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L(1,1)-Labeling of Direct Product of any Path and Cycle

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Abstract

Suppose that $[n] = \{0, 1, 2, ..., n\}$ *is a set of non-negative integers and* $h, k \in [n]$. *The* $L(h, k)$*-labeling of graph* G *is the function* $l : V(G) \rightarrow [n]$ *such that* $|l(u) - l(v)| \geq h$ *if the distance* $d(u, v)$ *between* u *and* v *is* 1 *and* $|l(u) - l(v)| \geq k$ *if* $d(u, v) = 2$. *Let* $L(V(G)) = \{l(v) : v \in V(G)\}$ *and let* p *be the maximum value of* $L(V(G))$. *Then* p *is called* λ_h^k-*number of* G *if* p *is the least possible member of* $[n]$ *such that* G *maintains an* $L(h, k)-$*labeling. In this paper, we establish* λ_1^1- *numbers of* $P_m \times P_n$ *and* $P_m \times C_n$ *graphs for all* $m, n \geq 2$.

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1. Introduction

Let $l : V(G) \rightarrow [n] = \{0, 1, 2, \cdots, n\}$ be a non negative function on the vertex set $V(G)$ of G. Given any two fixed non-negative integers h, k, the $L(h,k)$-labeling of G is defined such that for any edge $uv \in E(G)$, $|l(u) - l(v)| \geq h$ and if $d(u,v) = 2, u, v \in V(G)$, then $|l(u) - l(v)| \geq k$. The aim of $L(h,k)-$labeling is to obtain the smallest non negative integer $\lambda_h^k(G)$, such that there exists an $L(h,k)$-labeling of G with no $l(v) \in L(V(G))$ greater than $\lambda_h^k(G)$, where $L(V(G))$ is the set of all labels on $V(G)$.

In [13], Griggs and Yeh introduced the $l(h,k)-$labelling and particularly showed that any graph G with maximum degree $\Delta > 1$ has $\lambda_2^1(G) \leq \Delta^2 + 2\Delta$ and went further to put forward a conjecture that $\lambda_2^1(G) \leq \Delta^2$. Chang and Kuo, in [5] improved on Griggs and Yeh's bound by showing that $\lambda_2^1(G) \leq \Delta(\Delta+1)$, Kral' and Skrekovski [16] went another step showing that $\lambda_2^1(G) \leq \Delta(\Delta+1) - 1$ while Goncalves in [11] proved that $\lambda_2^1(G) \leq \Delta(\Delta+1)-2$. The interest in the Griggs-Yeh conjecture and in improving on the existing bounds have inspired a lot of work in the direction of $L(h,k)$-labeling, mostly on $h = 2$, $k = 1$. (See [5][6][10][12][18].) (An extensive review of all known results on $L(h,k)-$labeling can be seen in [3].) It is obvious that $L(2,1)-$labeling is an $L(1,1)-$labeling, therefore results on $L(2,1)$-labeling provide upper bound for $L(1,1)$-labeling of graphs and

$$\lambda_2^1(G) + 1 \geq \lambda_1^1(G) + 1 = \lambda(G^2)$$

where $\lambda(G^2)$ is the chromatic number of the square of G.

Finally, Georges and Mauro [8] obtained various results for the $L(h,k)-$number for path P_n and cycles C_n. Particularly among other results, they showed that $\lambda_h^k(P_n)$ is either $0, h, h+k, h+2k$, or $2h$.

Suppose that G and H are graphs. The Cartesian product and the direct product of G and H, $G\Box H$ and $G \times H$ respectively, have vertex set $V(G) \times V(H)$, while the edge sets are

$E(G\Box H)= \{((x_1,x_2),(y_1,y_2)) : (x_1,y_1) \in E(G)$ *and* $x_2 = y_2$ *or* $(x_2,y_2) \in E(H)$ *and* $x_1 = y_1\}$ and

$E(G \times H) = \{((x_1,x_2),(y_1,y_2)) : (x_1,y_1) \in E(G)$ *and* $(x_2,y_2) \in E(H)\}$ respectively.

The $L(h,k)-$labeling of the Cartesian product $G\Box H$ has been extensively investigated with $\lambda_h^k(G\Box H)$ obtained for various types of graphs G and H, while numerous upper and lower bounds have been suggested (see [8][7][16][18][20][22]). Most of the work on $L(h,k)$ labeling consider $h = 2$ and $k = 1$; although Chiang and Yan in [7] and Georges and Mauro in [10] worked on the $L(1,1)$labeling of Cartesian products of paths and cycles and Sopena and Wu in [20] worked on Cartesian products of cycles. In case of direct product graphs, Jha et al [15], established $\lambda_2^1(C_m \times C_n)$ for some values of m and n.

In this paper, we determine $\lambda_1^1(P_m \times P_n)$ and $\lambda_1^1(P_m \times C_n)$ where P_m and P_n are paths of length $m-1$ and $n-1$ respectively and C_n is a cycle of length n for all $m, n \geq 2$. We also deduce $\lambda_1^1(C_m \times C_n)$ for $m, n \equiv 0 \bmod 5$. Thus, we extend the results in [10] and [7] to direct product graphs among other results.

2. Preliminaries

The following results and definitions are necessary.

Let m be a non-negative integer. $P_m = u_0u_1u_2...u_{m-1}$ is a path of length $m-1$, where $u_i \in V(P_m)$, for all $i \in [m-1]$; $C_m = u_0u_1u_2...u_{m-1}u_0$ is a cycle of length m, where $u_i \in V(C_m)$, for all $i \in [m-1]$. Let $v \in V(G)$, we denote by $l(v)$ the label on v and let $U \subseteq V(G)$. Then $L(U)$ is a set of labels on U.

Suppose $P_m \times P_n$ is a direct product paths and G' is a component of $P_m \times P_n$. Then
$U_j = \{u_iv_j\} \subset V(G')$, for some $j \in [n-1]$, and for all $i \in [(m-1)(\epsilon)]$ or for all $i \in [(m-1)(o)]$.
$V_i = \{u_iv_j\} \subset V(G')$, for some $i \in [m-1]$, and for all $j \in [(n-1)(\epsilon)]$ or for all $j \in [(n-1)(o)]$.

Theorem 2.1. [22] Graph $G \times H$ is connected if and only if G and H are connected and at least one of G and H is non-bipartite.

Remark 2.2.

