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Orlicz-Lorentz Spaces and their Composition Operators

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Abstract

In a self-contained presentation, we discuss the Orlicz-Lorentz space. Also the boundedness of composition operators on Orlicz-Lorentz spaces are characterized in this paper.

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1. Introduction

Let f a complex-valued measurable function defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$. For $\lambda \geq 0$, define $D_f(\lambda)$ the distribution function of f as

$$D_f(\lambda) = \mu\left(\{x \in X : |f(x)| > \lambda\}\right). \tag{1.1}$$

Observe that D_f depends only on the absolute value $|f|$ of the function f and D_f may assume the value $+\infty$.

The distribution function D_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbf{R^n}$ and each of its translates have the same distribution function. It follows from (1.1) that D_f is a decreasing function of λ (not necessarily strictly) and continuous from the right.

Let (X, μ) be a measurable space and f and g be a measurable functions on (X, μ) then D_f enjoy the following properties for all $\lambda_1, \lambda_2 \geq 0$:

1. $|g| \leq |f|$ μ-a.e. implies that $D_g \leq D_f$;
2. $D_{cf}(\lambda) = D_f\left(\frac{\lambda}{|c|}\right)$ for all $c \in \mathbf{C}\{0\}$;
3. $D_{f+g}(\lambda_1 + \lambda_2) \leq D_f(\lambda_1) + D_g(\lambda_2)$;
4. $D_{fg}(\lambda_1\lambda_2) \leq D_f(\lambda_1) + D_g(\lambda_2)$.

For more details on distribution function see [5].
By f^* we mean the non-increasing rearrangement of f given as

$$f^*(t) = \inf\{\lambda > 0 : D_f(\lambda) \leq t\}, \quad t \geq 0$$

where we use the convention that $\inf \emptyset = \infty$. f^* is decreasing and right-continuous. Notice

$$f^*(0) = \inf\{\lambda > 0 : D_f(\lambda) \leq 0\} = \|f\|_\infty,$$

since

$$\|f\|_\infty = \inf\{\alpha \geq 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\}.$$

Also observe that if D_f is strictly decreasing, then

$$f^*(D_f(t)) = \inf\{\lambda > 0 : D_f(\lambda) \leq D_(f)t\} = t.$$

This fact demonstrates that f^* is the inverse function of the distribution function D_f. Let $\mathcal{F}(X, \mathcal{A})$ denote the set of all $\mathcal{A}$-measurable functions on X. Let $(X, \mathcal{A}_0, \mu)$ and $(Y, \mathcal{A}_1, \nu)$ be two measure spaces.

Two functions $f \in F(X, \mathcal{A}_0)$ and $g \in F(X, \mathcal{A}_1)$ are said to be equimeasurable if they have the same distribution function, that is, if

$$\mu\left(\{x \in X : |f(x)| > \lambda\}\right) = \nu\left(\{y \in Y : |g(y)| > \lambda\}\right), \quad \text{for all } \lambda \geq 0. \tag{1.2}$$

So then there exists only one right-continuous decreasing function f^* equimeasurable with f. Hence the decreasing rearrangement is unique.

In what follows, we gather some useful properties of the decreasing rearrangement function:

a) f^* is decreasing.

b) $f^*(t) > \lambda$ if and only if $D_f(\lambda) > t$.

c) f and f^* are equimeasurables, that is $\mathrm{D}_f(\lambda) = D_{f^*}(\lambda) \quad \text{for all } \lambda \geq 0$.

d) If $|f| \leq \liminf_{n\to\infty} |f_n|$ then $f^* \leq \liminf_{n\to\infty} f_n^*$.

e) If $E \in \mathcal{A}$, then $(\chi_E)^*(t) = \chi_{[0,\mu(E))}(t)$.

f) If $E \in \mathcal{A}$, then $(f\chi_E)^*(t) \leq f^*(t)\chi_{[0,\mu(E))}(t)$.

A weight is a nonnegative locally integrable function on $\mathbf{R^n}$ that takes values in $(0, \infty)$ almost everywhere. Therefore, weights are allowed to be zero or infinite only on a set of Lebesgue measure zero.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a convex function such that

1. $\varphi(x) = 0$ if and only if $x = 0$;
2. $\lim_{x\to\infty} \varphi(x) = \infty$.

Such as function is known as a Young function. A Young function is strictly increasing, in fact, let $0 < x < y$ then $0 < \frac{x}{y} < 1$ and hence, we might write

$$x = \left(1 - \frac{x}{y}\right) 0 + \frac{x}{y} y.$$

Since φ is convex, we have

$$\begin{aligned}\varphi(x) &= \varphi\left(\left(1 - \frac{x}{y}\right) 0 + \frac{x}{y} y\right) \\ &\leq \left(1 - \frac{x}{y}\right) \varphi(0) + \frac{x}{y}\varphi(y) \\ &< \varphi(y).\end{aligned}$$

A Young function is said to satisfy the Δ_2-condition if there exists a nonnegative constant x_0 and k such that

$$\varphi(2x) \leq k\varphi(x) \quad \text{for } x \geq x_0. \tag{1.3}$$

If $x_0 = 0$, we say that φ satisfy globally the Δ_2-condition. The smaller constant k which satisfy (1.3) is denoted by k_Δ.

