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On some I-convergent generalized difference sequence spaces associated with multiplier sequence defined by a sequence of modulli

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Abstract

In this article we introduce the sequence spaces $c^I(F, \Lambda, \Delta_m, p)$, ${c_0}^I(F, \Lambda, \Delta_m, p)$ *and* ${\ell_\infty}^I(F, \Lambda, \Delta_m, p)$, *associated with the multiplier sequence* $\Lambda = (\lambda_k)$, *defined by a sequence of modulli* $F = (f_k)$. *We study some basic topological and algebraic properties of these spaces. Also some inclusion relations are studied.*

Key words : *Ideal, I- convergence, modulus function, difference sequence.*

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1. Introduction and Preliminaries

The notion of I-convergence generalizes and unifies several notions of convergence for sequence spaces. The notion of I-convergence was studied at the initial stage by Kostyrko, Salat and Wilcznyski [23]. Later on it was studied by Tripathy et.al.[5-9, 15], B.Sarma [16], Debnath et.al.[25, 26], Khan et.al.[28]and many others. They used the notion of ideal I of subsets of the set N of natural numbers to define those concepts.

Let X be a non-empty set. Then a family of subsets $I \subset 2^X$ is said to be an ideal if I is additive, i.e, $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I$, $B \subset A \Rightarrow B \in I$. A non-empty family of subsets $F \subset 2^X$ is said to be a filter on X iff

i) $\emptyset \notin F$ ii) for all $A, B \in F \Rightarrow A \cap B \in F$ iii)$A \in F$, $A \subset B \Rightarrow B \in F$.

An ideal $I \subset 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal I is called admissible iff $I \supset \{\{x\} : x \in X\}$. A non-trivial ideal I is maximal if there does not exist any non-trivial ideal J$\neq I$, containing I as a subset. For each ideal I there is a filter $F(I)$ corresponding to I i.e $F(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

A sequence $x = (x_n)$ is said to be I-convergent to a number $L \in R$ if for each $\varepsilon > 0$, $A(\varepsilon)= \{n \in N : |x_n - L| \geq \varepsilon\} \in I$. The element L is called the I-limit of the sequence $x = (x_n)$.

The natural density of a subset A of N is denoted by $d(A)$ and is defined by

$$d(A) = \lim_{n\to\infty} \frac{1}{n} \mid \{k < n : k \in A\} \mid$$

Example: Let I=I_f=$\{A \subseteq N : A \text{ is finite}\}$. Then I_f is a nontrivial admissible ideal of N and the corresponding convergence coincides with ordinary convergence. If I=I_d=$\{A \subseteq N : d(A) = 0\}$, where $d(A)$ denotes the asymptotic density of the set A, then I_d is a non-trivial admissible ideal of N and the corresponding convergence coincide with statistical convergence.

The scope for the studies on sequence spaces was extended on introducing the notion of an associated multiplier sequence. S.Goes and G.Goes [18] defined the differentiated sequence space dE and the integrated sequence space $\int E$, for a sequence space E, by using the multiplier sequence (k^{-1})

and (k), respectively. We shall use a general multiplier sequence $\Lambda = (\lambda_k)$ for our study.

Throughout the article w, c, c_0, ℓ_∞ denote the spaces of all, convergent, null, bounded sequences respectively.

The notion of difference sequences was introduced by H.Kizmaz [19] and it was further generalized as follows:

$$Z(\Delta_m) = \{(x_k) \in w : (\Delta_m x_k) \in Z\}$$

for $Z = c, c_0, \ell_\infty$, where $\Delta_m x_k = x_k - x_{k+m}$, for all $k \in N$.

Throughout the article, $p = (p_k)$ denotes the sequence of positive real numbers. The notion of paranormed sequences was studied and investigated by Tripathy et.al.[10,12] and many others.

The notion of modulus function was introduced by Nakano [17]. It was further investigated with applications to sequences by Tripathy and Chandra [12], Khan et. al[28] and many others.

The following well-known inequality will be used throughout the article.

Let $p = (p_k)$ be any sequence of positive real numbers with $0 < p_k \leq \sup p_k = G$ and $D = max\{1, 2^{G-1}\}$. Then $(|a_k + b_k|)^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k})$ for all $k \in N$ and $a_k, b_k \in C$.