(i) Since P_m is bipartite for all $m \geq 2$, then for $P_m \times P_n$, there exist $G_1 \subset P_m \times P_n$ and $G_2 \subset P_m \times P_n$ such that G_1 and G_2 are components of $P_m \times P_n$.

(ii) From Theorem 2.1 and the Remark above, it is clear that $P_m \times P_n$ is not a connected graph. Suppose $P_m = u_0u_1u_2...u_{m-1}$ and $P_n = v_0v_1v_2...v_{n-1}$, then
$V(G_1) = \{u_i, v_j : i \in [(m-1)(\epsilon)], j \in [(n-1)(\epsilon)]$
or $i \in [(m-1)(o)]; j \in (n-1)[o]$
$V(G_2) = \{u_i, v_j : i \in [(m-1)(\epsilon)], j \in [(n-1)(o)]$

or $i \in [(m-1)(o)]; j \in (n-1)[\epsilon]$.

(iii) Suppose G is a graph such that $G = G' \cup G''$, where G', G'' are components of G, then, $\lambda_1^1(G) = max\left\{\lambda_1^1(G'), \lambda_1^1(G'')\right\}$.

(iv) For a direct product graph, $P_m \times P_2$, $m \geq 2$, its components G_1 and G_2 are paths P'_m and P''_m respectively such that
$P'_m = u_0v_0u_1v_1u_2v_0...u_{m-1}v_1(u_{m-1}v_0)$ (if m is even) and
$P''_m = u_0v_1u_1v_0u_2v_1...u_{m-1}v_0(u_{m-1}v_1)$ (if m is odd).

The following are known results for $L(1,1)$-labeling of paths, cycles and $L(h,k)$-labeling of stars, $k \leq h$.

Lemma 2.3. [1] Let P_m be a path of length $m-1$. $\lambda_1^1(P_m) = 1$, for $m = 2$ and $\lambda_1^1(P_m) = 2$ for all $m \geq 3$.

Lemma 2.4. [1] Let C_m be cycle of length m. Then $\lambda_1^1(C_m) = 2$ for $m \equiv 0 \bmod 3$ and $\lambda_1^1(C_m) = 3$ for $m \not\equiv 0 \bmod 3$.

The following result presents a general λ_h^k-value for stars for $k \leq h$.

Lemma 2.5. [4] Let $K_{1,\Delta}$ be a star of order $\Delta + 1$. Then, $\lambda_h^k(K_{1,\Delta}) = (\Delta - 1)k + h$ if $h \geq k$.

Henceforth we refer to direct product graph as product graph.

3. $L(1,1)$-Labeling of $P_m \times P_n$

Proposition 3.1. $\lambda_1^1(P_2 \times P_2) = 1$.

Proof. Clearly, G consists of connected components P'_2 and P''_2. By Lemma 2.3, $\lambda_1^1(P'_2) = \lambda_1^1(P''_2) = 1$. □

We extend the graph in Theorem 3.1 to $m \geq 3$.

Proposition 3.2. For $m \geq 3$, $\lambda_1^1(P_m \times P_2) = 2$.

Proof. $P_m \times P_2$ consists of two connected components P'_m and P''_m. By Lemma 2.3, $\lambda_1^1(P'_m) = \lambda_1^1(P''_m) = 2$ and the result follows from Remark 2.2 (iii). $\square$

The next results establish $\lambda_1^1(P_m \times P_n),\ m, n \geq 3$.

Lemma 3.3. Let $u_iv_j \in P_m \times P_n,\ n, m \geq 3$, Suppose $d_{u_i} = d_{v_j} = 2$ then $d_{u_iv_j} = 4$.

Proof. Let $u_{i-1}u_iu_{i+1} = P'_3,\ P'_3 \subseteq P_m,\ m \geq 3$ and let $v_{j-1}v_jv_{j+1} = P''_3,\ P''_3 \subseteq P_n,\ n \geq 3$. By the definition of direct product of graphs,
$V(P'_m \times P''_n)=$
$\{u_{i-1}v_{j-1}, u_{i-1}v_j, u_{i-1}v_{j+1}, u_iv_{j-1}, u_iv_j, u_iv_{j+1}, u_{i+1}v_{j-1}, u_{1+1}v_j, u_{i+1}v_{j+1}\}$ $\subseteq V(P_m \times P_n)$. Since $d_{u_i} = d_{v_j} = 2$, then by the definition of direct product of graphs, $u_iv_j \in V(P'_3 \times P''_3)$ is adjacent to all the members of $\{u_{i-1}v_{j-1}, u_{i+1}v_{j-1}, u_{i+1}v_{j+1}, u_{i-1}v_{j+1}\}$. Thus, $d_{u_iv_j} = 4$. $\square$

Proposition 3.4. Suppose $m, n \geq 3$. Then $\lambda_1^1(P_m \times P_n) = 4$ for all $m, n \geq 3$.

Proof. Let G_1 be a connected component of $P_m \times P_n$. By Lemma 3.3, there exists a star $K_{1,4} \subseteq G_1$. By Lemma 2.5, $\lambda_1^1(K_{1,4}) = 4$ and thus, $\lambda_1^1(P_m \times P_n) \geq 4$. Let $u_iv_j \in V(P_m \times P_n)$. For all $u_iv_j \in V(P_m \times P_n)$, $l(u_iv_j) = \left\lfloor \frac{i+3j}{2} \right\rfloor \mod 5$. Thus $\lambda_1^1(P_m \times P_n) \leq 4$ and then the equality follows. $\square$

Remark 3.5. By using $l(u_iv_j) = \left\lfloor \frac{i+3j}{2} \right\rfloor \mod 5$ as in the proof of Proposition 3.4, given both connected components of $P_m \times P_n$, for all $i \in [m(\epsilon)]$, *then* l($u_iv_{10}) = l(u_iv_0)$. Furthermore, for all
$u_iv_1 \in U_1,\ i \in \{3, 5, 7\}\ l(u_iv_1) \notin L(u_{i-2}v_9, u_iv_9, u_{i+2}v_9)$,
$\{u_{i-2}v_9, u_iv_9, u_{i+2}v_9\} \subset U_9$. We also notice that $l(u_1v_1) \notin L(u_1v_9, u_3v_9, u_9v_9)$, while $l(u_9v_1) \notin L(u_1v_9, u_7v_9, u_9v_9)$. Also, for all
$u_1v_j \in V_1, j \in \{3, 5, 7\},\ l(u_1v_j) \notin L(u_9v_{j-2}, u_9v_j, u_9v_{j+2})$,
$\{u_9v_{j-2}, u_9v_j, u_9v_{j+2}\} \subset V_9$ and $l(u_1v_1) \notin L(u_9v_1, u_9v_3, u_9v_9)$, while $l(u_1v_9) \notin L(u_9v_1, u_9v_7, u_9v_9)$.