Claim 1.1. *If φ is a Young function such that satisfy the Δ_2-condition, then for each $r \geq 0$ there exists a constant $k_\Delta(r)$ such that*

$$\varphi(rx) \leq k_\Delta(r)\varphi(x) \tag{1.4}$$

for $x > 0$ large enough.

Proof. [Proof of the claim.] If $r > 0$, we can choose $n \in \mathbf{N}$ such that $r \leq 2^n$. Then we can applied (1.3) n-times and use the fact that φ is increasing to obtain

$$\varphi(rx) \leq \varphi(2^n x) \leq k^n \varphi(x),$$

and hence we have (1.4). □

Example 1.2. *The function $\varphi_1(x) = \frac{x^p}{p}$ with $p > 1$ is a Young function which satisfy globally the Δ_2-condition with $k_\Delta = \frac{2^p}{p}$.*

Example 1.3. *The function $\varphi_2(t) = t^p \log(1+t)$ with $p \geq 1$ and $t \geq 0$ is a Young function which satisfy the Δ_2-condition, indeed, since*

$$\lim_{t\to\infty} \frac{\varphi_2(2t)}{\varphi_2(t)} = \lim_{t\to\infty} \frac{2^p t^p \log(1+2t)}{t^p \log(1+t)} = 2^{p-1}.$$

Also, φ_2 satisfy globally the Δ_2-condition.

In fact, since for each $t \geq 0$ we have $(1+t)^2 \geq 1+2t$, then

$$\begin{aligned}\varphi_2(2t) &= 2^p t^p \log(1+2t)\\ &\leq 2^{p+1} t^p \log(1+2t)\\ &\leq 2^{p+1} \varphi_2(2t).\end{aligned}$$

Lemma 1.4. *A Young function φ satisfy the Δ_2-condition if and only if there exist constants $\lambda > 1$ and $t_0 > 0$ such that*

$$\frac{tp(t)}{\varphi(t)} < \lambda$$

for all $t \geq t_0$, where p is the right derivate of φ.

Proof. Suppose that φ satisfy the Δ_2-condition, then there exists a constant $k > 0$ such that

$$k\varphi(t) \geq \varphi(2t) = \int_0^{2t} p(s)\, ds > \int_t^{2t} p(s)\, ds$$

for t large enough, since p is increasing, then we have

$$\int_t^{2t} p(s)\, ds > tp(t);$$

hence, for t large enough, we obtain

$$\frac{tp(t)}{\varphi(t)} \leq k.$$

Conversely, if

$$\frac{tp(t)}{\varphi(t)} < \lambda$$

for all $t \geq t_0$, then

$$\int_t^{2t} \frac{p(s)}{\varphi(s)}\, ds < \lambda \int_t^{2t} \frac{ds}{s} = \lambda \log 2.$$

Since $p(s) = \varphi'(s)$, we have

$$\log\left(\frac{\varphi(2t)}{\varphi(t)}\right) < \lambda \log 2,$$

which implies that

$$\varphi(2t) < 2^{\lambda}\varphi(t).$$

□ The following result show us that the Young functions which satisfy the Δ_2-condition have a cross rate less than the function t^p for some $p > 1$.

Theorem 1.5. *If φ is a Young function which satisfy the Δ_2-condition, then there exists constants $\lambda > 1$ and $C > 0$ such that*

$$\varphi(t) \leq Ct^{\lambda}$$

for t large enough.

Proof. By (1.4) we can write

$$\int_{t_0}^{t} \frac{p(s)}{\varphi(s)}\, ds < \lambda \int_{t_0}^{t} \frac{ds}{s}$$

where $t \geq t_0$. Then

$$\log\left(\frac{\varphi(t)}{\varphi(t_0)}\right) < \lambda \log\left(\frac{t}{t_0}\right),$$

therefore

$$\varphi(t) < \frac{\varphi(t_0)}{t_0^{\lambda}} t^{\lambda}.$$

And the proof is complete. □

Example 1.6. *The following are Young functions:*

1. $\varphi(x) = \frac{|x|^p}{p}$ *with* $p > 1$.
2. $\varphi(x) = e^{|x|} - |x| - 1$.
3. $\varphi(x) = e^{|x|^{\delta}} - 1$ *with* $\delta > 1$.

