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Spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ over the sequence space cs

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Abstract

Fine spectra of various matrix operators on different sequence spaces have been investigated by several authors. Recently, some authors have determined the approximate point spectrum, the defect spectrum and the compression spectrum of various matrix operators on different sequence spaces. Here in this article we have determined the spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ on the sequence space cs. In a further development, we have also determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $U(r, s)$ on the sequence space cs.

Keywords and Phrases : *Spectrum of an operator ; matrix mapping; sequence space.*

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1. Introduction

By w, we denote the space of all real or complex valued sequences. Throughout the paper c, c_0, bv, cs, bs, ℓ_1, ℓ_∞ represent the spaces of all convergent, null, bounded variation, convergent series, bounded series, absolutely summable and bounded sequences respectively. Also bv_0 denotes the sequence space $bv \cap c_0$.

Okutoyi [23] determined the spectrum of the Cesàro operator C_1 on the sequence space bv_0. The fine spectra of the Cesàro operator C_1 over the sequence space bv_p, $(1 \leq p < \infty)$ was determined by Akhmedov and Başar [2]. Altay and Başar [3, 4] determined the fine spectrum of the difference operator Δ and the generalized difference operator $B(r,s)$ on the sequence spaces c_0 and c. Furkan, Bilgiç and Kayaduman [14] have determined the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv. The spectrum and fine spectrum of the Zweier Matrix on the sequence spaces ℓ_1 and bv were studied by Altay and Karakuş [5]. Altun [6, 7] determined the fine spectra of triangular Toeplitz operators and tridiagonal symmetric matrices over some sequence spaces. Fine spectra of operator $B(r,s,t)$ over the sequence spaces ℓ_1 and bv and generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p, $(1 \leq p < \infty)$ were studied by Bilgiç and Furkan [11, 12]. Akhmedov and El-Shabrawy [1] determined the fine spectrum of the operator $\Delta_{a,b}$ on the sequence space c. Panigrahi and Srivastava [24, 25] studied the spectrum and fine spectrum of the second order difference operator Δ^2_{uv} on the sequence space c_0 and generalized second order forward difference operator Δ^2_{uvw} on the sequence space ℓ_1. Fine spectrum of the generalized difference operator Δ_v on the sequence space ℓ_1 was investigated by Srivastava and Kumar [28]. Fine spectra of upper triangular double-band matrix $U(r,s)$ over the sequence spaces c_0 and c were studied by Karakaya and Altun [20]. Recently, Karaisa and Başar [19] have determined the spectrum and fine spectrum of the upper traiangular matrix $A(r,s,t)$ over the sequence space ℓ_p, $(0 < p < \infty)$. In a further development, they have also determined the approximate point spectrum, defect spectrum and compression spectrum of the operator $A(r,s,t)$ on the sequence space ℓ_p, $(0 < p < \infty)$. The approximate point spectrum, defect spectrum and compression spectrum of the operator $B(r,s)$ on the sequence spaces c_0, c,ℓ_p and bv_p, $(1 < p < \infty)$ were studied by Başar, Durna and Yildirim [9].

The notion of matrix transformations over sequence space has been studied from various aspects. Banach algebra of matrix maps have been investigated by Rath and Tripathy [26]. Besides the above listed workers, the spectrum and fine spectrum for various matrix operators has been investigated by Tripathy and Pal [29, 30], Tripathy and Saikia [31] and many others in the recent years.

In this paper, we shall determine the spectrum and fine spectrum of the upper triangular matrix $U(r,s)$ on the sequence space cs. Also,we will determine the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $U(r,s)$ on the sequence space cs. Clearly,$cs = \{x = (x_n) \in w : \lim_{n\to\infty} \sum_{i=0}^{n} x_i \quad exists\}$ is a Banach space with respect to the norm $||x||_{cs} = \sup_n |\sum_{i=0}^{n} x_i|$.

2. Preliminaries and Background

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of T, i. e.