The implication of Remark 3.5 is expressed in the following results.

Corollary 3.6. Let C_m be a cycle of length m, then, $\lambda_1^1(C_{10} \times C_{10}) = 4$.

Corollary 3.7. For all $m, n \equiv 0 \mod 5$, $\lambda_1^1(C_m \times C_n) = 4$.

4. $L(1,1)$-Labeling of $P_m \times C_m$

Lemma 4.1. Let $G = P_m \times P_n$, where $n \geq 4$. Suppose that $\alpha_k \in [4]$, such that for some $v_i \in V(G)$, $l(v_i) = \alpha_k$, $v_j \in V(G)$ is the closest vertex in $V(G)$ to v_i, $i \neq j$ such that $l(v_j) = \alpha_k$. Then $3 \leq d(v_i, v_j) \leq 4$.

Proof. That $3 \leq d(v_i, v_j)$ follows directly from the definition of $L(1,1)$-labeling. Next, we show that $d(v_i, v_j) \leq 4$. Let S_n be a star of order $n+1$. Clearly, $diam(S_n) = 2$. Now, suppose that for two stars $S'_4 \subset G$ and $S''_4 \subset G$, there exits some vertex u_i such that $u_i \in V(S'_n)$ and also $u_i \in V(S''_n)$, making S'_n and S''_n to be neighbors. Then, $diam(H) = 4$, where $S'_4 \cup S''_4 = H \subset G$. Now, suppose $d(v_i, v_j) > 4$. Let $v_i \in V(S'_4)$ such that $l(v_i) = \alpha_k$. Also, let $L(S'_4) = [4]$. Then, $\alpha_k \neq l(v_k)$ for all $v \in V(S''_4)$ since $d(u_i, v_j) \geq 4$. Thus, there exits some $\alpha_j \notin [4]$ such that $\alpha_j \in L(S''_4)$. Then, $\lambda_1^1(H) \geq 5$, and consequently, $\lambda_1^1(G) \geq 5$. This is a contradiction. □

Lemma 4.2. Let $v_i, v_j \in V(G)$ be two center vertices of stars $S'_4, S''_4 \subset G$ respectively, and that $d(v_i, v_j) = 4$ if $\alpha_i = l(v_i)$ and $\alpha_j = l(v_j)$, $\alpha_i, \alpha_j \in [4]$, then $\alpha_i \neq \alpha_j$.

Proof. Suppose on the contrary that v_i, v_j are respective centers of S'_4, S''_4 such that $d(v_i, v_j) = 4$ and $\alpha_i = \alpha_j$. There exists a star $S'''_4 \subset G$ with $V(S'''_4) = u_q v_r, u_{q+2} v_r, u_{q+1} v_{r+1}, u_q v_{r+2}, u_{q+2} v_{r+2}$, where $0 \leq q$, $q + 2 \leq m$ and $r \leq 2, r+2 \leq n-3$, such that $v_i = u_{q+1} v_{r-1}$ and $v_j = u_{q+1} v_{r+3}$. Therefore v_i is adjacent to $u_q v_r$ and $u_{q+2} v_r$ and $d(v_i, u_{q+1} v_{r+1}) = 2$. Likewise, v_j is adjacent to both $u_q v_{r+2}, u_{q+2} v_{r+2}$ and $d(v_j, u_{q+1} v_{r+1}) = 2$. Thus there exists no vertex $v_l \in V(S''')$ such that $l(v_l) = \alpha_i \in [4]$. This contradicts the fact that $\lambda_1^1(G) \leq 4$, for all $m, n \geq 2$. □

Lemma 4.3. Let $G' \subset G$ with
$V(G') = \{u_q v_r, u_{q+2} v_r, u_{q+1} v_{r+1}, u_q v_{r+2}, u_{q+2} v_{r+2}, u_{q+1} v_{r+3}, u_q v_{r+4}, u_{q+2} v_{r+4}\}$, $q, r \geq 0$. Suppose that $l(u_q v_r), l(u_{q+2} v_r)$ are α_0, α_1 respectively, then $l(u_q v_{r+4}), l(u_{q+2} v_{r+4})$ are both neither α_0 nor α_1.

Proof. The vertex set $\{u_q v_r, u_{q+2} v_r, u_{q+1} v_{r+1}, u_q v_{r+2}, u_{q+2} v_{r+2}\} \subset V'(G')$ induces a star $S_4 \subset G$. Since $\lambda_1^1(S_4) = 4$, we have
$l(u_{q+1} v_{r+1}) = \alpha_2, l(u_q v_{r+2}) = \alpha_3, l(u_{q+2} v_{r+2}) = \alpha_4$. Set

$\{u_q v_{r+2}, u_{q+2} v_{r+2}, u_{q+1} v_{r+3}, u_q v_{r+4}, u_{q+2} v_{r+4}\} \subset V(G')$ induces another star $S_4' \subset G'$. Clearly, S_4 and S_4' are adjacent and $S_4 \cup S_4' = G'$ Now, suppose $l(u_q v_{r+4}) = \alpha_0, l(u_{q+2} v_{r+4}) = \alpha_1$, or vice versa without the loss of generality. Since $l(u_q v_{r+2}) = \alpha_3$, and $l(u_{q+2} v_{r+2}) = \alpha_3$ from the labeling on S_4, the only label left in [4] for $u_{q+1} v_{r+3}$ is α_2. This however is a contradiction since $d(u_{q+1} v_{r+1}, u_{q+1} v_{r+3}) = 2$. □

Remark 4.4.

(i) By theorem 2.1, $P_m \times C_n$ is connected if n is odd and not connected if n is even. This is because when n is odd, cycle C_n is non bipartite and when n is even, C_n is bipartite. Now, Let $P_m \times C_n = G = G_1 \cup G_2$, where n is even. Then
$V(G_1) = \{(u_i, v_j) : i \in [(m-1)(\epsilon)], j \in [n(\epsilon)] \text{ or } i \in [(m-1)(o)], j \in [n(o)]\}$
and
$V(G_2) = \{(u_i, v_j) : i \in [(m-1)(\epsilon)], j \in [n(o)] \text{or } i \in [(m-1)(o)], j \in [n(\epsilon)]\}$.

(ii) G_1 and G_2 above are isomorphic since C_n is a cycle and they are both components of G.