Related with the Young function φ, we define, for $t \geq 0$ the complementary function of Young function as

$$\psi(t) = \sup\{ts - \varphi(s) : s \geq 0\}.$$

Example 1.7. *If* $\varphi(t) = \frac{1}{p}t^p$ *with* $p > 1$ *and* $t \geq 0$*, then its complementary function is* $\psi(t) = \frac{1}{q}t^q$ *where* $\frac{1}{p} + \frac{1}{q} = 1$*.*

Indeed, by definition we have

$$\psi(t) = \sup\left\{ts - \frac{1}{p}s^p : s \geq 0\right\},$$

next, for $t > 0$ *fixed, we can consider the function*

$$g(s) = ts - \frac{1}{p}s^p, \quad \textit{with } s \geq 0.$$

It is not hard to check that g *achieves its maximum at* $s = t^{\frac{1}{p-1}}$ *which is given by*

$$g\left(t^{\frac{1}{p-1}}\right) = \frac{1}{q}t^q.$$

Hence

$$\psi(t) = \sup\left\{ts - \frac{1}{p}s^p : s \geq 0\right\} = \frac{1}{q}t^q.$$

Proposition 1.8. *If* φ *is a Young function, then its complementary function* ψ *is also a Young function.*

Proof. It is clear that $\psi(0) = 0$ if and only if $x = 0$. Now, we just need to show that ψ is a convex function. To this end, let us choose $t_1, t_2 \in [0, +\infty)$ and $\lambda \in [0, 1]$. Then, by definition of ψ we have

$$\psi(\lambda t_1 + (1-\lambda)t_2) = \sup\{s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) : s \geq 0\}.$$

On the other hand

$$\lambda\psi(t_1) = \lambda \sup\{st_1 - \varphi(s) : s \geq 0\} \geq \lambda(st_1 - \varphi(s)) \ \forall \ s \geq 0$$

and

$$(1-\lambda)\psi(t_2) = (1-\lambda)\sup\{st_2 - \varphi(s) : s \geq 0\} \geq (1-\lambda)(st_2 - \varphi(s)) \ \forall \ s \geq 0.$$

From the last two inequalities, we have

$$\begin{aligned} s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) &= \lambda(st_1 - \varphi(s)) + (1-\lambda)(st_2 - \varphi(s)) \\ &\leq \lambda\psi(t_1) + (1-\lambda)\psi(t_2) \end{aligned}$$

for all $s \geq 0$. Which means that $\lambda\psi(t_1) + (1-\lambda)\psi(t_2)$ is an upper bound of the set

$$\{s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) : s \geq 0\},$$

then

$$\psi(\lambda t_1 + (1-\lambda)t_2)) \leq \psi(t_1) + (1-\lambda)\psi(t_2),$$

and so ψ is convex. □

Theorem 1.9 (Young's Inequality). *Let ψ be the complementary function of φ. Then*

$$ts \leq \varphi(s) + \psi(t)$$

where $t, s \in [0, +\infty)$.

Proof. Let $t, s \in [0, +\infty)$. Then $\psi(t) = \sup\{st - \varphi(s) : s \geq 0\}$ $\geq st - \varphi(s) \quad \forall\, s \geq 0$, then

$$\psi(t) + \varphi(s) \geq st,$$

and the proof is complete. □ For more details on Young functions see [13].

2. Weighted Lorentz-Orlicz Spaces

The aim of this section is to present basic results about Lorentz-Orlicz spaces. We have tried to make the proofs as self-contained and synthetic as possible.

Definition 2.1 (Luxemburg norm). *Let φ be a Young function. For any measurable function f on X,*

$$\|f\|_{\varphi,w} = \inf\left\{\varepsilon > 0 : \int_0^\infty \varphi\left(\frac{f^*(t)}{\varepsilon}\right) w(t)\, dt \leq 1\right\} \in [0, \infty),$$

where it is understood that $\inf(\emptyset) = +\infty$.

Remark 2.2. *In this article, we will not always require that the Luxemburg norm actually be a norm. $\|\cdot\|_{\varphi,w}$ is indeed a quasinorm. A quasinorm is a functional that is like a norm except that it does only satisfy the triangle inequality with a constant $C \geq 1$, that is, $\|f+g\| \leq C(\|f\| + \|g\|)$ where $C \geq 1$.*

Lemma 2.3. *For any measurable function f on X, $\|f\|_{\varphi,w} = 0$ if and only if $f = 0$ μ-almost everywhere.*

Proof. Clearly $\|f\|_{\varphi,w} = 0$ if and only if $\int_0^\infty \varphi\left(\frac{f^*(t)}{\varepsilon}\right) w(t)\, dt \leq 1 \,\forall\, \varepsilon > 0$. It follows that

$$\|f\|_{\varphi,w} = 0 \text{ if and only if } \int_0^\infty \varphi\left(\alpha f^*(t)\right) w(t)\, dt = 0 \;\forall\; \alpha > 0$$

$$\text{if and only if } \varphi\left(\alpha f^*(t)\right) w(t) = 0 \;\mu-\text{a.e.} \;\forall\; \alpha > 0$$

$$\text{if and only if } f^*(t) = 0 \;\mu-\text{a.e.}$$

$$\text{if and only if } D_f(\lambda) = 0 \;\mu-\text{a.e.}$$

$$\text{if and only if } f = 0 \;\mu-\text{a.e.}$$

□

Identification of almost everywhere equal functions. As with L_p spaces, one identifies the function which are μ-almost everywhere equal. This means that one works with the equivalence classes of the equivalence relation defined by the μ-almost everywhere equality. From now on, this will be done without further mention. Consequently, one write:

$$\|f\|_{\varphi,w} = 0 \text{ if and only if } f = 0. \tag{2.1}$$

Lemma 2.4. *If* $0 < \|f\|_{\varphi,w} < \infty$ *then* $\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\, dt \leq 1$. *In particular,* $\|f\|_{\varphi,w} \leq 1$ *is equivalent to* $\int_0^\infty \varphi\left(f^*(t)\right) w(t)\, dt \leq 1$.

