

Square Sum Labeling of Class of Planar Graphs

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Received : November 2013. Accepted : February 2015

Abstract

A (p, q) graph G is said to be square sum, if there exists a bijection $f : V(G) \rightarrow \{0, 1, 2, \dots, p - 1\}$ such that the induced function $f^* : E(G) \rightarrow N$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. In this paper we proved that the planar graphs $Pl_{m,n}, TBL(n, \alpha, k, \beta)$ and higher order level joined planar grid admits square sum labeling. Also the square sum properties of several classes of graphs with many odd cycles are studied.

Key Words : *Square sum graphs, $Pl_{m,n}, TBL(n, \alpha, k, \beta)$.*

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¹First author is indebted to the University Grants Commission(UGC) for granting her Teacher Fellowship under UGC's Faculty Development Programme during XI plan. footnote

1. Introduction

Unless mentioned otherwise, by a *graph* we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [7]. Labeling of a graph G is an assignment of integers either to the vertices or edges or both subject to certain conditions. A dynamic survey to know about the numerous graph labeling method is regularly updated by J.A Gallian [5]. Acharya and Germina defined a square sum labeling of a (p, q) -graph G [1, 2] as follows.

Definition 1.1. *A (p, q) -graph G is said to be square sum, if there exists a bijection $f : V(G) \rightarrow \{0, 1, 2, \dots, p - 1\}$ such that induced function $f^* : E(G) \rightarrow N$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective.*

A connected graph with one cycle is called unicyclic. We proved that all unicyclic graphs are square sum [6]. In this paper we proved that the planar graphs $Pl_{m,n}, TBL(n, \alpha, k, \beta)$ and higher order level joined planar grid admits square sum labeling. Also the square sum properties of several classes of graphs with many odd cycles are studied.

Theorem 1.2. [6] *Complete graph K_n is square sum If and only if $n \leq 5$.*

A graph is planar if it has an embedding on plane. $K_n, n \leq 4$ are planar. But K_n for $n \geq 5$ is non-planar. The complete bipartite graph $K_{m,n}$ is square sum if $m < 4$ and for all n [2]. However an attempt is made to find the maximal square sum bipartite planar graph and we show that the planar graph whose definition are based on complete bipartite graphs are square sum. In [9], Babujee defines a class of planar graphs which is obtained from the complete bipartite graph $K_{m,n}, m, n \geq 3$ by removing some edges to make it planar graphs, which is called a bipartite planar class and it is denoted by $Pl_{m,n}$.

Definition 1.3. [9] *Let $K_{m,n}(V_m, U_n)$ be the complete bipartite graph on $V_m = \{v_1, v_2, \dots, v_m\}$ and $U_n = \{u_1, u_2, \dots, u_n\}$. The class of graphs $Pl_{m,n}(V, E)$ has the vertex set $V_m \cup U_n$ and the edge set $E = E(K_{m,n}(V_m, U_n)) \setminus \{(v_l, u_p) : 3 \leq l \leq m \text{ and } 2 \leq p \leq n - 1\}$.*

The graph shown in Figure 1 is a bipartite planar graph with $2m + 2n - 4$ edges and $m + n$ vertices. We now describe the embedding used for our

proofs. Place the vertices u_1, u_2, \dots, u_n in that order along a horizontal line segment with u_1 as the left endpoint and u_n as the right endpoint as shown in Figure 1. Place the vertices $v_m, v_{m-1}, \dots, v_3, v_1$ in that order along a vertical line segment with v_m as the top end point and v_1 as the bottom end point so that this entire line segment is above the horizontal line segment where the vertices u_1 through u_n are placed. Finally place v_2 below the horizontal line segment so that the vertices v_1, u_k, v_2, u_{k+1} form a face of length 4 for $1 \leq k \leq n - 1$. Notice that though we talk about placement along a line segment, no edges other than those mentioned in the definition are to be added.

Here we show that the planar graph $Pl_{m,n}$ has square sum labeling.