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$,we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$, for all $f \in X^*$ and $x \in X$. Let $X \neq \{\theta\}$ be a complex normed linear space, where θ is the zero element and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator

$$T_\lambda = T - \lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1},$$

and call it the *resolvent* operator of T. A *regular value* λ of T is a complex number such that

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded,

(R3) T_λ^{-1} is defined on a set which is dense in X i. e. $\overline{R(T_\lambda)} = X$.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its complement $\sigma(T, X) = \mathbf{C} - \rho(T, X)$ in the complex plane $\mathbf{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point(discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an eigenvalue of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and satisfies $(R3)$, but not $(R2)$, that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not), but does not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X.

From Goldberg [17], if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1}:

(I) $R(T) = X$,

(II) $R(T) \neq \overline{R(T)} = X$

(III) $\overline{R(T)} \neq X$

and

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

Applying Goldberg [17] classification to T_λ , we have three possibilities for T_λ and T_λ^{-1};

(I) T_λ is surjective,

(II) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$,

(III) $\overline{R(T_\lambda)} \neq X$,

and

(I) T_λ is injective and T_λ^{-1} is continuous,

(II) T_λ is injective but T_λ^{-1} is discontinuous,

(III) T_λ is not injective.

If these possibilities are combined in all possible ways, nine different states are created which may be shown as in the Table 2.1.

These are labelled by: I_1,I_2,I_3,II_1, II_2, II_3, III_1, III_2 and III_3. If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in I_2$,then λ is in the resolvent set $\rho(T,X)$ of T. The further classification gives rise to the fine spectrum of T. If an operator is in state II_2 for example, then $R(T) \neq \overline{R(T)} = X$ and T^{-1} exists but is discontinuous and we write $\lambda \in II_2\sigma(T,X)$. The state II_1 is impossible as if T_λ is injective, then from Kreyszig [[22], Problem 6, p. 290] T_λ^{-1} is bounded and hence continuous if and only if $R(T_\lambda)$ is closed.

Again, following Appell et. al. [8], we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $||x_k|| = 1$ and $||Tx_k|| \to 0$ as $k \to \infty$.

The *approximate point spectrum* of T ,denoted by $\sigma_{ap}(T,X)$, is defined as the set

(2.1) $\sigma_{ap}(T,X) = \{\lambda \in \mathbf{C} : \textit{there exists a Weyl sequence for} \quad T - \lambda I\}$

The *defect spectrum* of T,denoted by $\sigma_\delta(T,X)$,is defined as the set

(2.2) $$\sigma_\delta(T,X) = \{\lambda \in \mathbf{C} : T - \lambda I \quad \textit{is not surjective}\}$$

The two subspectra given by equations (2.1) and (2.2) form a (not necessarily disjoint) subdivisions

(2.3) $$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X)$$

of the spectrum. There is another subspectrum

$$\sigma_{co}(T,X) = \{\lambda \in \mathbf{C} : \overline{R(T-\lambda I)} \neq X\}$$

which is often called the *compression spectrum* of T. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X) \tag{2.4}$$

Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$. Moreover, it is easy to verify that $\sigma_r(T,X) = \sigma_{co}(T,X) \setminus \sigma_p(T,X)$ and $\sigma_c(T,X) = \sigma(T,X) \setminus [\sigma_p(T,X) \cup \sigma_{co}(T,X)]$.

By the definitions given above, we can illustrate the subdivisions spectrum in the Table 2.2.

Proposition 2.1. [Appell et al. [8], Proposition 1.3, p. 28] Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:

a) $\sigma(T^*,X^*) = \sigma(T,X)$.

b) $\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$.

c) $\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$.

d) $\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$.

e) $\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$.

f) $\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$.

g) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum. The equality (g) implies, in particular, that $\sigma(T,X) = \sigma_{ap}(T,X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (Appell et al. [8]).

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbf{N}_0 = \{0,1,2,...\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, n \in \mathbf{N}_0. \tag{2.5}$$

By $(\lambda : \mu)$, we denote the class of all matrices such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right hand side of equation (2.5) converges for each $n \in \mathbf{N}_0$ and every $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbf{N}_0} \in \mu$ for all $x \in \lambda$.