Proof. For all $b > \|f\|_{\varphi,w}$, we have

$$\int_0^\infty \varphi\left(\frac{f^*(t)}{b}\right) w(t)\, dt \leq 1.$$

Letting b decrease to $\|f\|_{\varphi,w}$, one obtains the first result by monotone convergence. The second statement follows from this and lemma 2.8. □

Proposition 2.5. *The gauge* $\|\cdot\|_{\varphi,w}$ *is a quasinorm on the vector space of all the measurable functions* f *such that* $\|f\|_{\varphi,w} < \infty$.

Proof. It is already seen that (2.1) holds under identification of a.e. equal functions.

It is clear that for all real λ, $\|\lambda f\|_{\varphi,w} = |\lambda|\|f\|_{\varphi,w}$.

It remains to prove the triangle inequality. Let f and g be two measurable functions such that $0 < \|f\|_{\varphi,w} + \|g\|_{\varphi,w} < \infty$. Then

$$\begin{aligned}
&\int_0^\infty \varphi\left(\frac{(f+g)^*(t)}{2(\|f\|_{\varphi,w}+\|g\|_{\varphi,w})}\right) w(t)\,dt \\
&\leq \int_0^\infty \varphi\left(\frac{f^*(t/2)+g^*(t/2)}{2(\|f\|_{\varphi,w}+\|g\|_{\varphi,w})}\right) w(t)\,dt \\
&= \int_0^\infty \varphi\left(\frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w}+\|g\|_{\varphi,w})}\frac{f^*(t/2)}{\|f\|_{\varphi,w}} + \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w}+\|g\|_{\varphi,w})}\frac{g^*(t/2)}{\|g\|_{\varphi,w}}\right) w(t)\,dt \\
&\leq \frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w}+\|g\|_{\varphi,w})}\int_0^\infty \varphi\left(\frac{f^*(t/2)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&\quad + \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w}+\|g\|_{\varphi,w})}\int_0^\infty \varphi\left(\frac{g^*(t/2)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&= \frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w}+\|g\|_{\varphi,w})} 2\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(2t)\,dt \\
&\quad + \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w}+\|g\|_{\varphi,w})} 2\int_0^\infty \varphi\left(\frac{g^*(t)}{\|f\|_{\varphi,w}}\right) w(2t)\,dt \\
&\leq \frac{\|f\|_{\varphi,w}}{\|f\|_{\varphi,w}+\|g\|_{\varphi,w}}\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&\quad + \frac{\|g\|_{\varphi,w}}{\|f\|_{\varphi,w}+\|g\|_{\varphi,w}}\int_0^\infty \varphi\left(\frac{g^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&\leq 1,
\end{aligned}$$

where the last but one inequality follows from the convexity of φ and the fact that w is nonincreasing and the last inequality from lemma 2.4. Therefore

$$\|f+g\|_{\varphi,w} \leq 2\left(\|f\|_{\varphi,w} + \|g\|_{\varphi,w}\right).$$

As a consequence, the set of all measurable functions f such that $\|f\|_{\varphi,w} < \infty$ is a vector space. □

Definition 2.6. *Let φ be a Young function. We define the weighted Lorenz-Orlicz spaces*

$$L_{\varphi,w} = \left\{ f : X \to \mathbf{C} \text{ measurable} : \int_0^\infty \varphi(\alpha f^*(t)) w(t)\, dt < \infty, \text{ for some } \alpha > 0 \right\}.$$

It follows from proposition 1.8 that if $L_{\varphi,w}$ is a weighted Lorentz-Orlicz space, then $L_{\psi,w}$ is also a weighted Lorenz-Orlicz space.

Proposition 2.7 (Hölder's type inequality). *For $f \in L_{\varphi,1}$ and $g \in L_{\psi,1}$*

$$\int_X |fg|\, d\mu \leq 2\|f\|_{\varphi,1}\|g\|_{\psi,1}.$$

In particular, $fg \in L_1$.

Proof. If $\|f\|_{\varphi,1} = 0$ or $\|g\|_{\psi,1} = 0$, one concludes with lemma 2.3.