Definition 1.1: A modulus function f is a mapping from $[0, \infty)$ into $[0, \infty)$ such that
(i) $f(x) = 0$ if and only if $x = 0$
(ii) $f(x + y) \leq f(x) + f(y)$
(iii) f is increasing
(iv) f is continuous from the right at 0

Hence f is continuous everywhere in $[0, \infty)$

Let X be a sequence space. Then the sequence space $X(f)$ is defined as

$$X(f) = \{x = (x_k) : f(x_k) \in X\},$$

for a modulus function.

Definition 1.2: A sequence space E is said to be solid (or normal) if $(y_k) \in E$ whenever $(x_k) \in E$ and $|y_k| \leq |x_k|$ for all $k \in N$.

Definition 1.3: A sequence space E is said to be monotone if it contains the canonical preimages of all its step spaces.

Lemma 1.1: A sequence space E is normal implies that it is monotone.

Definition 1.4: A sequence space E is said to be symmetric if $(x_{\pi(n)}) \in E$, whenever $(x_n) \in E$, where π is a permutation of N.

Definition 1.5: A sequence space E is said to be convergence free if $(y_n) \in E$, whenever $(x_n) \in E$ and $x_n = 0$ implies $y_n = 0$.

2. Main result

Definition 2.1: Let $F = (f_k)$ be a sequence of modulli, then for a given multiplier sequence $\Lambda = (\lambda_k)$, we introduce the following sequence spaces:

$c^I(F, \Lambda, \Delta_m, p) = \{(x_k) \in w : \{n \in N : (f_k(| \lambda_k(\Delta_m x_k - L) |))^{p_k} \geq \varepsilon\} \in I$, for some $L \in R\}$ ${c_0}^I(F, \Lambda, \Delta_m, p) = \{(x_k) \in w : \{n \in N : (f_k(| \lambda_k(\Delta_m x_k) |))^{p_k} \geq \varepsilon\} \in I\}$ ${\ell_\infty}^I(F, \Lambda, \Delta_m, p)$=$\{(x_k) \in w :$ there exist $M > 0$ such that $\{n \in N : (f_k(| \lambda_k(\Delta_m x_k - L) |))^{p_k} \geq M\} \in I$ $\}$

When $f_k(x) = f(x)$, for all $k \in N$, then the above spaces are denoted by $c^I(f, \Lambda, \Delta_m, p), {c_0}^I(f, \Lambda, \Delta_m, p)$, ${\ell_\infty}^I(f, \Lambda, \Delta_m, p)$.

When $I = I_f$ and $f_k(x) = f(x)$, for all $k \in N$, then the above spaces become $c(f, \Lambda, \Delta_m, p), c_0(f, \Lambda, \Delta_m, p), \ell_\infty(f, \Lambda, \Delta_m, p)$, which was studied by Tripathy and Chandra [12].

When $\lambda_k = p_k = 1$, for all $k \in N$ and $m = 1$, then the above spaces are denoted by $c^I(F, \Delta), {c_0}^I(F, \Delta), {\ell_\infty}^I(F, \Delta)$, studied by Khan et. al[28].

When $I = I_f$, $\lambda_k = p_k = 1$, for all $k \in N$ and $f_k(x) = x$, for all $k \in N$ then the above spaces are denoted by $c(\Delta_m), c_0(\Delta_m), \ell_\infty(\Delta_m)$, studied by Tripathy et.al.

When $I = I_f$, $\lambda_k = p_k = 1$, for all $k \in N$ and $f_k(x) = x$, for all $k \in N$ and m=1, then the above spaces reduce to $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$, studied by Kizmaz [19].

Theorem 2.2: The classes of sequences $c^I(F, \Lambda, \Delta_m, p), {c_0}^I(F, \Lambda, \Delta_m, p)$, and ${\ell_\infty}^I(F, \Lambda, \Delta_m, p)$ are linear spaces.

Proof: We prove the theorem for the class of sequences ${c_0}^I(F, \Lambda, \Delta_m, p)$. The other cases can be proved similarly.