The upper triangular matrix $U(r,s)$ is an infinite matrix of the form

$$U(r,s) = \begin{bmatrix} r & s & 0 & 0 & \cdots \\ 0 & r & s & 0 & \cdots \\ 0 & 0 & r & s & \cdots \\ 0 & 0 & 0 & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $s \neq 0$.

The following results will be used in order to establish the results of this article.

Lemma 2.2. **[Wilansky [32] Example 6B, Page 130]** The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(cs)$ from cs to itself if and only if:

(i) $\sup_m \sum_k | \sum_{n=1}^{m} (a_{nk} - a_{n,k-1})| < \infty$.

(ii) $\sum_n a_{nk}$ is convergent for each k.

Lemma 2.3. **[Goldberg [17], Page 59]** T has a dense range if and only if T^* is one to one.

Lemma 2.4. **[Goldberg [17], Page 60]** T has a bounded inverse if and only if T^* is onto.

3. Spectrum and fine spectrum of the operator $U(r,s)$ over the sequence space cs

The following result will be used for establishing some results of this section.

Lemma 3.1. [Akhmedov and El-Shabrawy [1], Lemma 2.1] Let (c_n) and (d_n) be two sequences of complex numbers such that $\lim_{n\to\infty} c_n = c$ and $|c| < 1$. Define the sequence (z_n) of complex numbers such that $z_{n+1} = c_{n+1}z_n + d_{n+1}$ for all $n \in \mathbf{N}_0$. Then

(i) if (d_n) is bounded, then (z_n) is bounded.

(ii) if (d_n) is convergent then (z_n) is convergent.

(iii) if $(d_n)$is a a null sequence, then (z_n) is a null sequence.

Theorem 3.2. $U(r,s) : cs \to cs$ is a bounded linear operator and

$$\| U(r,s) \|_{(cs:cs)} \leq |r| + |s|$$

Proof. From Lemma 2.2, it is easy to show that $U(r,s) : cs \to cs$ is a bounded linear operator.

Now,

$$\begin{aligned} |Ux| = |U(r,s)x| &= |\sum_{i=1}^{n}(rx_{i-1} + sx_i)| \\ &\leq |r||\sum_{i=1}^{n} x_{i-1}| + |s||\sum_{i=1}^{n} x_i| \\ &\leq (|r| + |s|) \| x \|_{cs} \end{aligned}$$

And hence,$\| U(r,s) \|_{(cs:cs)} \leq |r| + |s|$ $\square$

Theorem 3.3. The point spectrum of the operator $U(r,s)$ over cs is given by

$$\sigma_p(U(r,s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| < |s|\}.$$

Proof. Let α be an eigenvalue of the operator $U(r,s)$. Then there exists $x \neq \theta = (0,0,0,0,\ldots)$ in cs such that $U(r,s)x = \alpha x$. Then, we have

$$\begin{aligned} rx_0 + sx_1 &= \alpha x_0 \\ rx_1 + sx_2 &= \alpha x_1 \\ rx_2 + sx_3 &= \alpha x_2 \\ &\cdots \\ rx_n + sx_{n+1} &= \alpha x_n, \quad n \geq 0 \end{aligned}$$

Then, we have

$$\begin{aligned} x_1 &= \frac{\alpha - r}{s} x_0 \\ x_2 &= \frac{\alpha - r}{s} x_1 = \left(\frac{\alpha - r}{s}\right)^2 x_0 \\ x_3 &= \frac{\alpha - r}{s} x_2 = \left(\frac{\alpha - r}{s}\right)^3 x_0 \\ &\cdots \\ x_n &= \left(\frac{\alpha - r}{s}\right)^n x_0, \quad n \geq 1 \end{aligned}$$

