'''(t) + k_1(x(t), x'(t))x''(t) + k_2(x(t), x'(t), x''(t)) = 0$$

that has gained some attention in the literature (see, Chlouverakis and Sprott [7], Eichhorn et al. [9], Elhadj and Sprott [10] and Linz [14]). Besides, qualitative properties of solutions of third order differential equations such as stability, instability, boundedness, oscillation, and periodicity of solutions have been studied by many authors; in this regard, we refer the reader to the monograph by Reissig et al. [17], and the recent papers of Adams et al. [1], Ademola and Arawomo [2], Afuwape and Adesina [3], Bai and Guo [5], Ogundare and Okecha [15], Rauch [16], Sadek [18], Tunç ([20]-[27]), Zhang and Yu [29] and the references cited therein. However, to the best of our knowledge, there exist few results on the mentioned qualitative behaviors of solutions for the non-autonomous third order differential equations of retarded type (see, the references of this paper). Motivated by the above discussions, the main purpose of this paper is to give some sufficient conditions for the stability, uniformly stability, boundedness and uniformly boundedness of solutions of equation (1.1). Our results complement and improve some recent ones.

One tool to be used here is the stability and boundedness theorems.

Let us consider non-autonomous delay differential equation

$$x' = F(t, x_t), x_t = x(t+\theta), \quad -r \le \theta \le 0, \quad t \ge 0, \tag{1.3}$$

where $F : \Re_+ \times C_H \to \Re^n$, $\Re_+ = [0, \infty)$, is a continuous mapping, $F(t, 0) = 0$, and we suppose that F takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \ \|.\|)$ is the Banach space of continuous function $\phi : [-r, \ 0] \to \Re^n$ with supremum norm; C_H is the open H -ball in C; $C_H := \{\phi \in (C[-r, 0], \ \Re^n) : \|\phi\| < H\}$. Let S be the set of $\phi \in C$ such that $\|\phi\| \ge H$ denote by $S^\bullet$ the set of all functions $\phi \in C$ such that $|\phi(0)| \ge H$, where H is large enough.

First, we will give some basic definitions.

Definition 1.1. *(Burton [6]) Let $F(t, 0) = 0$. The zero solution of equation (1.3) is stable if for each*

$\varepsilon > 0$ there is a $\delta > 0$ such that $[t \ge 0, \ \ \|\phi\| < \delta, \ \ t \ge t_0]$ implies that $|x(t, t_0, \phi)| < \varepsilon$.

Definition 1.2. *(Burton [6]) A continuous functions $W : \Re_+ \to \Re_+$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)*

Definition 1.3. *(Burton [6]) Let D be an open set in $\Re^n$ with $0 \in D$. A function $V : \Re_+ \times D \to \Re_+$ is called*
(i) positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \ge W_1(|x|)$;
(ii) decresent if there is a wedge W_2 with $V(t, x) \le W_2(|x|)$.

The following theorems are basic tools for our results.

Theorem 1.1. *(Burton [6]) Let* $V : \Re_+ \times C_H \to \Re_+$ *be continuous, (where* V *is Lyapunov functional for equation (1.3). If*
(i) If $W_1(|\phi(0)|) \leq V(t,\phi), V(t,0) = 0,$
and
$V'(t,x_t) \leq 0,$ *then the zero solution of equation (1.3) is stable;*
(ii) If $W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(\|\phi\|), V(t,0) = 0,$
and
$V'(t,x_t) \leq 0,$ *(where* W_1 *and* W_2 *are wedges), then the zero solution of equation (1.3) is uniformly stable.*

Theorem 1.2. *(Yoshizawa [28]) Suppose that there exists a continuous Lyapunov functional* $V(t,\phi)$ *defined for all* $t \in \Re_+$ *and* $\phi \in S^\bullet$*, which satisfies the following conditions;*
(i) $a(|\phi(0)|) \leq V(t,\phi) \leq b_1(|\phi(0)|) + b_2(\|\phi\|),$
where $a(r)$, $b_1(r)$, $b_2(r) \in CI$, *(*CI *denotes the families of continuous increasing functions), and are positive for* $r > H$ *and* $a(r) - b_2(r) \to \infty$ *as* $r \to \infty$*;*
(ii) $V'(t,\phi) \leq 0.$
Then, the solutions of equation (1.3) are uniformly bounded.