Let $(x_k), (y_k) \in {c_0}^I(F, \Lambda, \Delta_m, p)$, then

$A = \{k \in N : (f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{p_k} \geq \frac{\varepsilon}{2D([\alpha]+1)}\} \in I$
and $B = \{k \in N : (f_k(\mid \lambda_k(\Delta_m y_k) \mid))^{p_k} \geq \frac{\varepsilon}{2D([\beta]+1)}\} \in I$

Our aim is to show that $(\alpha x_k + \beta y_k) \in {c_0}^I(F, \Lambda, \Delta_m, p)$, for scalars α, β.

We have

$(f_k(\mid \lambda_k(\Delta_m(\alpha x_k + \beta y_k)) \mid))^{p_k}$
$\leq (f_k(\mid \alpha \mid\mid \lambda_k(\Delta_m x_k) \mid) + f_k(\mid \beta \mid) \mid)\lambda_k(\Delta_m x_k) \mid)^{p_k}$
$\leq D([\alpha] + 1)(f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{p_k} + D([\beta] + 1)(f_k(\mid \lambda_k(\Delta_m y_k) \mid))^{p_k}$

Now, $C = \{k \in N : (f_k(\mid \lambda_k(\Delta_m(\alpha x_k + \beta y_k)) \mid))^{p_k} \geq \varepsilon\}$
$\subseteq \{k \in N : D([\alpha] + 1)(f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{p_k} \geq \frac{\varepsilon}{2}\} \cup \{k \in N : D([\beta] + 1)(f_k(\mid \lambda_k(\Delta_m y_k) \mid))^{p_k} \geq \frac{\varepsilon}{2}\}$
$= \{k \in N : (f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{p_k} \geq \frac{\varepsilon}{2D([\alpha]+1)}\} \cup \{k \in N : (f_k(\mid \lambda_k(\Delta_m y_k) \mid))^{p_k} \geq \frac{\varepsilon}{2D([\beta]+1)}\}$
$= A \cup B$
i.e, $C \subseteq A \cup B$

But $A, B \in I$, hence $A \cup B \in I$, therefore $C \in I$.

Theorem 2.3: The classes of sequences $c^I(F, \Lambda, \Delta_m, p), {c_0}^I(F, \Lambda, \Delta_m, p)$, and ${\ell_\infty}^I(F, \Lambda, \Delta_m, p)$ are paranormed spaces paranormed by g,

$$g(x) = sup_k(f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{\frac{p_k}{M}},$$

where M=max(1,sup p_k).

Proof: Clearly $g(x) \geq 0, g(-x) = g(x), g(x+y) \leq g(x) + g(y)$.

Next we show the continuity of the product.
Let α be fixed and $g(x) \to 0$. Then it is obvious that $g(\alpha x) \to 0$.

Next let $\alpha \to 0$ and x be fixed. Since f_k are continuous, we have $f_k(\mid \alpha \mid\mid \lambda_k \Delta_m x_k \mid) \to 0$, as $\alpha \to 0$.

Thus we have

$$sup_k(f_k(\mid \lambda_k(\Delta_m x_k) \mid))^{\frac{p_k}{M}} \to 0, \text{ as } \alpha \to 0.$$

Hence $g(\alpha x) \to 0$, as $\alpha \to 0$.
Therefore g is a paranorm.

Proposition 2.1: ${c_0}^I(F, \Lambda, \Delta_m, p) \subset c^I(F, \Lambda, \Delta_m, p) \subset {\ell_\infty}^I(F, \Lambda, \Delta_m, p)$ and the inclusion is proper.

Example: Let $I = I_f$, $f_k(x_k) = x_k = (-1)^k, \lambda_k = p_k = 1, m = 1$, then $(x_k) \in {\ell_\infty}^I(F, \Lambda, \Delta_m, p)$ but $(x_k) \notin {c_0}^I(F, \Lambda, \Delta_m, p)$ or $c^I(F, \Lambda, \Delta_m, p)$.

Theorem 2.4: The spaces $c^I(F, \Lambda, \Delta_m, p), {c_0}^I(F, \Lambda, \Delta_m, p)$, and ${\ell_\infty}^I(F, \Lambda, \Delta_m, p)$ are neither solid nor monotone in general, but the spaces ${c_0}^I(F, \Lambda, p)$, and ${\ell_\infty}^I(F, \Lambda, p)$ are solid and as such are monotone.

Proof: Let (x_k) be a given sequence and (α_k) be a sequence of scalars such that $\mid \alpha_k \mid \leq 1$, for all $k \in N$.