Since, $x = (x_n) \in cs$, so $\lim_{n\to\infty} \sum_{i=0}^{n} x_i = \lim_{n\to\infty} \sum_{i=0}^{n} (\frac{\alpha-r}{s})^n x_0$ exists if and only if
$|\alpha - r| < |s|$. Hence, $\sigma_p(U(r,s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| < |s|\}$. $\square$

If $T : cs \to cs$ is a bounded linear operator represented by a matrix A, then it is known that the adjoint operator $T^* : cs^* \to cs^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space cs^* of cs is isometrically isomorphic to the Banach space bv of all bounded variation sequences normed by $\| x \|_{bv} = \sum_{n=0}^{\infty} |x_{n+1} - x_n| + \lim_{n\to\infty} |x_n|$.

Theorem 3.4. The point spectrum of the operator $U(r,s)^*$ over cs^* is given by

$$\sigma_p(U(r,s)^*, cs^* \cong bv) = \phi.$$

Proof. Let α be an eigenvalue of the operator $U(r,s)^*$. Then there exists $x \neq \theta = (0,0,0,...)$ in bv such that $U(r,s)x = \alpha x$. Then, we have

$$rx_0 = \alpha x_0$$

$$\begin{aligned} sx_0 + rx_1 &= \alpha x_1 \\ sx_1 + rx_1 &= \alpha x_2 \\ &\cdots \\ sx_n + rx_{n+1} = \alpha x_n, &\quad n \geq 1 \end{aligned}$$

If x_{n_0} is the first non-zero entry of the sequence (x_n), then $\alpha = r$. Then from the relation $sx_{n_0} + rx_{n_0+1} = \alpha x_{n_0+1}$, we have $sx_{n_0} = 0$. But $s \neq 0$ and hence, $x_{n_0} = 0$, a contradiction. Hence,$\sigma_p(U(r,s)^*, cs^* \cong bv) = \phi$. □

Theorem 3.5. For any $\alpha \in \mathbf{C}$, $U(r,s) - \alpha I$ has a dense range.

Proof. By Theorem 3.4, $\sigma_p(U(r,s)^*, cs^* \cong bv) = \phi$.

Hence, $U(r,s)^* - \alpha I$ i.e. $(U(r,s) - \alpha I)^*$ is one to one for all $\alpha \in \mathbf{C}$. So, by applying Lemma 2.3, we get the required result. □

Corollary 3.6. The residual spectrum of the operator $U(r,s)$ over cs is given by

$$\sigma_r(U(r,s), cs) = \phi.$$

Proof. Since,$U(r,s)-\alpha I$ has a dense range for all $\alpha \in \mathbf{C}$, so $\sigma_r(U(r,s), cs) = \phi$. □

Theorem 3.7. The continuous spectrum and the spectrum of the operator $U(r,s)$ over cs are respectively given by

$$\sigma_c(U(r,s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| = |s|\}$$

and

$$\sigma(U(r,s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| \leq |s|\}.$$

Proof. Let $y = (y_n) \in bv$ be such that $(U(r,s) - \alpha I)^* x = y$ for some $x = (x_n)$. Then we have following system of linear equations:

$$\begin{aligned} (r-\alpha)x_0 &= y_0 \\ sx_0 + (r-\alpha)x_1 &= y_1 \\ sx_1 + (r-\alpha)x_2 &= y_2 \end{aligned}$$

$$\begin{aligned} &\cdots \\ sx_{n-1} + (r-\alpha)x_n &= y_n + 1, \quad n \geq 1 \end{aligned}$$

Solving these equations we get,

$$\begin{aligned}
x_0 &= \frac{1}{r-\alpha} y_0 \\
x_1 &= \frac{1}{r-\alpha}(y_1 - sx_0) = \frac{1}{r-\alpha} y_1 - \frac{s}{(r-\alpha)^2} y_0 \\
x_2 &= \frac{1}{r-\alpha}(y_2 - sx_1) = \frac{1}{r-\alpha} y_2 - \frac{s}{(r-\alpha)^2} y_1 + \frac{s^2}{(r-\alpha)^3} y_0 \\
x_3 &= \frac{1}{r-\alpha}(y_3 - sx_2) = \frac{1}{r-\alpha} y_3 - \frac{s}{(r-\alpha)^2} y_2 + \frac{s^2}{(r-\alpha)^3} y_1 - \frac{s^3}{(r-\alpha)^4} y_0 \\
\cdots & \\
x_k &= \frac{1}{r-\alpha} \sum_{i=0}^{k} \left(- \frac{s}{r-\alpha} \right)^{k-i} y_i
\end{aligned}$$