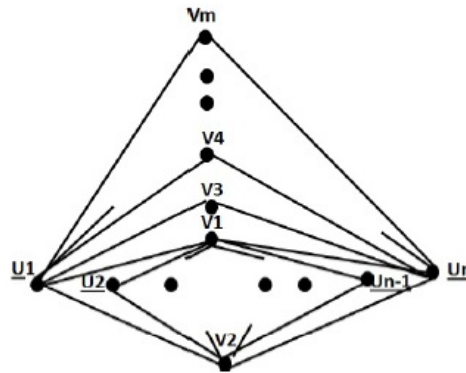


Figure 1 : The class $Pl_{m,n}$

Theorem 1.4. *The graph $Pl_{m,n}, m, n, \geq 3$, is a square sum graph.*

Proof. Consider the graph $Pl_{m,n}(V, E)$, where
 $V = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$. Define the injective function
 $f : V(Pl_{m,n}) \rightarrow \{0, 1, \dots, m + n - 1\}$ by
 $f(u_i) = i - 1; 2 \leq i \leq n - 1;$
 $f(u_1) = n - 1;$
 $f(u_n) = n;$
 $f(v_2) = 0;$

$$\begin{aligned} f(v_1) &= n + 1; \\ f(v_i) &= f(v_1) + i - 2; 3 \leq i \leq m \end{aligned}$$

Clearly f is injective. It is enough to check the labeling of edges with end vertices having labels $(a, a+x)$ and $(a+1, a+y)$ where $a = f(u_1)$, $x > y$. If $a^2 + (a+x)^2 = (a+1)^2 + (a+y)^2$, then $k^2 + 2ak + 2yk = 2a + 1$, where $x - y = k$, has no integer solution for any positive integer value of k . Hence all the edge labels are distinct. \square

Definition 1.5. Let u be a vertex of $P_m \times P_n$ such that $\deg(u) = 2$. Introduce an edge between every pair of distinct vertices v, w with $\deg(v), \deg(w) \neq 4$, if $d(u, v) = d(u, w)$, where $d(u, v)$ is the distance between u and v . The graph so obtained is defined as the level joined planar grids and it is denoted by $LJ_{m,n}$.

An example of $LJ_{4,5}$ is illustrated in figure 2.

Theorem 1.6. The graph $LJ_{m,n}$ is square sum for all $m, n \geq 2$.

Proof. Denote the vertex at the i^{th} row and j^{th} column of $P_m \times P_n$ as $v_{i,j}$. Without loss of generality we can assume $m \leq n$. Construct the graph $LJ_{m,n}$ as illustrated in Figure 2. Let V be the vertex set of $LJ_{m,n}$ with p vertices. Define a function $f : V \rightarrow \{0, 1, 2, \dots, p-1\}$ such that

$$\begin{cases} f(v_{1,j}) = \frac{j(j+1)-2}{2} ; 1 \leq j \leq m \\ f(v_{1,j}) = \frac{(m+2mj-m^2)-2}{2} ; m < j \leq n \\ f(v_{i,j}) = f(v_{i-1,j+1}) - 1 ; 2 \leq i \leq m, 1 \leq j \leq n+1-i \\ f(v_{i,j}) = f(v_{i-1,j}) + (m+n+1-j-i) ; 2 \leq i \leq m, n+1-i < j \leq n \end{cases}$$

Here f is injective. With the above labeling no two of the edge labels are same as the edge labels can be arranged in an increasing order. Hence f is square sum labeling of $LJ_{m,n}$ for all $m, n \geq 2$.

\square

For $m=1, n=2$ we get $P_1 \times P_2 \equiv P_2$ which is square sum. A square sum labeling of level joined planar grid $LJ_{4,5}$ using theorem 1.6 is exhibited in Figure 2.