(iii) Suppose $G = P_m \times C_n$, n odd. Then G is equivalent to G', where G' is one of the two components of $P_m \times C_{2n}$.

(iv) G' above is equivalent to the connected component of $P_m \times P_{2n+1}$ such that $u_i v_0$ coincides with $u_i v_{2n}$, for all $i \in [(m-1)(\epsilon)]$ or for all $i \in [(m-1)(o)]$.

Lemma 4.5. [2] $\lambda_1^1(C_m) = \begin{cases} 2 & \text{if } m \equiv 0 \bmod 3 \\ 3 & m \not\equiv 0 \bmod 3; m \neq 5 \\ 4 & m = 5. \end{cases}$

Theorem 4.6. $\lambda_1^1(P_2 \times C_m) = \begin{cases} 2 & \text{if } m \equiv 0 \mod 3 \\ 3 & \text{otherwise.} \end{cases}$

Proof. By Remark 4.4 (iii), if m is odd, then $P_2 \times C_m \equiv C_{2m}$. If m is even, then $P_2 \times C_m$ is a union of m-cycles, C_m' and C_m'' are $m - cycles$ which are its components. By Lemma 4.5, for m odd, $\lambda_1^1(P_2 \times C_m) = \lambda_1^1(C_{2m}) = q$, where $q = 2$ for $2m \equiv 0 \mod 3$ and $q = 3$ if otherwise. Also $\lambda_1^1(P_2 \times C_2) = \lambda_1^1(C_n) = p$, where $p = 2$ if $n \equiv 0 \bmod 3$ and $p = 3$ otherwise. □

Theorem 4.7. For any $m \in \mathbf{N}$, $m \geq 3$, $\lambda_1^1(P_m \times C_3) = 5$.

Proof. By Remarks 4.4 (iii) and (iv), and $P_m \times C_3$ is congruent to a connected component G' of $P_m \times P_7$ with $u_iv_0 \equiv u_iv_6$, $u_iv_0, u_iv_6 \in V(G')$. Thus, $L(u_iv_0) = L(u_iv_6)$ for all $i \in [(m-i)(\epsilon)]$. Now, let G'' be a subgraph of G' induced by the vertex subset
$\{u_iv_0, u_{i+2}v_0, u_{i+1}v_1, u_iv_2, u_{i+2}v_2, u_{i+1}v_3, u_iv_4, u_{i+2}v_4, u_{i+1}v_5, u_iv_6, u_{i+1}v_6\} \subseteq V(G')$, for any $i \in [(m-1)(\epsilon)]$. Suppose $\lambda_1^1(G') = 4$ and $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [4]$. Let $l(u_iv_0) = \alpha_0$ and $l(u_{i+2}v_0) = \alpha_1$. Then, $l(u_iv_6) = \alpha_0$ and $l(u_{i+2}v_6) = \alpha_1$. Now, suppose $l(u_{i+1}v_1) = \alpha_2$. Since $d(u_{i+1}v_1, u_{i+1}v_5) = 2$, then for some $\alpha_k \in [4]$, $\alpha_k = l(u_{i+1}v_5) \neq \alpha_2$. In fact, $\alpha_k \notin \{\alpha_0, \alpha_1, \alpha_2\}$. Set $\alpha_k = \alpha_3$. The vertex subset $\{u_iv_0, u_{i+2}v_0, u_{i+1}v_1, u_iv_2, u_{i+2}v_2\} \subset V(G'')$ induces a star $S_4 \subset G'$ with center $u_{i+1}v_1$. Since $\lambda_1^1(S_4) = 4$, if $l(u_iv_2) = \alpha_3$, then $l(u_{i+2}v_2) = \alpha_4$. Let A and B be vertex subsets of $V(G')$, such that $A = \{u_iv_4, u_{i+2}v_4\}$ and $B = \{u_iv_2, u_{i+2}v_2, u_{i+1}v_5, u_1v_6, u_{i+2}v_6\}$. Clearly, $d(u,v) \leq 2$ for all $u \in A$ and $v \in B$. Then, $l(u_iv_4), l(u_{i+2}v_4) \notin \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \}$. Therefore, since $\lambda_1^1(S_4) = 4$, $l(u_iv_4) = \alpha_3 = l(u_{i+2}v_4)$. But $d(u_iv_4, u_{i+2}v_4) = 2$. This a contradiction and hence, $\lambda_1^1(P_m \times C_3) \geq 5$.
Claim: Let α_k $L(V_i)$, then $\alpha_k \notin V_{i+2}$, for V_i,$V_{i+2} \in V(G')$.
Reason: For all $v \in V_i$, $u \in V_{i+2}$, $d(u,v) \leq 2$.
Now, let $U_i = \{u_iv_0, u_iv_2, u_1v_4\}$, $U_{i+1} = \{u_{i+1}v_1, u_{i+1}v_3, u_{i+1}v_5\}$, $U_i, U_{i+1} \subset V(G'')$. $l(u_iv_j)$ labels u_{i+1} for all v_j, u_k in U_iU_{i+1} respectively where $|k-j| = 3$ since $d(u_iv_j, u_iv_k) = 3$. Therefore without loss of generality, we say $L(U_i) = L(U_{i+1}) = \{\alpha_0, \alpha_1, \alpha_2\} \subset [5]$. Likewise, let
$U_{i+2} = \{u_{i+2}v_0, u_{i+2}v_2, u_{i+2}v_4\}$ and
$U_{i+3} = \{u_{i+3}v_1, u_{i+3}v_3, u_{i+3}v_5\}$, $U_{i+2}, U_{i+3} \subset V(G'')$. $l(u_{i+2}v_l)$ labels $u_{i+3}v_p$ for all v_l, v_p in U_{i+2}, U_{i+3} respectively, where $|l-p| = 3$. Thus $L(U_{i+2}) = L(U_{i+3}) = \{\alpha_3, \alpha_4, \alpha_5\} \subset [5]$. Based on the last scheme, we have $L(U_a) = L(U_{a+4})$ for any $a \in [i, i+3]$, where $i \in [(m-1)(\epsilon)]$. Thus there exists a $5 - L(1,1)-$labeling of $P_m \times C_3$ and thus $\lambda_1^1(P_m \times C_3) \leq 5$ and then the equality holds. □

Corollary 4.8. If $m \geq 3$, then, $\lambda_1^1(P_m \times C_6) = 5$.