Also $x_{k+1} = \frac{1}{r-\alpha} \sum_{i=0}^{k+1} \left(- \frac{s}{r-\alpha} \right)^{k+1-i} y_i$.

Now,

$$\begin{aligned}
|x_{k+1} - x_k| &= \frac{1}{|r-\alpha|} \left| \sum_{i=0}^{k+1} \left(- \frac{s}{r-\alpha} \right)^{k+1-i} y_i - \sum_{i=0}^{k} \left(- \frac{s}{r-\alpha} \right)^{k-i} y_i \right| \\
&= \frac{1}{|r-\alpha|} \left| \left\{ \left(- \frac{s}{r-\alpha} \right)^{k+1} y_0 + \left(- \frac{s}{r-\alpha} \right)^{k} y_1 + \cdots + y_{k+1} \right\} \right. \\
&\quad \left. - \left\{ \left(- \frac{s}{r-\alpha} \right)^{k} y_0 + \left(- \frac{s}{r-\alpha} \right)^{k-1} y_1 + \cdots + y_k \right\} \right| \\
&= \frac{1}{|r-\alpha|} \left| \left(- \frac{s}{r-\alpha} \right)^{k+1} y_0 + \left(- \frac{s}{r-\alpha} \right)^{k} (y_1 - y_0) + \right. \\
&\quad \left. \left(- \frac{s}{r-\alpha} \right)^{k-1} (y_2 - y_1) + \cdots + (y_{k+1} - y_k) \right| \\
&\leq \frac{1}{|r-\alpha|} \left[\left| \frac{s}{r-\alpha} \right|^{k+1} |y_0| + \left| \frac{s}{r-\alpha} \right|^{k} |y_1 - y_0| + \right.
\end{aligned}$$

$$\left|\frac{s}{r-\alpha}\right|^{k-1} |y_2 - y_1| + \cdots + |y_{k+1} - y_k|\Bigg]$$

Since $y = (y_n) \in bv$, therefore $\lim_{n\to\infty} \sum_{i=0}^{n} |y_{i+1} - y_i|$ exists and

$$\| y \|_{bv} = \sum_{n=0}^{\infty} |y_{n+1} - y_n| + \lim_{n\to\infty} |y_n|.$$

Therefore $|y_{n+1} - y_n| \leq \| y \|_{bv}$ and $|y_n| \leq \| y \|_{bv}$ for all $n \in \mathbf{N}_0$.
Hence,

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=0}^{n} |x_{k+1} - x_k| \\
&\leq \frac{1}{|r-\alpha|} \left[\begin{array}{l} \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \left|\frac{s}{r-\alpha}\right|^{k+1} |y_0| + \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \left|\frac{s}{r-\alpha}\right|^{k} |y_1 - y_0| \\ + \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \left|\frac{s}{r-\alpha}\right|^{k-1} |y_2 - y_1| + \cdots + \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} |y_{k+1} - y_k| \end{array} \right] \\
&\leq \frac{1}{|r-\alpha|} \left[\begin{array}{l} \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \left|\frac{s}{r-\alpha}\right|^{k+1} + \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \left|\frac{s}{r-\alpha}\right|^{k} + \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \left|\frac{s}{r-\alpha}\right|^{k-1} \\ + \cdots + \left|\frac{s}{r-\alpha}\right| + 1 \end{array} \right] \|y\|_{bv}
\end{aligned}$$

Let $\alpha \in \mathbf{C}$ such that $|r - \alpha| > |s|$. Then $\lim_{n\to\infty} \sum_{k=0}^{n} |x_{k+1} - x_k| < \infty$ and hence $x = (x_n)$ is in bv. Therefore the operator $(U(r,s) - \alpha I)^*$ is onto if $|r - \alpha| > |s|$. Then by Lemma 2.4, $U(r,s) - \alpha I$ has a bounded inverse if $|r - \alpha| > |s|$.