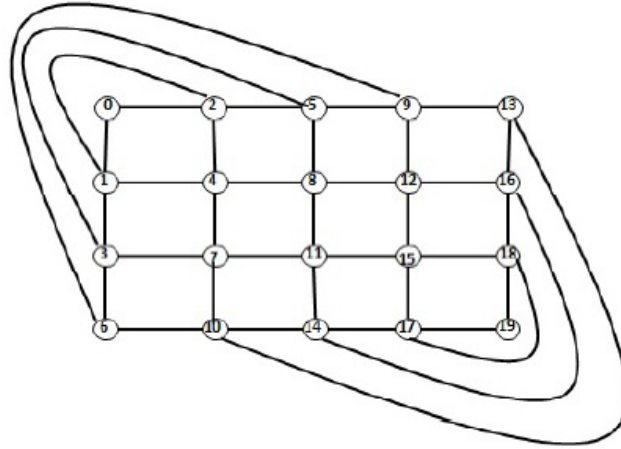


Figure 2 : Square sum labeling of $LJ_{4,5}$

Theorem above motivate us to construct a higher order square sum graph. Let $v_{i,j}^x$ denote the vertex $v_{i,j}$ in the x^{th} copy of $LJ_{m,n}$. For any integer $t > 1$, construct a graph by joining the vertex $v_{m-1,n}^x$ to the vertices $v_{1,1}^{x+1}, v_{1,2}^{x+1}, v_{2,1}^{x+1}$; $1 \leq x < t$ and denote the resulting graph as $LJ_{m,n}^t$ (see Figure 3). The following result is the general form of Theorem 1.6.

Theorem 1.7. *The graph $LJ_{m,n}^t$ is square sum for all $m, n \geq 2$ and $t \geq 1$.*

Proof. Denote the vertex of the graph $LJ_{m,n}^t$ as illustrated. Here $|V(LJ_{m,n}^t)| = mnt$. That is $p = mnt$. Define a function $f : V(LJ_{m,n}^t) \rightarrow \{0, 1, 2, \dots, mnt - 1\}$ for $1 \leq x \leq t$, by

$$\begin{cases} f(v_{1,j}^x) = \frac{j(j+1)-2}{2} + mn(x-1) ; 1 \leq j \leq m \\ f(v_{1,j}^x) = \frac{(m+2mj-m^2)-2}{2} + mn(x-1) ; m \leq j \leq n \\ f(v_{i,j}^x) = f(v_{i-1,j+1}^x) - 1 ; 2 \leq i \leq m, 1 \leq j \leq n+1-i \\ f(v_{i,j}^x) = f(v_{i-1,j}^x) + (m+n+1-j-i) ; 2 \leq i \leq m, n+1-i < j \leq n \end{cases}$$

Here f is injective. With the above defined vertex labeling the edge labels can be arranged in an increasing order. Hence f so defined is a square sum labeling of graph $LJ_{m,n}^t$ for all $m, n \geq 2$ and $t \geq 1$. \square

For example, a square sum labeling of $LJ_{m,n}^t$ when $m = 4, n = 5, t = 3$ using theorem 1.7 is illustrated in Figure 3.

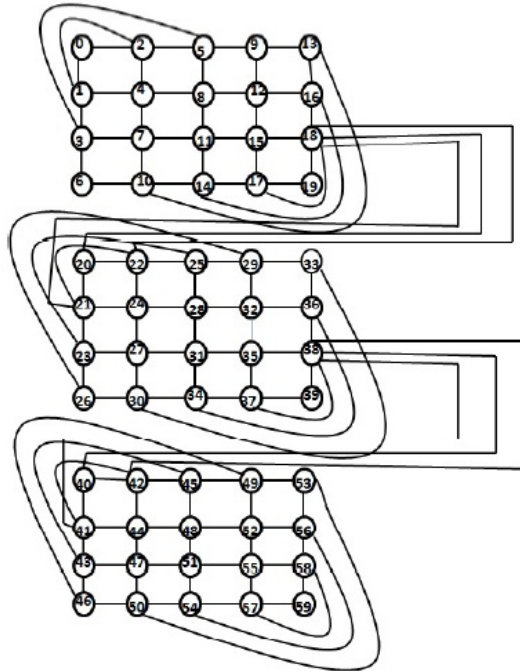


Figure 3 : Square sum labeling of $LJ_{4,5}^3$

For $n \geq 2$ let L_n be the Cartesian product $P_n \times P_2$ of a path on n vertices with a path on two vertices. In [2] it is proved that L_n is square sum for $n \geq 2$.

Let $S = \{\uparrow, \downarrow\}$ be the symbol representing the position of the block (Figure 4).