Assume now that $0 < \|f\|_{\varphi,1}, \|g\|_{\psi,1}$. Because of Young's inequality: $st \leq \varphi(s) + \varphi(t)$ we have

$$\int_X \frac{|fg|}{\|f\|_{\varphi,1}\|g\|_{\psi,1}}\, d\mu \leq \int_0^\infty \frac{f^*(t)g^*(t)}{\|f\|_{\varphi,1}\|g\|_{\psi,1}}\, dt$$

$$\leq \int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,1}}\right) dt + \int_0^\infty \psi\left(\frac{g^*(t)}{\|g\|_{\psi,1}}\right) dt$$

$$\leq 2.$$

Therefore

$$\int_X |fg|\, d\mu \leq 2\|f\|_{\varphi,1}\|g\|_{\psi,1}.$$

□

Lemma 2.8. *Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence in $L_{\varphi,w}$. Then, the following assertions are equivalent:*

a) $\lim_{n\to\infty} \|f_n\|_{\varphi,w} = 0$;

b) For all $\alpha > 0$, $\limsup_{n\to\infty} \int_0^\infty \varphi(\alpha f_n^*(t)) w(t)\, dt \leq 1$;

c) For all $\alpha > 0$, $\lim_{n\to\infty} \int_0^\infty \varphi(\alpha f_n^*(t)) w(t)\, dt = 0$.

Proof. The equivalence $(a) \Leftrightarrow (b)$ is a direct consequence of the definition of $\|\cdot\|_{\varphi,w}$. Off course $(c) \Rightarrow (b)$ is obvious. As φ is convex and $\varphi(0) = 0$ for all $t \geq 0$ and $0 < \varepsilon \leq 1$, we have

$$\varphi(t) = \varphi\left((1-\varepsilon)0 + \varepsilon\frac{t}{\varepsilon}\right) \leq (1-\varepsilon)\varphi(0) + \varepsilon\varphi\left(\frac{t}{\varepsilon}\right),$$

that is

$$\varphi(t) \leq \varepsilon\varphi\left(\frac{t}{\varepsilon}\right) \quad t \geq 0, 0 < \varepsilon \leq 1.$$

From which $(b) \Rightarrow (c)$ follows easily. □

Theorem 2.9. *The space $L_{\varphi,w}$ is a quasi-Banach space.*

Proof. Let $\{f_n\}_{n\in\mathbf{N}}$ be a Cauchy sequence in $L_{\varphi,w}$. Let us choose $\tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon}\varphi^{-1}\left(\frac{\varepsilon}{k_0}\right) < \frac{1}{n+m}$ for $n, m \in \mathbf{N}$ and $\varepsilon > 0, k_0 > 0$. For such $\tilde{\varepsilon}$ there exists $n_0 \in \mathbf{N}$ such that

$$\|f_n - f_m\|_{\varphi,w} < \tilde{\varepsilon}.$$

If $n, m \geq n_0$. By the definition of the Luxemburg quasi-norm we can use $k_0 > 0$ in such a way that $k_0 < \tilde{\varepsilon}$ and

$$\int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt \leq 1.$$

Let $E = \{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}$, then

$$\varepsilon\chi_E(x) \leq |f_n(x) - f_m(x)|.$$

Hence

$$\varepsilon\chi_E^*(t) \leq (f_n - f_m)^*(t),$$

$$\varepsilon\chi_{(0,\mu(E))}(t) \leq (f_n - f_m)^*(t).$$

Therefore

$$\int_0^\infty \varphi\left(\frac{\varepsilon}{k_0}\chi_{(0,\mu(E))}(t)\right) w(t)\,dt \leq \int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt.$$

Then

$$\int_0^{\mu(E)} \varphi\left(\frac{\varepsilon}{k_0}\right) w(t)\,dt \leq \int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt$$

$$\Rightarrow \tilde{\varepsilon} \int_0^{D_{f_n - f_m(\varepsilon)}} w(t)\, dt \le \tilde{\varepsilon} \varphi^{-1}\left(\frac{\varepsilon}{k_0}\right) \int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\, dt$$

$$\Rightarrow \tilde{\varepsilon} \int_0^{D_{f_n - f_m(\varepsilon)}} w(t)\, dt \le \frac{1}{n+m}$$

$$\Rightarrow \tilde{\varepsilon} \lim_{n,m\to\infty} \int_0^{D_{f_n - f_m(\varepsilon)}} w(t) = 0.$$

Since $w > 0$, we must have $\lim_{n,m\to\infty} D_{f_n - f_m}(\varepsilon) = 0$ which means that $\{f_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in measure. Then some subsequence $\{f_{n_k}\}_{k\in\mathbf{N}}$ converges almost everywhere to a measurable function f, that is, $f_{n_k} \to f$ μ-a.e.

Let $\alpha > 0$. By lemma 2.8 there exists a large enough integer $n(\alpha)$ such that

$$\int_0^\infty \varphi\left(\alpha(f_n - f_m)^*(t)\right) w(t)\, dt \le 1, \quad \forall\ m, n \ge n(\alpha).$$

With Fatou's lemma this gives

$$\int_0^\infty \varphi\left(\alpha(f_n - f)^*(t)\right) w(t)\, dt \le \liminf \int_0^\infty \varphi\left(\alpha(f_n - f_m)^*(t)\right) w(t)\, dt \le 1$$

$\forall\ m \ge n(\alpha)$. Therefore $f_n - f$ belongs to $L_{\varphi,w}$, but $f_n \in L_{\varphi,w}$, so that $f \in L_{\varphi,w}$.