Hence,

$$\sigma_c(U(r,s), cs) \subseteq \{\alpha \in \mathbf{C} : |r - \alpha| \leq |s|\}.$$

Now

$$\sigma(U(r,s), cs) =$$

$$\sigma_p(U(r,s), cs) \cup \sigma_r(U(r,s), cs) \cup \sigma_c(U(r,s), cs) \subseteq \{\alpha \in \mathbf{C} : |r - \alpha| \leq |s|\}.$$

By Theorem 3.3, we have

$$\{\alpha \in \mathbf{C} : |r - \alpha| < |s|\} = \sigma_p(U(r,s), cs) \subset \sigma(U(r,s), cs)$$

Since $\sigma(U(r,s), cs)$ is a compact set, so it is closed and hence,

$$\overline{\{\alpha \in \mathbf{C} : |r - \alpha| < |s|\}} \subset \overline{\sigma(U(r,s), cs)} = \sigma(U(r,s), cs)$$

and therefore, $\{\alpha \in \mathbf{C} : |r-\alpha| \leq |s|\} \subset \sigma(U(r,s), cs)$.

Hence, $\sigma(U(r,s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| \leq |s|\}$.

Since $\sigma(U(r,s), cs)$ is disjoint union of $\sigma_p(U(r,s), cs)$,$\sigma_r(U(r,s), cs)$ and $\sigma_c(U(r,s), cs)$,

therefore $\sigma_c(U(r,s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| = |s|\}$.
□

Theorem 3.8. If $|\alpha - r| < |s|$, then $\alpha \in I_3\sigma(U(r,s), cs)$ (See Table 2.2).

Proof. Let $|\alpha - r| < |s|$. Then by Theorem 3.3, $\alpha \in \sigma_p(U(r,s), cs)$. So α satisfies Goldberg's condition 3.

To get the result we need to show that $U(r,s) - \alpha I$ is onto when $|\alpha - r| < |s|$.

Let $y = (y_n) \in cs$ be such that $(U(r,s) - \alpha I)x = y$ for some $x = (x_n)$.

Then $(r-\alpha)x_n + sx_{n+1} = y_n; \quad n \geq 0$.

Now,

$$\begin{aligned}
sx_{n+1} &= (\alpha - r)x_n + y_n \\
&\Rightarrow x_{n+1} = \frac{(\alpha - r)}{s}x_n + \frac{1}{s}y_n \\
&\Rightarrow \sum_{n=0}^{k} x_{n+1} = \frac{(\alpha - r)}{s}\sum_{n=0}^{k} x_n + \frac{1}{s}\sum_{n=0}^{k} y_n \\
&\Rightarrow \sum_{n=0}^{k+1} x_n = \frac{(\alpha - r)}{s}\sum_{n=0}^{k} x_n + (\frac{1}{s}\sum_{n=0}^{k} y_n + x_0)
\end{aligned}$$

Let $c_k = \frac{\alpha - r}{s}$, $z_k = \sum_{n=0}^{k} x_n$ and $d_{k+1} = \frac{1}{s}\sum_{n=0}^{k} y_n + x_0$.

Then $z_{k+1} = c_{k+1}z_k + d_{k+1}$. Now, $lim_{k\to\infty} c_k = \frac{\alpha - r}{s}$ and $|\frac{\alpha - r}{s}| < 1$.

Also, as $y = (y_n) \in cs$, so $lim_{n\to\infty}\sum_{n=0}^{k} y_n$ exists and hence, $(d_k) \in c$.