Proof. Follows from Remark 4.4 (iii) and Theorem 4.7. □

Theorem 4.9. If $m \geq 3$, then $\lambda_1^1(P_m \times C_4) = 5$.

Proof. From Remarks 4.4 (ii) and (iii), $P_m \times C_4 = G_1 \cup G_2$, where G_1, G_2 are isomorphic connected components of $P_m \times C_4$. Let $u_iv_0, u_iv_4 \in V(G_1)$, say, for all $i \in [(m-1)(\epsilon)]$, such that $u_iv_0 \equiv u_iv_4$ then by Remark 4.4 (iv) , G_1 is equivalent to a connected component of $P_m \times P_5$. Now, let $G_1' \subseteq G_1$ be a subgraph of G_1 with
$V(G_1')=\{u_rv_0, u_{r+2}v_0, u_{r+1}v_1, u_rv_2, u_{r+2}v_2, u_{r+1}v_3, u_rv_4, u_{r+2}v_4\}$, where $r \leq m-4$. Obviously, $u_rv_0 \equiv u_rv_4$ and $u_{r+2}v_0 \equiv v_4$. Thus, $l(u-rv_0)=l(u_rv_4) = \alpha_i$ and $l(u_{r+2}v_0) = l(u_{r+2}v_4) = \alpha_j$, $\alpha_i, \alpha_j \in [4]$. By Lemma 4.3, there exists a vertex $v \in V(G_1')$ such that $l(v) \notin [4]$. Thus $\lambda_1^1(G_1') \geq 5$ and therefore, $\lambda_1^1(G_1) \geq 5$ and finally, $\lambda_1^1(P_m \times C_4) \geq 5$. Now, for any pair $v_a, v_b \in V(G,_1)$, $d(v_a, v_b) \leq 2$. Thus $L(V_i) \cap L(V_{i+1}) = \emptyset$ and $L(V_i) \cap L(V_{i+2}) = \emptyset$. However, $L(V_i)$ labels $L(Vi+3)$ since $d(v_a, v_c) = 3$ for all $v_a \in V_i$ and $v_c \in V_{i+3}$. Thus, $L(V_i) = L(V_{i+3k})$, $L(V_{i+1}) = L(V_{i+4k})$ and $L(V_{i+2}) = L(V_{i+5k})$ for all $k \in \mathbf{N}$. since $|V(G_1')| = 6$, then $\lambda_1^1(P_m \times C_4) \leq 5$ and therefore, the equality follows. □

Theorem 4.10. If $m \geq 3$, then $\lambda_1^1(P_m \times C_5) = 4$.

Proof. Clearly, $P_m \times C_5 \equiv G_1$, where G_1 is a connected component of $P_m \times C_{10}$.

Therefore, $\lambda_1^1(P_m \times C_5) \leq \lambda_1^1(P_m \times C_{10}) \leq \lambda_1^1(C_{10m'} \times C_{10n'}) = 4$, for all $m', n' \in \mathbf{N}$. Now, since there exists a star $S_4 \subset P_m \times C_5$, then $\lambda_1^1(P_m \times C_5) \geq 5$. □

The last theorem clearly yields the next corrolary.

Corollary 4.11. For all $m \geq 3, n' \in \mathbf{N}$, $\lambda_1^1(P_m \times C_{5n'}) = 4$.

Lemma 4.12. Suppose G' is a connected component of $P_3 \times P_n$, $n \geq 9$, such that $u_iv_j, u_iv_k \in V(G')$. If $d(u_iv_j, u_iv_k) = 8$, then $l(u_iv_j) \neq l(u_iv_k)$.

Proof. Suppose $\alpha_j, \alpha_k \in [4]$ and $\alpha_j = l(u_1v_j)$, $\alpha_k = l(u_1v_k)$, while $d(u_1v_j, u_1v_k) = 8$. The next vertex, according to Lemmas 4.1 and 4.2, that α_j labels is either u_0v_{j+3} and u_2v_{j+3}. Now, since $d(u_0v_{j+3}, u_1v_k) = 5$, then by Lemma 4.1, $\alpha_j \neq l(u_1v_k)$. Thus, $\alpha_k \neq \alpha_j$. □

Theorem 4.13. For $m \geq 3$, $\lambda_1^1(P_m \times C_7) = 5$.

Proof. Suppose $\lambda_1^1(P_m \times C_7) = 4$. Clearly from an earlier remark, $P_m \times C_7 \equiv G'$ where $G'$is a connected component of $P_m \times C_{14}$ Also, $G' \equiv G''$, where G'' is the connected component of $P_m \times P_{15}$, with $u_iv_0 \equiv u_iv_{14}$ for all $i \in [(m-1)(\epsilon)]$. Suppose $\bar{G}$ is a subgraph of G'' induced by the vertex set U_i, U_{i+1} and U_{i+2} such that $u_iv_0 \in U_i$, and $u_{i+2}v_0 \in U_{i+2}$. Let $\{\alpha_i\}_{i=0}^4 = [4]$ and suppose $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$, labels $u_iv_0, u_{i+2}v_0\ u_{i+1}v_1, u_{i+2}v_0\ u_{i+2}v_2$. Then $l(u_0v_{14}) = \alpha_0$ and $l(u_2v_{14}) = \alpha_1$. Since $d(u_{i+1}v_1, u_{i+2}v_{13}) = 2$, then $l(u_{i+1}v_{13}) \in \{\alpha_3, \alpha_4\}$. Without loss of generality, let $l(u_iv_{13}) = \alpha_3$. Then $L(u_0v_{12}, u_2v_{12}) = \{\alpha_2, \alpha_4\}$. Now, $d(u_{i+j}v_k, u_{i+1}v_7) = 5$ for all $j \in \{0, 2\}$, $k \in \{2, 12\}$ Thus, by Lemma 4.1, $l(u_{i+1}v_7) \in A = \{\alpha_2, \alpha_3, \alpha_4\}$. Also, by the reason of distance, $l(u_{i+1}v_3) \in A$. thus, $l(u_{i+1}v_3)$ is either α_0 or α_1. Again without loss of generality, suppose $l(u_{i+1}v_3) = \alpha_0$. By Lemma 4.2, $l(u_{i+1}v_7) \neq \alpha_0$. Thus, $l(u_{i+1}v_7) = \alpha_1$. Since $l(u_{i+1}v_7) = \alpha_1$, then $l(u_{i+1}v_{11}) \neq \alpha_1$. therefore, $l(u_{i+1}v_{11}) \notin \{\alpha_1 \cup A\}$ and hence, $l(u_{i+1}v_{11}) = \alpha_0$. But this is a contradiction of Lemma 4.12 since $d(u_{i+1}v_3, u_{i+1}v{+}11) = 8$ and it is assumed that $\lambda_1^1(P_m \times C_7) = 4$. Thus, $\lambda_1^1(P_m \times C_7) \geq 5$. Conversely, for each $i \in [m-1]$, $|V_i| = 7$, where $V_i \subset V(G')$. Therefore, suppose $|L(V_i)| = 6$, then there exists a pair $v_1, v_2 \in V_i$ such that $l(v_i) = l(v_2) = \alpha_k$ for some $\alpha_k \in [5]$. Now, set $u_1 = u_iv_j$ and $u_2 = u_iv_{j+4}$ such that $d(u_iv_j, v_{j+4})\ d(u_1, u_2) = 4$. Let $\bar{V_1} = V_i \backslash \{u_iv_j\}$. Set $\alpha_j = l(u_kv_l) = l(u_{k+1}v_{l+3})$ for all u $u_kv_j \in \bar{V_1}$. Now, there exists $u_3 = u_{k+3}v_{j+3} \in V_{i+1}$ such that u_3 is not yet labeled. Let $u_4 = u_{k+1}v_{j-1}$ and set $l(u_{k+1}v_{j-1}) = l(u_{k+1}v_{j+3})$. Obviously, $d(u_3, u_4) = 4$ and $u_3, u_4 \in V_{i+1}$. Repeat the above scheme between V_{i+1} and V_{i+2}, V_{i+2} and $V_{i+3}, ..., V_{m-2}, V_{m-1}$. Thus $\lambda_1^1(P_m \times C_7) \leq 5$ and then the equality follows. □