2. Main results

We consider the nonlinear third order differential equation with constant delay τ,

$$x'''(t) = a(t)\varphi(x''(t-\tau))x''(t)+b(t)\psi(x'(t))+h(x(t))+f(x(t),x'(t-\tau))+p(t), \tag{2.1}$$

where $\Re = (-\infty,\infty)$, $\Re_+ = [0,\infty)$, $a(t)$ and $b(t)$ are negative and continuous functions in $\Re_+ = [0,\infty)$, $\varphi : \Re \to \Re$, $\psi : \Re \to \Re$, $h : \Re \to \Re$, $f : \Re^2 \to \Re$ and $p : \Re_+ \to \Re$ are continuous with $\psi(0) = 0$, $h(0) = 0$, $f(0,0) = 0$, and τ is a positive constant.The continuity of the functions $a(t)$, $b(t)$, φ, ψ, h, f and p guarantees the existence of the solutions, and we assume that φ, ψ, h and f satisfy local Lipschitz conditions so that we have uniqueness of solutions to initial value problems as well (see, Èl'sgol'ts [12]), and the functions h, f and $b(t)$ are differentiable.

We can write equation (2.1) as the system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \end{aligned}$$

$$\begin{aligned} z'(t) &= a(t)\varphi(z(t-\tau))z(t) + b(t)\psi(y(t)) \\ &\quad + h(x(t)) + f(x(t), y(t-\tau)) + p(t). \end{aligned}$$

Let g(x(t),y(t-τ)) $= h(x(t)) + f(x(t), y(t-\tau))$.
Hence, we have

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \\ z'(t) &= a(t)\varphi(z(t-\tau))z(t) + b(t)\psi(y(t)) + g(x(t), y(t-\tau)) + p(t), \end{aligned}$$

which implies that

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \\ z'(t) &= a(t)\varphi(z(t-\tau))z(t) + b(t)\psi(y(t)) + g(x(t), y(t)) \\ &\quad - \int_{-\tau}^{0} g_y(x(t), y(t+\sigma))z(t+\sigma)d\sigma + p(t), \end{aligned} \tag{2.2}$$

where

$$g_y = \frac{\partial g}{\partial y}.$$

Assume that:
(C1) $a_0 \geq a(t) \geq 1$, $b_0 \geq b(t) \geq 1$, $b'(t) \geq 0$, where $a_0,\ b_0 \in \Re$;
(C2) $v(t+\tau)[f(x(t), 0) + h'(x(t))x(t)] \leq 0$;
(C3) $\int_0^{v(t+\tau)} [f(x(t), u)du - f(x(t), 0)]v(t+\tau) < 0$ for $v(t+\tau) \neq 0$;
(C4) $v(t+\tau)[h'(x)v(t+\tau) + \int_0^{v(t+\tau)} f_x(x(t), u)du] \geq 0$;
(C5) $|f_v(x, v)| < P < \infty$;
(C6) $a + \tau\alpha < 0$, $a,\ \tau,\ \alpha \in \Re$, $a < 0,\ \tau > 0,\ \alpha > 0$;
(C7) $4\alpha(a + \tau\alpha) + \tau\, P^2 < 0$.

Define the function $H(x, y)$ by

$$H(x, y) = -\int_0^y g(x, u)du - \frac{1}{2}by^2, b \in \Re, b < 0, (\ x,\ y) \in \Omega_0,$$

$$\Omega_0 = \{(x(t), y(t)) : (x(t), y(t+\tau)) \in \Omega, \quad t \geq 0\},$$

and Ω_0 is a domain of the two dimensional Euclidean space $\Re^2$.

Lemma 2.1. *Assume that*

(D1) $\psi(0) = 0$, $\frac{\psi(y)}{y} \leq b$ *for* $y \neq 0$, *where* $b \in \Re$, $b < 0$;

(D2) $yg(x, 0) \leq 0$ *for* x, y, *and* $\int\limits_0^y g(x, u)du - g(x, 0)y < 0$ *for* x, $y \neq 0$.