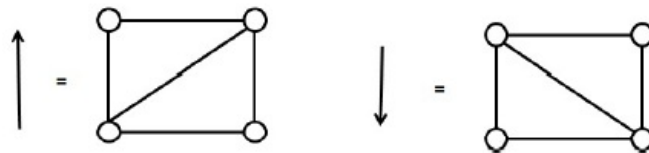


Figure 4 :

Let α be a sequence of n symbols of S , i.e., $\alpha \in S^n$. We will construct a graph by tiling n -blocks side by side, with their positions indicated by α . We will denote the resulting graph by $TB(\alpha)$ and refer to it as a triangular belt. For simplicity we denote $(\uparrow, \uparrow, \uparrow, \dots, \uparrow)$ by \uparrow^n and $(\downarrow, \downarrow, \dots, \downarrow)$ by \downarrow^n . Example 1:

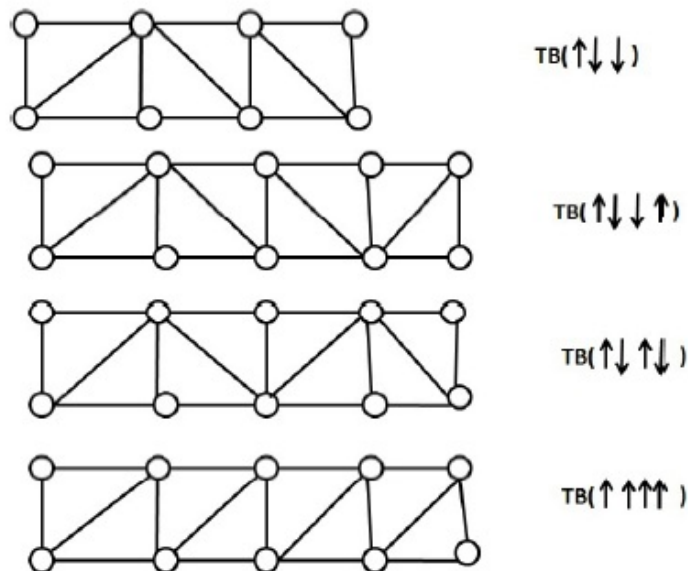


Figure 5 :

Theorem 1.8. For any $\alpha \in S^n, n > 1$, the triangular belt $TB(\alpha)$ is square sum.

Proof. The following algorithm indicates a labeling pattern. Label the top vertices of belt by numbers $\{0, 2, \dots, 2n\}$ from left to right respectively. Then assign the bottom vertices of the belt from left to right by all odd numbers $\{1, 3, 5, \dots, 2n + 1\}$. With the above defined labeling pattern the edge labels can be arranged in increasing order. Figure 6 illustrates the algorithm for $TB(\alpha), TB(\beta), TB(\gamma)$ and $TB(\delta)$ respectively in example 1. \square

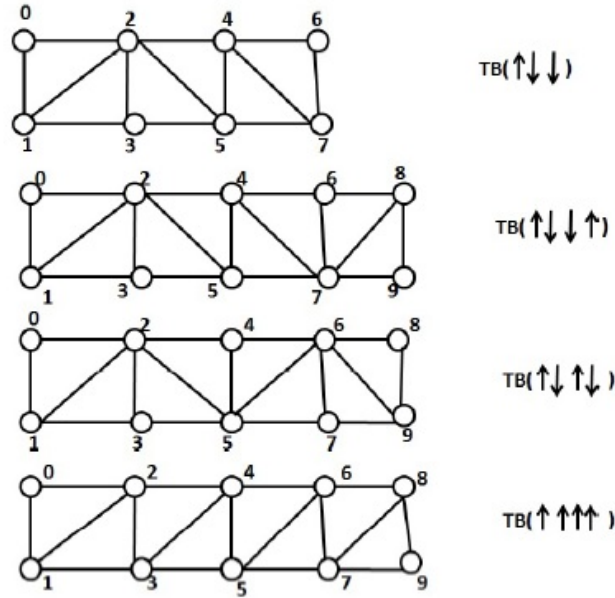


Figure 6 :