The proof of the next results follow the last theorem and some remarks made earlier.

Corollary 4.14. For $m \geq 3$, $\lambda_1^1(P_m \times C_{14}) = 5$.

Theorem 4.15. Let $m \geq 3$. Then $\lambda_1^1(P_m \times C_8) = 5$.

Proof. That $\lambda_1^1(P_m \times C_8) \geq 5$ follows from Lemma 4.12 and $\lambda_1^1(P_m \times C_8) \leq 5$ follows from repeating the $L(1,1)-$labeling of $P_m \times C_4$. □

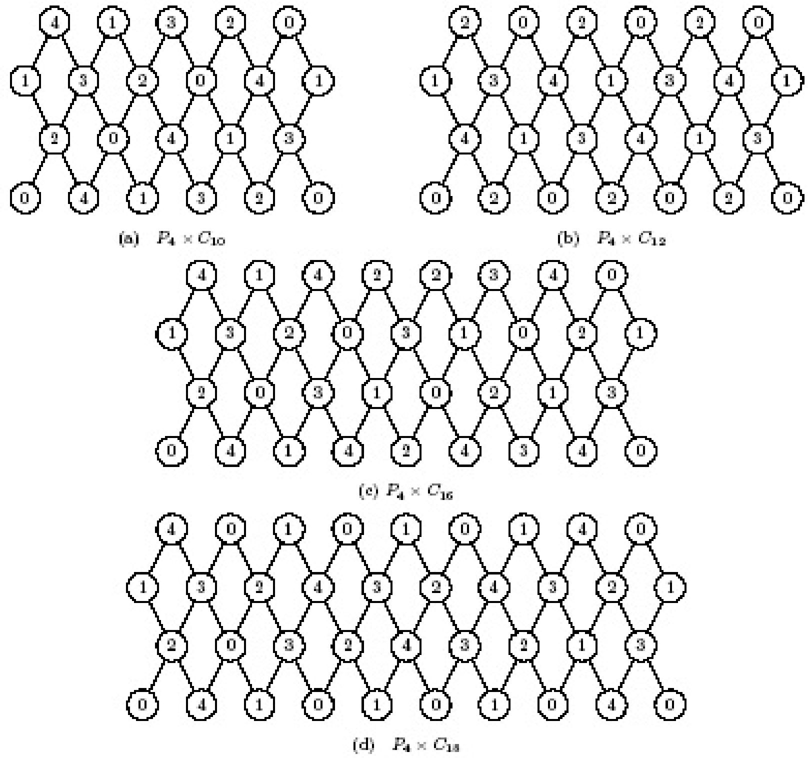


Fig. 1 $4-L(1,1)$–Labeling of $P_4 \times C_n$, $n = 10, 12, 16, 18$

Theorem 4.16. Given that $n \geq 9$, $n \neq 14$, then $\lambda_1^1(P_4 \times C_n) = 4$.

Proof. From (b),(c),(d) of Fig. 1, we notice that $\lambda_1^1(P_4 \times C_{n'}) = 4$, for all $m' \in \{12, 16, 18\}$.

Now, by combining each of (b),(c),(d) with (a), we see that $\lambda_1^1(P_m \times C_{n'+10}) = 4$, for each $n' \in \{12, 16, 18\}$. Therefore, $\lambda_1^1(P_4 \times C_{km'+p}) = 4$ $\forall k \geq 0$ and $p \in \{0, 10\}$. Thus by an earlier remark, $\lambda_1^1(P_4 \times C_n) = 4$ for all $n \geq 9$, $n \neq 14$. $\square$

Corollary 4.17. Given that $n \geq 9$, $n \neq 14$, and that $m \in \{3, 4\}$ then $\lambda_1^1(P_m \times C_n) = 4$.

Theorem 4.18. For $m \geq 3$, $\lambda_1^1(P_m \times C_{14}) = 4$.

Proof. It follows directly from Remark 4.4 (iii) and Theorem 4.13. □

Next, we derive the general lower bound for the $L(1,1)-$ labelling of $P_m \times C_n$, where $m \geq 5$, $n \not\equiv 0 \bmod 5$. That $\lambda_1^1(P_m \times C_n) = 4$, where m, n are both multiples of 5, has already been established. We need the next lemma to prove the theorem that follows.

Lemma 4.19. If $\lambda_1^1(P_m \times C_n) = 4$ for $n \not\equiv 0 \bmod 5$, $n \geq 9$. Then, for all $V_j \subset V(P_m \times C_n)$, $0 \leq j \leq n-2$, there exist $v_a, v_b \in V_j$, such that $l(v_a) = l(v_b)$ and $d(v_a, v_b) = 6$.