Then, the function $H(x, y) = Lx^2 + 2Mxy + Ny^2$ *is positive definite and decrescent, where*

$$L = L(x, y) = \frac{1}{x^2}\left[-\int\limits_0^y g(x, u)du + \int\limits_0^y g(x, 0)du\right],$$

$$M = M(x) = -\frac{1}{2x}g(x, 0), and N = -\frac{1}{2}b.$$

Proof. By noting the assumptions of Lemma 2.1, it follows that

$$L = \frac{1}{x^2}\left[-\int\limits_0^y g(x, u)du + \int\limits_0^y g(x, 0)du\right] > 0,$$

$$2Mxy = -yg(x, 0) \geq 0$$

and

$$Ny^2 = -\frac{1}{2}by^2 \geq 0.$$

Then, we can conclude that

$$H(x, y) \geq K(x^2 + y^2),$$

where $K = \min\{[\inf L(x, y)] \text{ for all } x, y \in \Omega_0, N\}, K > 0$. This means that $H(x, y)$ is positive definite. It is also clear that the quadratic form $H(x, y)$ can be rearranged as

$$H(x, y) = [x, y]\ T(x, y)\ \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$\text{T(x,y)}=\begin{bmatrix} \frac{1}{x^2}\left[-\int\limits_0^y g(x,u)du+\int\limits_0^y g(x,0)du\right] & -\frac{1}{2x}g(x,0) \\ -\frac{1}{2x}g(x,0) & -\frac{1}{2}b \end{bmatrix}.$$

Let $\lambda_1(x,y)$ and $\lambda_2(x,y)$ denote the characteristic roots of the matrix $T(x,y)$. Then, it is clear that

$$H(x,y)\leq K^{\frac{1}{2}}(x^2+y^2),$$

where $K=\sup[\lambda_1^2(x,y)+\lambda_2^2(x,y)]$ for all $x,y\in\Omega_0$, and $K>0$. Thus, the function $H(x,y)=Lx^2+2Mxy+Ny^2$ is decrescent. This completes the proof of Lemma 2.1. □

Theorem 2.1. *Assume that* $p(t)\equiv 0$, *conditions* $(C1)-(C7)$ *hold, and* $\psi(0)=0, \frac{\psi(y)}{y}\leq b\ for\ y\neq 0, b\in\Re, b<0;\ \varphi(z(t-\tau))\leq a\ for\ z(t-\tau), a\in\Re, a<0.$

Then, the zero solution of equation (2.1) is stable.

Proof. We define the Lyapunov-Krasovskii functional [13] $V=V(t,x_t,y_t,z_t)$ by

$$(2.3)\, V=-\int\limits_0^y g(x,u)du-b(t)\int\limits_0^y\psi(u)du+\frac{1}{2}z^2+\alpha\int\limits_{-\tau}^0\left[\int\limits_\theta^0 z^2(\sigma)d\sigma\right]d\theta,$$

where

g(x,y(t-τ)) $=f(x,y(t-\tau))+h(x)$, (x,y)$\in\Omega_0, z=\{z(t):z(t)=y'(t),\quad t\geq 0\}$, and α is a certain positive constant.

Consider the terms

$-\int\limits_0^y g(x,u)du-b(t)\int\limits_0^y\psi(u)du$, which are involved in (2.3).

It is clear that

$$-\int\limits_0^y g(x,u)du-b(t)\int\limits_0^y\psi(u)du\ =\ -\int\limits_0^y g(x,u)du-b(t)\int\limits_0^y\frac{\psi(u)}{u}udu$$

$$\geq -\int_0^y g(x,u)du - \int_0^y budu$$
$$= -\int_0^y g(x,u)du - \frac{1}{2}by^2$$

by the assumptions of Theorem 2.1.
Then, it is clear that the Lyapunov-Krasovskii functional $V = V(t, x_t, y_t, z_t)$ is positive definite, and

$$V \geq -\int_0^y g(x,u)du - \frac{1}{2}by^2 + \frac{1}{2}z^2 + \alpha \int_{-\tau}^{0} \left[\int_{\theta}^{0} z^2(\sigma)d\sigma \right] d\theta.$$

In view of the above discussion and the functional V, we can conclude that

$$K(x^2 + y^2) + \frac{1}{2}z^2 + \alpha \int_{-\tau}^{0} \left[\int_{\theta}^{0} z^2(\sigma)d\sigma \right] d\theta \leq V. \tag{2.4}$$

Differentiating the functional V with respect to t, we have

$$V' = -y \int_0^y g_x(x,u)du + a(t)\varphi(z(t-\tau))z^2 - b'(t) \int_0^y \psi(u)du - \int_{-\tau}^{0} g_y(x(t), y(t+\sigma))z(t)z(t+\sigma)d\sigma + \alpha \int_{-\tau}^{0} [z^2(t) - z^2(t+\sigma)]d\sigma.$$

By the assumptions of Theorem 2.1, we have

$$V' \leq -y \int_0^y g_x(x,u)du + \int_{-\tau}^{0} \left[\left(\frac{a}{\tau} + \alpha \right) z^2(t) - \alpha z^2(t+\sigma) \right] d\sigma - \int_{-\tau}^{0} g_y(x(t), y(t+\sigma))z(t)z(t+\sigma)d\sigma,$$

where

$g_x = \frac{\partial g}{\partial x}, g_y = \frac{\partial g}{\partial y}$.