Moreover, as $\limsup_{m\to\infty} \int_0^\infty \varphi\left(\alpha(f_m - f)^*(t)\right) w(t)\, dt \le 1$ for all $\alpha > 0$, we have $\lim_{m\to\infty} \|f_m - f\|_{\varphi,w} = 0$. This proves that $L_{\varphi,w}$ is complete. □

Theorem 2.10. *Simple functions are dense in* $L_{\varphi,w}$.

Proof. Suppose $f \in L_{\varphi,w}$. We may assume that $f \ge 0$. Note that if $D_f(\lambda) = \infty$, then $\lim_{t\to\infty} f^*(t) = 0$. It follows that $D_f(\lambda) < \infty$.

Hence, given $\varepsilon, \delta > 0$, we can find a simple function $s_n \ge 0$ such that $s_n(x) = 0$ when $f(x) \le \varepsilon$ and $f(x) - \varepsilon \le s_n(x) \le f(x)$ when $f(x) > \varepsilon$ except on a set of measure less than δ. It follows that

$$\mu\left(\{x \in X : |f(x) - s_n(x)| > \varepsilon\}\right) < \delta.$$

Next, choose $n \in \mathbf{N}$ such that $n \ge \frac{1}{\varepsilon}$, then

$$(f - s_n)^*(t) = \inf\{\varepsilon > 0 : D_{f-s_n}(\varepsilon) < \delta \le t\}.$$

Thus

$$(f - s_n)^*(t) \le \frac{1}{n} \quad \text{for } t \ge \delta,$$

since $s_n \leq f$, then $s_n^*(t) \leq f^*(t)$, for each $t > 0$. Since $n > \frac{1}{\varepsilon}$, we have

$$(f - s_n)^*(t) \leq \frac{1}{n} < \varepsilon,$$

next,

$$\int_0^\infty \varphi\left(\frac{(f-s_n)^*(t)}{k}\right) w(t)\,dt \leq \int_0^\infty \varphi\left(\frac{1}{nk}\right) w(t)\,dt.$$

Let $a = \int_0^\infty w(t)\,dt$, then

$$\begin{aligned}\|f - s_n\|_{\varphi,w} &= \inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(f-s_n)^*(t)}{k}\right) w(t)\,dt \leq 1\right\} \\ &= \frac{1}{n\varphi^{-1}\left(\frac{1}{a}\right)} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

□

3. Composition Operator

Let $(X, \mathcal{A}, \mu)$ be a σ-finite complete measure space and let $T : X \to X$ be a measurable transformation, that is, $T^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$.

If $\mu\left(T^{-1}(A)\right) = 0$ for all $A \in \mathcal{A}$ with $\mu(A) = 0$, then T is said to be nonsingular. This condition means that the measure $\mu \circ T^{-1}$, defined by $\mu \circ T^{-1}(A) = \mu\left(T^{-1}(A)\right)$ for $A \in \mathcal{A}$ is absolutely continuous with respect to μ (it is usually denoted $\mu \circ T^{-1} \ll \mu$). Then the Radon-Nikodym theorem ensure the existence of a non-negative locally integrable function f_T on X such that

$$\mu \circ T^{-1}(A) = \int_A f_T\,d\mu \quad \text{for } A \in \mathcal{A}.$$

Any measurable nonsingular transformation T induces a linear operator (composition operator) C_T from $F(X, \mathcal{A}, \mu)$ into itself defined by

$$C_T(f)(x) = f\left(T(x)\right), x \in X, f \in F(X, \mathcal{A}, \mu),$$

where $F(X, \mathcal{A}, \mu)$ denotes the linear space of all equivalence classes of $\mathcal{A}$-measurable functions on X, where we identify any two functions that are equal μ-almost everywhere on X.

Here the nonsingularty of T guarantees that the operator C_T is well defined as a mapping of equivalence classes of functions into itself since $f = g$ μ-a.e. implies $C_T(f) = C_T(g)$ μ-a.e.

Example 3.1. *Let $([0,1],\mathcal{B},m)$ be a Lebesgue measure space, $\mathcal{B}$ stand for the Borel's σ-algebra and $T:[0,1]\to[0,1]$ a transformation defined by*

$$T(x)=\begin{cases} 2x, & \text{if } 0\le x\le \frac{1}{2} \\ 1, & \text{if } \frac{1}{2}<x\le 1. \end{cases}$$

It is not hard to see that T is $\mathcal{B}$-measurable, also, observe that T is not nonsigular, indeed

$$T^{-1}(\{1\})=\left(\frac{1}{2},1\right],$$

hence $m(T^{-1}(\{1\}))=\frac{1}{2}$ but $m(\{1\})=0$.