Corollary 1.9. *The 2^{nd} power of P_n is square sum since it is isomorphic to $TB(\uparrow, \uparrow, \uparrow, \dots, \uparrow)$ or $TB(\downarrow, \downarrow, \downarrow, \dots, \downarrow)$.*

We now consider a class of planar graphs that are formed by amalgamation of triangular belts. For each $n > 1$ and $\alpha \in S^n$, we take the triangular belt $TB(\alpha)$ and triangular belt $TB(\beta)$ where $k > 0$, k blocks with last block is \uparrow . We rotate $TB(\beta)$ by 90 degrees counter clockwise and amalgamate last block with the first block of $TB(\alpha)$ by sharing an edge. The resulting graph is denoted by $TBL(n, \alpha, k, \beta)$. The graph $TBL(n, \alpha, k, \beta)$ has $2(n + k + 1)$ vertices with $V(TBL(n, \alpha, k, \beta)) = \{x_{1,1}, x_{1,2}, \dots, x_{1,n+1}, x_{2,1}, x_{2,2}, x_{2,n+1}, y_{3,1}, y_{3,2}, \dots, y_{3,k}, y_{4,1}, y_{4,2}, \dots, y_{4,k}\}$ (figure 7).

Theorem 1.10. *The graph $TBL(n, \alpha, k, \beta)$ is square sum for all α in S^n and β in S^k with last block is being \uparrow for all $k > 0$.*

Proof. Define $f : V(TBL(n, \alpha, k, \beta)) \rightarrow \{0, 1, \dots, 2n + 2k + 1\}$ as follows.

$$\begin{aligned}
 x_{1,1} &= 2n \\
 x_{1,j} &= 2n - (2j - 2); 2 \leq j \leq n + 1 \\
 x_{2,1} &= 2n + 1 \\
 x_{2,j} &= 2n + 1 - (2j - 2); 2 \leq j \leq n + 1 \\
 y_{3,1} &= 2(n + k) + 1 \\
 y_{3,j} &= 2(n + k) + 1 - (2j - 2); 2 \leq j \leq k \\
 y_{4,1} &= 2(n + k) \\
 y_{4,j} &= 2(n + k) - (2j - 2); 2 \leq j \leq k.
 \end{aligned}$$

Clearly f is injective. Here the only possible case, where there may be edges having same label is edges with end vertices $(a, a+3)$ and $(a+1, a+2)$, where $a = 2n - 1$. If $a^2 + (a + 3)^2 = (a + 1)^2 + (a + 2)^2$, we get $9 = 5$, is not possible. Hence all the edge labels are distinct. \square

Example 1.11. Figure 7 illustrates $k = 1, 3$ for Theorem 1.10

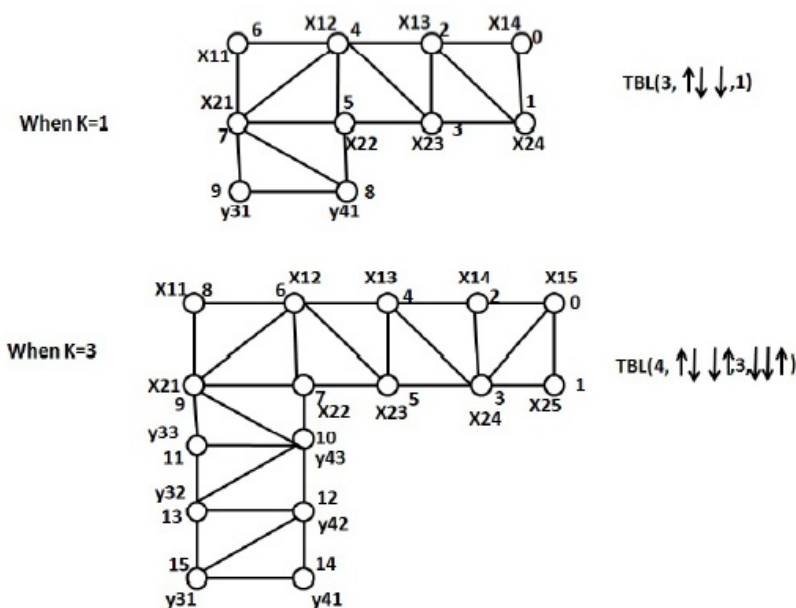


Figure 7 :

Theorem 1.12. For $k \geq 1$, the planar graph $(P_2 \cup kK_1) + N_2$ is square sum.