Hence, by Lemma 3.1 (ii), the sequence $(z_k) = (\sum_{n=0}^{k} x_n)$ is also convergent and so,
$x = (x_n) \in cs$. Therefore, $U(r,s) - \alpha I$ is onto. So, $\alpha \in I$ (See Table 2.2).

Hence the result.
□

Theorem 3.9. The approximate point spectrum of the operator $U(r, s)$ over cs is given
by

$$\sigma_{ap}(U(r, s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| \leq |s|\}.$$

Proof. From Table 2,

$$\sigma_{ap}(U(r, s), cs) = \sigma(U(r, s), cs) \backslash III_1\sigma(U(r, s), cs).$$

Also $\sigma_r(U(r, s), cs) = III_1\sigma(U(r, s), cs) \cup III_2\sigma(U(r, s), cs)$ (See Table 2)

By Corollary 3.6, $\sigma_r(U(r, s), cs) = \phi$ and so, $III_1\sigma(U(r, s), cs) = \phi$.

Hence, from Theorem 3.6, $\sigma_{ap}(U(r, s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| \leq |s|\}$. □

Theorem 3.10. The compression spectrum of the operator $U(r, s)$ over cs is given by

$$\sigma_{co}(U(r, s), cs) = \phi.$$

Proof. By Proposition 3.6(e), we get

$$\sigma_p(U(r, s)^*, cs^*) = \sigma_{co}(U(r, s), cs).$$

Using Theorem 3.4, we get the required result. □

Theorem 3.11. The defect spectrum of the operator $U(r, s)$ over cs is given by

$$\sigma_\delta(U(r, s), cs) = \{\alpha \in \mathbf{C} : |\alpha - r| = |s|\}.$$

Proof. From Table 2.2,we have

$$\sigma_\delta(U(r, s), cs) = \sigma(U(r, s), cs) \backslash I_3\sigma(U(r, s), cs).$$

Using Theorem 3.7 and Theorem 3.8, we get the required result. □

Corollary 3.12. The following statements hold:

(i) $\sigma_p(U(r, s)^*, cs^* \cong bv) = \{\alpha \in \mathbf{C} : |\alpha - r| = |s|\}$.

(ii) $\sigma_\delta(U(r, s)^*, cs^* \cong bv) = \{\alpha \in \mathbf{C} : |\alpha - r| \leq |s|\}$.

Proof. Using Proposition **??**(c) and (d), we get

$$\begin{aligned}\sigma_p(U(r,s)^*, cs^* \cong bv) &= \sigma_\delta(U(r,s), cs)\\ \sigma_\delta(U(r,s)^*, cs^* \cong bv) &= \sigma_{ap}(U(r,s), cs)\end{aligned}$$

Using Theorem 3.9 and Theorem 3.11, we get the required results. □

4. Conclusion

In this article we have determined the spectrum and fine spectrum of the matrix operator $U(r,s)$ over the sequence space cs. In a further development, we have introduced the concept of the approximate point spectrum, defect spectrum and compression spectrum of the operator $U(r,s)$ on the sequence space $U(r,s)$. This is a new development and the work can be applied to investigate the spectra and fine spectra of some other matrix operators too.

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	I	II	III
1	$\rho(T,X)$		$\sigma_r(T,X)$
2	$\sigma_c(T,X)$	$\sigma_c(T,X)$	$\sigma_r(T,X)$
3	$\sigma_p(T,X)$	$\sigma_p(T,X)$	$\sigma_p(T,X)$

Table 2.1: Subdivisions of spectrum of a linear operator

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is not bounded	T_λ^{-1} does not exist
I	$R(T-\lambda I)=X$	$\lambda\in\rho(T,X)$	$\cdots$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$
II	$\overline{R(T-\lambda I)}=X$	$\lambda\in\rho(T,X)$	$\lambda\in\sigma_c(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$
III	$\overline{R(T-\lambda I)}\neq X$	$\lambda\in\sigma_r(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$	$\lambda\in\sigma_r(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$

Table 2.2: Subdivisions of spectrum of a linear operator