Now, let us consider $f=\chi_{[0,1)}$ and $g=\chi_{[0,1]}$ note $f=g$ μ-a.e., but

$$\begin{aligned} C_T(f) &= C_T\left(\chi_{[0,1)}\right) \\ &= \chi_{[0,1)}\circ T \\ &= \chi_{[0,\frac{1}{2})} \end{aligned}$$

and

$$\begin{aligned} C_T(g) &= C_T\left(\chi_{[0,1]}\right) \\ &= \chi_{[0,1]}\circ T \\ &= \chi_{[0,1]}. \end{aligned}$$

Then $C_T(f)\neq C_T(g)$, which means that C_T is not well defined.

In other words, the nonsingularity of T is a necessary condition in order to T induces a composition operator on $F(X,\mathcal{A},\mu)$.

Composition operators are relatively simple operators with a wide range of applications in areas such a partial differential equations, group representation theory, ergodic theory or dynamical systems, etc. For details on composition operator see [7, 10, 11, 12, 14, 15] and the references given therein.

In what follows, we will consider the transformation C_T from $L_{\varphi,w}$ into the space of all complex-valued measurable functions on X as

$$(C_T f)(x) == \begin{cases} f(T(x)), \text{if } x\in Y \\ 0, \text{otherwise} \end{cases}$$

where Y is a measurable subset of X.

Next, a necessary and sufficient condition for the boundedness of composition mapping C_T is given.

If $(X,\mathcal{A},\mu)$ is a σ-finite measure space and $T:X\to X$ is a non-singular measurable transformation and w is a weight function, define a measure ν on the σ-algebra $\mathcal{A}$ as

$$\nu(A)=\int_0^{\mu(A)} w(t)\,dt.$$

Next, for $A\in\mathcal{A}$,
$\|\chi_A\|_{\varphi,w}=\inf\left\{k>0:\int_0^\infty \varphi\left(\frac{\chi_A^*(t)}{k}\right)w(t)\,dt\le 1\right\}$

$$=\inf\left\{k>0:\int_0^\infty \varphi\left(\frac{\chi_{(0,\mu(A))}(t)}{k}\right)w(t)\,dt\le 1\right\}$$
$$=\inf\left\{k>0:\int_0^{\mu(A)} \varphi\left(\frac{1}{k}\right)w(t)\,dt\le 1\right\}.$$

Now, observe that if $k=\frac{1}{\varphi^{-1}\left(\frac{1}{\nu(A)}\right)}$, then

$$\varphi\left(\frac{1}{\frac{1}{\varphi^{-1}(\nu(A))}}\right)=\varphi\left(\varphi^{-1}\left(\frac{1}{\nu(A)}\right)\right)=\frac{1}{\nu(A)},$$

thus

$$\int_0^{\mu(A)}\varphi\left(\frac{1}{\frac{1}{\varphi^{-1}(\nu(A))}}\right)w(t)\,dt=\int_0^{\mu(A)}\varphi\left(\varphi^{-1}\left(\frac{1}{\nu(A)}\right)\right)w(t)\,dt$$
$$=\int_0^{\mu(A)}\frac{w(t)}{\nu(A)}\,dt$$
$$=\frac{1}{\nu(A)}\int_0^{\mu(A)} w(t)\,dt$$
$$=\frac{1}{\nu(A)}\cdot\nu(A)$$
$$=1.$$

Therefore

$$\|\chi_A\|_{\varphi,w} = \frac{1}{\varphi^{-1}\left(\frac{1}{\nu(A)}\right)}.$$

Theorem 3.2. *Let $T : X \to X$ be a non-singular measurable transformation. Then C_T induced by T is bounded on $L_{\varphi,w}$ if and only if there exists $M \geq 1$ such that*

$$\nu\left(T^{-1}(A)\right) \leq M\nu(A) \quad \forall\ A \in \mathcal{A}. \tag{3.1}$$

Moreover

$$\|C_T(f)\| = \sup_{0<\nu(A)<\infty} \left(\frac{\nu(T^{-1}(A))}{\nu(A)}\right). \tag{3.2}$$

Proof. Let C_T be a bounded transformation on $L_{\varphi,w}$, then we can find $M \geq 1$ such that

$$\|C_T f\|_{\varphi,w} \leq M\|f\|_{\varphi,w} \quad \forall\ f \in L_{\varphi,w}.$$

If $A \in \mathcal{A}$ is such that $\nu(A) = \infty$, then (3.1) holds. Suppose $A \in \mathcal{A}$ is such that $\nu(A) < \infty$, thus

$$\int_0^\infty \varphi\left(\alpha\chi_A^*(t)\right) w(t)\, dt = \int_0^\infty \varphi\left(\alpha\chi_{0,\mu(A)}(t)\right) w(t)\, dt$$

$$= \int_0^{\mu(A)} \varphi(\alpha) w(t)\, dt$$

$$= \varphi(\alpha)\nu(A) < \infty.$$

Hence

$$\|C_T\chi_A\|_{\varphi,w} \leq M\|\chi_A\|_{\varphi,w}. \tag{3.3}$$

Note

$$(\chi_A \circ T)(x) = \chi_A(T(x)) = \begin{cases} 1, & \text{if } \mathrm{T(x)} \in A \\ 0, & \text{if } \mathrm{T(x)} \notin A \end{cases}$$