Proof. Let the vertex set of $(P_2 \cup kK_1)$ be $\{z_1, z_2, x_1, x_2, \dots, x_k\}$ and $V(N_2) = \{y_1, y_2\}$.
 $|V(P_2 \cup kK_1) + N_2| = k + 4$.

Define a labeling $f : V(P_2 \cup kK_1) + N_2 \rightarrow \{0, 1, 2, \dots, k+3\}$ by $f(z_1) = 0, f(y_1) = 1, f(y_2) = 2, f(z_2) = 3$
 $f(x_i) = f(z_2) + i; 1 \leq i \leq k$

Clearly f is injective. It is enough to check the labeling of edges with end vertices having labels $(a, a+x)$ and $(a+1, a+y)$ where $a = f(y_1), x > y$. If $a^2 + (a+x)^2 = (a+1)^2 + (a+y)^2$, then $k^2 + 2ak + 2yk = 2a + 1$, where $x - y = k$, has no integer solution for any positive integer value of k . Hence all the edge labels are distinct. \square

Example 1.13. Square sum labeling of $(P_2 \cup 3K_1) + N_2$

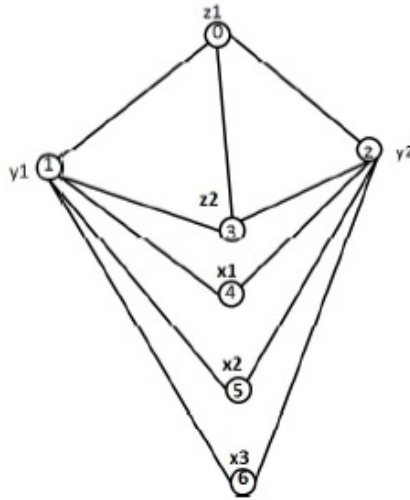


Figure 8 :

2. Square sum graph with many odd cycles

We would like to consider the graphs with many cycles. Let $P_n(+)N_m$ be the graph with $p = n + m$ and $q = n + 2m - 1$.

$V(P_n(+)N_m) = \{v_1, v_2, \dots, v_n, y_1, y_2, \dots, y_m\}$ where $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $V(N_m) = \{y_1, y_2, \dots, y_m\}$

$E(P_n(+)N_m) = E(P_n) \cup \{(v_1, y_1)(v_1, y_2), \dots, (v_1, y_m)(v_n, y_1)(v_n, y_2), \dots, (v_n, y_m)\}$.

The graph has m -cycles of length $n + 1$ and many 4-cycles.

Theorem 2.1. $P_n(+)N_m$ is square sum for all $n, m \geq 1$.

Proof. Define $f : V(P_n(+)N_m) \rightarrow \{0, 1, \dots, m + n - 1\}$ as follows.

Case 1. When n is odd.

$$f(v_{\lceil \frac{n}{2} \rceil}) = 0$$

Start from $v_{\lceil \frac{n}{2} \rceil}$ apply BFS algorithm and label the vertices as $0, 1, \dots, n + m - 1$ in increasing order, the order in which they are visited. It is enough to check the labeling of edges with end vertices having labels $(a, a + x)$ and $(a + 1, a + y)$ where $a = f(v_1), x > y$. If $a^2 + (a + x)^2 = (a + 1)^2 + (a + y)^2$, then $k^2 + 2ak + 2yk = 2a + 1$, where $x - y = k$, has no integer solution for any positive integer value of k . Hence all the edge labels are distinct.

Case2. When n is even.