Then we have

$$(f_k(\mid \lambda_k\alpha_k x_k \mid))^{p_k} \leq (f_k(\mid \lambda_k x_k \mid))^{p_k}, \text{ for all } k \in N.$$

The solidness of ${c_0}^I(F,\Lambda,p)$, and ${\ell_\infty}^I(F,\Lambda,p)$ follows from this inequality. The monotonicity follows by lemma 2.1

The first part of the proof follows from the following example:

Example: Let $I = I_f$, $f_k(x) = x$, for all $x \in [0,\infty]$, $m = 1, \lambda_k = 1$ for all $k \in N$, $p_k = 1$ for k odd, $p_k = 3$ for k even, $x_k = k$, for all $k \in N$ belongs to $c^I(\Delta,p)$ and ${\ell_\infty}^I(\Delta,p)$. For E, a sequence space, consider its step space E_J defined by $(y_k) \in E_J$ implies $y_k = 0$ for all k odd and $y_k = x_k$ for k even. Then (y_k) neither belongs to $(c^I(\Delta,p))_J$ nor to ${\ell_\infty}^I(\Delta,p)_J$. Hence the spaces are not monotone. Hence are not solid.

Theorem 2.5: The spaces $c^I(F,\Lambda,\Delta_m,p), {c_0}^I(F,\Lambda,\Delta_m,p)$ and ${\ell_\infty}^I(F,\Lambda,\Delta_m,p)$ are not symmetric in general.

Proof: The result follows from the following example:

Example: Let $I = I_f$, $f_k(x) = x$, for all $x \in [0,\infty]$, $m = 0, \lambda_k = k$ for all $k \in N$, $p_k = 1$ for k odd, $p_k = 4$ for k even, $x_k = k^{-2}$, for all $k \in N$. Then (x_k) belongs to $c^I(F,\Lambda,p), {c_0}^I(F,\Lambda,p)$. Consider its rearrangement (y_k) defined as follows:

$$(y_n) = (x_1, x_3, x_4, x_2, x_6, x_7, x_8, \ldots.., x_{24}, x_5, x_{26}, x_{27}, \ldots., x_{624}, x_{25}, x_{626}, \ldots).$$

Then (y_n) neither belongs to $c^I(F,\Lambda,p)$ nor to ${c_0}^I(F,\Lambda,p)$. Hence the spaces $c^I(F,\Lambda,\Delta_m,p), {c_0}^I(F,\Lambda,\Delta_m,p)$ and ${\ell_\infty}^I(F,\Lambda,\Delta_m,p)$ are not symmetric in general.

Theorem 2.6: The spaces $c^I(F,\Lambda,\Delta_m,p), {c_0}^I(F,\Lambda,\Delta_m,p)$, and ${\ell_\infty}^I(F,\Lambda,\Delta_m,p)$ are not convergence free.

Example: Let $I = I_f$, $f_k(x) = x$, for all $x \in [0, \infty]$, $m = 1, \lambda_k = 1$ for all $k \in N$, $p_k = 1$ for k odd, $p_k = 2$ for k even, consider the sequence (x_k) defined by $x_k = k^{-1}$, for all $k \in N$, then (x_k) belongs to each of $c^I(\Delta, p), {c_0}^I(\Delta, p)$, and ${\ell_\infty}^I(\Delta, p)$. Consider the sequence (y_k) defined by $y_k = k^2$, for all $k \in N$. Then (y_k) neither belongs to $c^I(\Delta, p)$ nor to ${c_0}^I(\Delta, p)$ nor to ${\ell_\infty}^I(\Delta, p)$. Hence the spaces are not convergence free.

References

[1] A. Esi and M. Et. : Some new spaces defined by Orlicz functions; Indian J. Pure and Appl. Math., 31 (8), pp. 967-972, (2000).

[2] B. C. Tripathy, A. Baruah, M. Et and M. Gungor: On almost statistical convergence of new type of generalized difference sequence of fuzzy numbers, Iranian J. Sci. and Techno., Transactions A: Science, 36 (2), pp. 147-155, (2012).

[3] B. C. Tripathy and A. J. Dutta : On I-acceleration convergence of sequences of fuzzy real numbers; Math. Modell. Analysis, 17 (4), pp. 549-557, (2012).