Proof. Let $G = P_m \times C_n$. Suppose, without loss of generality, that n is even since by Remark 4.4 (iii), if n is odd then G is equivalent to one of the two components of $P_m \times C_{2n}$. Let G' be the connected component of G. Let $V_j' \subset V(G')$ such that $V_j' \subset V_j$. Let $v_a \in V_j'$ such that $l(v_a) = \alpha_k \in [4]$. Since n is not a multiple of 5, and $n \geq 9$, then $\left|V_j'\right| = \frac{n}{2} > 5$. Since $\lambda_1^1(G) = 4$, then there exists at least some vertex $v_b \in V_j'$ such that $l(v_b) = \alpha_k$. By the definition of $L(1,1)-$ labeling, $d(v_a, v_b) \neq 2$. Likewise by Lemmas 4.2 and 4.12, $d(v_a, v_b) \notin \{4, 8\}$ thus, $d(v_a, v_b) = 6$. □

Theorem 4.20. Let $m \geq 5$, $n \not\equiv 0 \bmod 5$ and $n \geq 9$. Then, $\lambda_1^1(P_m \times C_n) \geq 5$.

Proof. Let $m \geq 5$, $n \not\equiv 0 \bmod 5$ and $n \geq 9$. Suppose $\lambda_1^1(P_m \times C_n) = 4$. Let $G = P_m \times C_n$. Suppose n is even. Then there exists G', a connected component of $P_m \times C_n$. (If n is odd, we know from an earlier result that G is a connected component of $P_m \times C_{2n}$.) We defined an arbitrary vertex set $V(G'') = \{u_iv_j, u_iv_{j+2}, u_{i+1}v_{j+1}, u_{i+2}v_j, u_{i+2}v_{j+2}, u_{i+3}v_{j+1}, u_{i+4}v_j, u_{i+4}v_{j+2}\}$, with $V(G'') \subset V(G')$. Clearly, $V(G'')$ induces a subgraph G'' of G' such that $G'' = S_4' \cup S_4''$ where S_4', S_4'' are stars with
$V(S_4') = \{u_iv_j, u_iv_{j+2}, u_{i+1}v_{j+1}, u_{i+2}v_j, u_{i+2}v_{j+2}, \}$ and
$S_4'' = u_{i+2}v_j, u_{i+2}v_{j+2}, u_{i+3}v_{j+i}, u_{i+4}v_j, u_{i+4}v_{j+2}$ respectively. Now, by 4.19 above, for all $V_i \subset V(G'), 0 \leq i \leq m-2$ there exist at least a vertex pair

$v_a, v_b \in V_i$ such that for some $\alpha_i \in L(V_i) \subseteq [4]$, $l(v_a) = l(v_b) = \alpha_i$ and $d(v_a, v_b) = 6$.

Suppose $u_{i+2}v_{j-2}, u_{i+2}v_{j+4} \in V_{i+2}$ such that $l(u_{i+2}v_{j-2}) = l(u_{i+2}v_{j+4}) = \alpha_i$.

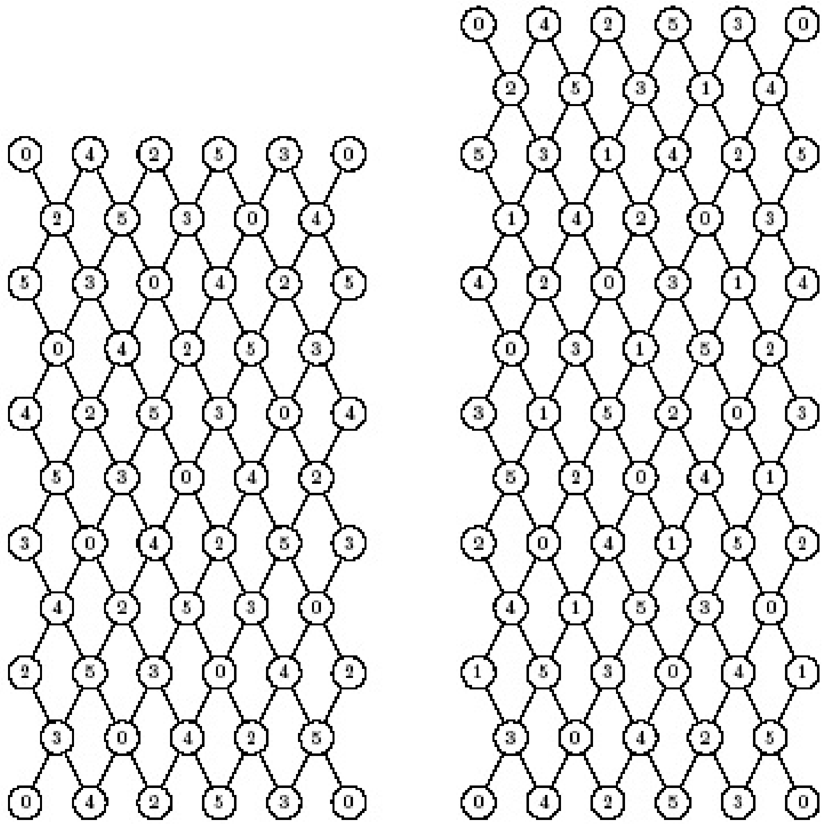
There exist vertices $u_{i+1}v_{j+1} \in V_{i+1}$ and $u_{i+3}v_{j+1} \in V_{i+3}$.

By Lemma 4.1, $l(u_{i+1}v_{j+1}) = \alpha_i$ or $l(u_{i+3}v_{j+1}) = \alpha_i$. Suppose $l(u_{i+1}v_{j+1}) = \alpha_i$, then $d(u_a, u_b) \leq 2$ for any $u_a \in V(S_4'')$ and $u_b \in \{u_{i+1}v_{j+1}, u_{i+2}v_{j-2}, u_{i+2}v_{j+4}\}$.

Thus there is no such vertex as $u_a \in S_4'$ such that $l(u_a) = \alpha_i \in V(S_4')$. Likewise, $d(u_a', u_b) \leq 2$ for any $u_a' \in V(S_5')$ and $u_b \in \{u_{i+3}v_{j+1}, u_{i+2}v_{j-2}, u_{i+2}v_{j+4}\}$.