In view of the assumption (C4), it follows that $y\int\limits_0^y g_x(x,u)du \geq 0$. Consider the terms $\int\limits_{-\tau}^{0} \left[\left(\frac{a}{\tau}+\alpha\right) z^2(t) - \alpha z^2(t+\sigma)\right] d\sigma - \int\limits_{-\tau}^{0} g_y(x(t), y(t+\sigma))z(t)z(t+\sigma)d\sigma$.

By noting assumptions (C4)-(C6), it can be seen that

$$-\alpha^2 - \frac{a\alpha}{\tau} - \frac{1}{4}g_y^2(x(t), y(t+\sigma)) = -\frac{4\tau\alpha^2 + 4a\alpha + \tau g_y^2(x(t), y(t+\sigma))}{\tau}$$

$$\geq -\frac{4\tau\alpha^2 + 4a\alpha + \tau P^2}{\tau}.$$

Therefore, if $4\tau\alpha^2 + 4a\alpha + \tau P^2 < 0$, then the quadratic form

$$\alpha z^2(t+\sigma) + g_y(x(t), y(t+\sigma))z(t+\sigma)z(t) - \left(\frac{a}{\tau}+\alpha\right) z^2(t)$$

$$= [z(t+\sigma), z(t)] \begin{bmatrix} \alpha & \frac{1}{2}g_y(x(t), y(t+\sigma)) \\ \frac{1}{2}g_y(x(t), y(t+\sigma)) & -\left(\frac{a}{\tau}+\alpha\right) \end{bmatrix} \begin{bmatrix} z(t+\sigma) \\ z(t) \end{bmatrix}$$

is positive for any $z(t+\sigma)$ and $z(t)$.
Then, we have $V' \leq 0$.

Thus, in view of the discussion made and Theorem 1.1, we can conclude that the zero solution of equation (2.1) is stable. □

Remark 2.1. *2.1 If the assumptions of Theorem 2.1 hold, then*

$$K(x^2+y^2) + \frac{1}{2}z^2 + \alpha \int\limits_{-\tau}^{0} \left[\int\limits_{\theta}^{0} z^2(\sigma)d\sigma\right] d\theta \leq V$$

$$\leq K^{\frac{1}{2}}(x^2+y^2) + \frac{1}{2}z^2 + \alpha \int\limits_{-\tau}^{0} \left[\int\limits_{\theta}^{0} z^2(\sigma)d\sigma\right] d\theta.$$

Hence, we can conclude that the zero solution of equation (2.1) is uniformly stable.

Finally, for the case $p(t) \neq 0$, we prove the following theorem.

Theorem 2.2. *In addition to conditions $(C1)-(C7)$, assume that $p \in L^1(0,\infty)$. Then, all solutions of equation (2.1) are bounded.*

Proof. For the case $p(t) \neq 0$, it is easy to see from V, which is given in (2.3), that

$$V' \leq zp(t).$$

Then, we have

$$V' \leq |z|\,|p(t)| \leq (1+z^2)\,|p(t)|\,.$$

From the discussion made for (2.3), it follows that
$\mathrm{K}(\mathrm{x}^2+y^2)+\frac{1}{2}z^2 \leq V$.

Hence, $\mathrm{V}' \leq (1+2V)\,|p(t)|\,$, and an application of Gronwall's inequality [4] bounds V.

Thus, all solutions of (2.1) are bounded. □

Remark 2.2. *If the assumptions of Theorem 2.1 hold, then*

$$K(x^2+y^2)+\frac{1}{2}z^2+\alpha\int_{-\tau}^{0}\left[\int_{\theta}^{0} z^2(\sigma)d\sigma\right]d\theta \leq V$$

$$\leq \quad K^{\frac{1}{2}}(x^2+y^2)+\frac{1}{2}z^2+\alpha\int_{-\tau}^{0}\left[\int_{\theta}^{0} z^2(\sigma)d\sigma\right]d\theta.$$

Hence, we can conclude that all solutions of equation (2.1) are uniformly bounded.

3. Conclusion

A kind of functional differential equations of third order with retarded argument has been considered. Defining an appropriate Lyapunov-Krasovskii functional [13], stability, uniformly stability, boundedness and uniformly boundedness of solutions have been investigated. Our results complement and improve some recent ones.

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