[4] B. C. Tripathy, B. Choudhury and B. Sarma : On some new type of generalized difference sequence spaces; Kyungpook Math. Jour., 48 (4), pp. 613-622, (2008).

[5] B. C. Tripathy and B. Hazarika : Some I-convergent sequence spaces defined by Orlicz functions; Acta Math. Appl. Sin., 27 (1), pp. 149-154 (2011).

[6] B. C.Tripathy and B.Hazarika : Paranorm I-convergent sequence spaces; Math. Slovaca, 59 (4), pp. 485-494, (2009).

[7] B. C.Tripathy and B. Hazarika : I-convergent sequence spaces associated with multiplier sequence spaces; Math. Ineq. Appl.,11 (3), pp. 543-548, (2008).

[8] B. C. Tripathy and B. Hazarika : I-monotonic and I-convergent sequences; Kyungpook Math. J., 51 (2), pp. 233-239, (2011).

[9] B. C. Tripathy and B. Sarma : On I-convergent double sequences of fuzzy real numbers, Kyungpook Math. J.,52 (2), pp. 189-200, (2012).

[10] B. C. Tripathy and H. Dutta: On some new paranormed difference sequence spaces defined by Orlicz functions; Kyungpook Math. J., 50 (1), pp. 59-69, (2010).

[11] B. C. Tripathy, M. Sen and S. Nath: I-convergent in probabilistic n-normed space; Soft Comput., 16, pp. 1021-1027, (2012).

[12] B. C.Tripathy and P.Chandra: On some generalized difference paranormed sequences spaces associated with multiplier sequence defined by modulus function; Anal. Theory and Appl, 27 (1), pp. 21-27, (2011).

[13] B. C. Tripathy and S. Debnath: On generalized difference sequence spaces of fuzzy numbers, Acta Sinet. Tech., Maringa, 35 (1), pp. 117-121, (2013).

[14] B. C. Tripathy and S. Mahanta: On a class of vector valued sequences associated with multiplier sequences; Acta Math. Appl. Sinica, 20 (3), pp. 487-494, (2004).

[15] B. C. Tripathy and S. Mahanta: On I-acceleration convergence of sequences; J. Frank. Inst., 347, pp. 591-598, (2010).

[16] B. Sarma: I-convergent sequences of fuzzy real numbers defined by Orlicz function; Mathematical Sciences, 6: 53, doi : 10.1186/2251-7456-6-53, (2012).

[17] E. Savas and P. Das: A generalized statistical convergence via ideals; Appl. Math. Lett, 24, pp. 826-830, (2011).

[18] G. Goes and S. Goes: Sequences of bounded variation and sequences of fourier coefficients; Math. Zeift., 118, pp. 93-102, (1970).

[19] H. Kizmaz : On certain sequence spaces; Canad. Math. Bull., 24 (2), pp. 169-176, (1981).

[20] H. Nakano : Concave Modulars; J.Math Soc. Japan, 5, pp. 29-49, (1953).

[21] I. J. Schoenburg : The integrability of certain functions and related summability methods; Amer. Math. Month, 66, pp. 361-375, (1951).

[22] J. Lindenstrauss and L. Tzafriri: On Orlicz sequence spaces; Israel J. Math, 101, pp. 379-390, (1971).

[23] P. Kostyrko, T.Salat and W.Wilczynski: I-convergence; Real Anal. Exchange, 26 (2), pp. 669-686, (2000/2001).

[24] R. C Buck: The measure theoretic approach to density; Amer. J. Math., 68, pp. 560-580, (1946).

[25] S. Debnath and J.Debnath: Some ideal convergent sequence spaces of fuzzy real numbers; Palestine J. Math, 3 (1), pp. 27-32, (2014).

[26] S. Debnath and J.Debnath: On I-statistically convergent sequence spaces defined by sequences of Orlicz functions using matrix transformation; Proyecciones J. Math, 33 (3), pp. 277-285, (2014).

[27] T. Salat: On statistically convergent sequences of real numbers; Math. Slovaka, 30, pp. 139-150, (1980).

[28] V. A. Khan and K. Ebadullah : I-convergent difference sequence spaces defined by a sequence of modulli; J. Math. Comput. Sci. 2 (2), pp. 265-273, (2012).

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