$$= \begin{cases} 1, & \text{if } \mathrm{x} \in T^{-1}(A) \\ 0, & \text{if } \mathrm{x} \notin T^{-1}(A) \end{cases}$$

$$= \chi_{T^{-1}(A)}(x).$$

Then

$$
\begin{aligned}
\|C_T\chi_A\|_{\varphi,w} &= \|\chi_{T^{-1}(A)}\|_{\varphi,w}\\
&= \frac{1}{\varphi^{-1}\left(\frac{1}{\nu(T^{-1}(A))}\right)},
\end{aligned}
$$

and

$$
\|\chi_A\|_{\varphi,w} = \frac{1}{\varphi^{-1}\left(\frac{1}{\nu(A)}\right)}.
$$

Hence, we can write (3.3) as follows

$$
\frac{1}{\varphi^{-1}\left(\frac{1}{\nu(T^{-1}(A))}\right)} \leq \frac{M}{\varphi^{-1}\left(\frac{1}{\nu(A)}\right)}
$$

and so

$$
\varphi^{-1}\left(\frac{1}{\nu(A)}\right) \leq \varphi^{-1}\left(\frac{1}{\nu(T^{-1}(A))}\right).
$$

Since φ^{-1} is concave and $0 = \varphi^{-1}(\varphi(0)) = \varphi^{-1}(0)$ thus φ^{-1} is increasing, then

$$
\begin{aligned}
\frac{1}{\nu(A)} &\leq M\frac{1}{\nu\left(T^{-1}(A)\right)}\\
\nu(T^{-1}(A)) &\leq M\nu(A).
\end{aligned}
$$

Conversely, if inequality (3.1) holds for all $A \in \mathcal{A}$, then
Therefore

$$
(f\circ T)^*(t) \leq Mf^*(t) \quad \text{a.e.}
$$

Since $\varphi(\alpha t) \leq \alpha\varphi(t)$ for $\alpha < 1$, then

$$
\begin{aligned}
\int_0^\infty \varphi\left(\frac{(f\circ T)^*(t)}{M\|f\|_{\varphi,w}}\right) w(t)\,dt &\leq \frac{1}{M}\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt\\
&\leq \int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \leq 1.
\end{aligned}
$$

Finally

$$
\|f\circ T\|_{\varphi,w} \leq M\|f\|_{\varphi,w},
$$

that is

$$\|C_T f\|_{\varphi,w} \leq M \|f\|_{\varphi,w}.$$

On the one hand, let us prove (3.2). Indeed, let

$$N = \sup_{0<\nu(A)<\infty} \left(\frac{\nu\left(T^{-1}(A)\right)}{\nu(A)} \right),$$

then

$$\nu\left(T^{-1}(A)\right) \leq N\nu(A)$$

and thus

$$\|C_T f\|_{\varphi,w} \leq N \|f\|_{\varphi,w}, \quad \forall\, f \in L_{\varphi,w}$$

hence

$$\frac{\|C_T f\|_{\varphi,w}}{\|f\|_{\varphi,w}} \leq N, \quad \text{for all } 0 \neq f \in L_{\varphi,w}.$$

Therefore

$$\|C_T\| = \sup_{f \neq 0} \frac{\|C_T(f)\|_{\varphi,w}}{\|f\|_{\varphi,w}}$$

$$< N = \sup_{0<\nu(A)<\infty} \left(\frac{\nu\left(T^{-1}(A)\right)}{\nu(A)} \right).$$

That is

(3.4) $$\|C_T\| \leq \sup_{0<\nu(A)<\infty} \left(\frac{\nu\left(T^{-1}(A)\right)}{\nu(A)} \right).$$

On the other hand, let us consider

$$M = \|C_T\| = \sup_{f \neq 0} \frac{\|C_T(f)\|_{\varphi,w}}{\|f\|_{\varphi,w}},$$

then

$$\frac{\|C_T(f)\|_{\varphi,w}}{\|f\|_{\varphi,w}} \leq M \quad \forall\, 0 \neq f \in L_{\varphi,w}.$$

In particular, if $f = \chi_A$ such that $0 < \mu(A) < \infty$, $A \in \mathcal{A}$, then

$$\frac{\|C_T(\chi_A)\|_{\varphi,w}}{\|\chi_A\|_{\varphi,w}} = \left(\frac{\nu\left(T^{-1}(A)\right)}{\nu(A)} \right) \leq M,$$

therefore

(3.5) $$\sup_{0<\nu(A)<\infty} \left(\frac{\nu\left(T^{-1}(A)\right)}{\nu(A)} \right) \leq M = \|C_T\|.$$

Combining 3.4 and 3.5 we have

$$\|C_T\| = \sup_{0<\nu(A)<\infty} \left(\frac{\nu\left(T^{-1}(A)\right)}{\nu(A)} \right).$$

□

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