Define $f(v_{\frac{n}{2}+1}) = 0$. Start from $v_{\frac{n}{2}+1}$, apply BFS algorithm and label the vertices as $0, 1, \dots, n + m - 1$, in increasing order, the order in which they are visited. It is enough to check the labeling of edges with end vertices having labels $(a, a + x)$ and $(a + 1, a + y)$ where $a = f(v_1), x > y$. If $a^2 + (a + x)^2 = (a + 1)^2 + (a + y)^2$, then $k^2 + 2ak + 2yk = 2a + 1$, where $x - y = k$, has no integer solution for any positive integer value of k . Hence all the edge labels are distinct. \square

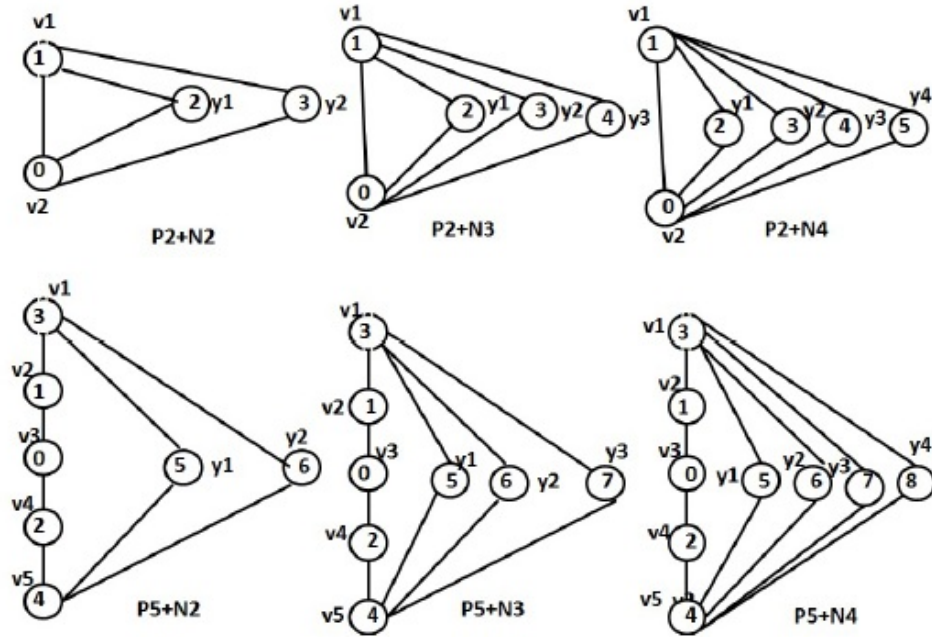


Figure 9 : Square sum labeling of $P_2(+)$ N_m and $P_5(+)$ N_m , for $m = 2, 3$ and 4

For integers $m, n \geq 0$, we consider the graph $J(m, n)$ with vertex set $V(J(m, n)) = \{u, v, x, y\} \cup \{x_1, x_2, \dots, x_m\} \cup \{y_1, y_2, \dots, y_n\}$ and edge set $E(J(m, n)) = \{(u, x), (u, v), (u, y), (v, x), (v, y)\} \cup \{(x_i, x) : i = 1, 2, \dots, m\} \cup \{(y_i, y) : i = 1, 2, \dots, n\}$.

We will refer to $J(m, n)$ as a Jellyfish graph.

Theorem 2.2. *The Jellyfish graph $J(m, n)$ is square sum for all $m, n \geq 0$.*

Proof. We define a labeling $f : V(J(m, n)) \rightarrow \{0, 1, \dots, m + n + 3\}$ as follows.

$$\begin{aligned} f(x_i) &= i; 1 \leq i \leq m \\ f(u) &= m + 1, f(x) = 0 \\ f(v) &= m + 2; f(y) = m + 3 \\ f(y_i) &= m + 3 + i; 1 \leq i \leq n. \end{aligned}$$

With the above defined labeling we can arrange the edge labels in an increasing order. \square

Example 2.3. Square sum labeling of Jellyfish graph $J(3, 4)$ is shown in Figure 10.

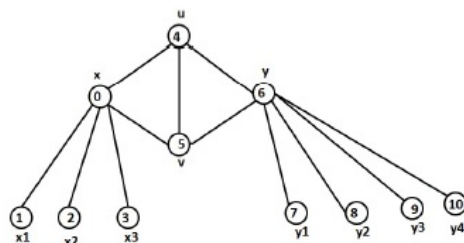


Figure 10 :

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