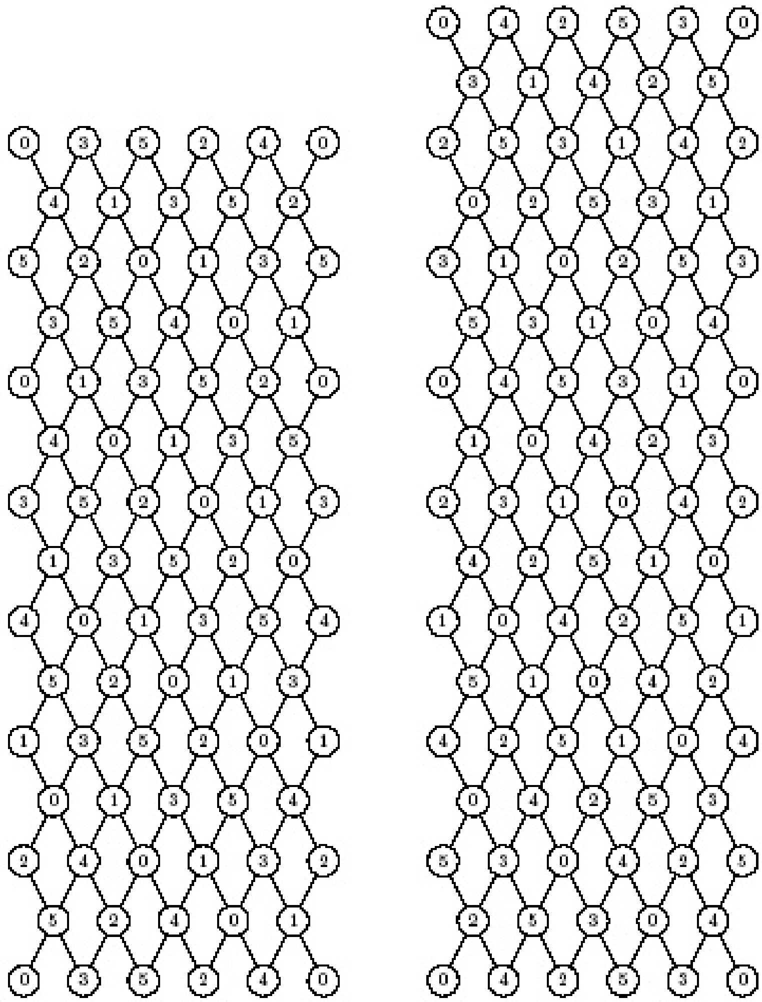
Thus, there exists no vertex $u_a' \in V(S_5')$, such that $l(u_a') = \alpha_j \in [4]$ and therefore, a contradiction. □

By the result obtained in Theorem 4.20, we see that the $\lambda_1^1(P_m \times C_n) \geq 5$ for all $m \geq 5$ and $n \geq 9$, where n is not a multiple of 5. In the subsequent results, we obtain the λ_1^1-number for the remaining $P_m \times C_n$ graphs.

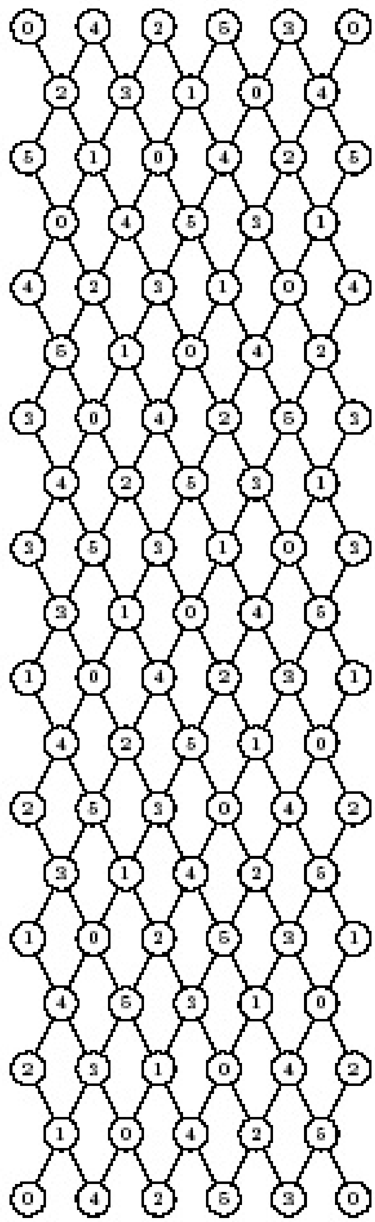


(a) $C_{10} \times C_{10}$ (b) $C_{10} \times C_{12}$

Fig.2.5 − $L(1,1)$−Labeling of $C_{10} \times C_n$, $m = 10, 12$



(a) $C_{10} \times C_{14}$ (b) $C_{10} \times C_{16}$



(c) $C_{10} \times C_{18}$

Fig.3 5 – $(L(1,1))$ Labeling of $C_{10} \times C_n, n = 14, 16, 18$

Theorem 4.21. Let $k \in A$. For all k, m', n', $\lambda_1^1(C_{10m'} \times C_{k+10n'}) = 5$, where m' is any positive integer, n' a non-negative integer and $A = \{12, 14, 16, 18\}$.

Proof. The result follows by combining the $5-$labeling of $C_{10} \times C_{10n'}$ which is obtainable from $n'-$times repeat of Fig.5 a, with the $5-$labeling of $C_{10} \times C_{12}$, $C_{10} \times C_{14}$, $C_{10} \times C_{16}$ and $C_{10} \times C_{18}$ in Fig.5 b and of Fig.3 a, b and c respectively along with C_n and then m'-copy the resultant graph along with C_m. □

Corollary 4.22. For all $P_m \times C_n$, where $m \geq 5$ and $n \geq 6$, $n \not\equiv 0 \bmod 5$ then $\lambda_1^1(P_m \times C_m) = 5$.

Proof. Let h be a positive even integer with $h \geq 12$. Let $k \in A = \{12, 14, 16, 18\}$. Then, for all h, $h \equiv 0 \bmod k + 10n'$ for some $k \in A$. The result thus follows from Remarks 4.4 (iii) and (iv) and the fact that $P_m \times C_n \subset P_{10m} \times C_n$. □

5. Conclusion

The following summarizes the results obtained in this work:

For $G = P_m \times P_n$:

m	n	$\lambda_1^1(P_m \times P_n)$
2	2	1
≥ 3	2	2
≥ 3	≥ 3	4

For $G = P_m \times C_n$:

m	n	$\lambda_1^1(P_m \times C_n)$
2	$\equiv 0 \bmod 3$	2
2	$\not\equiv 0 \bmod 3$	3
≥ 3	$\in \{3, 4, 6, 7, 8, 14\}$	5
≥ 3	$\equiv 0 \bmod 5$	4
$3, 4$	$\geq 9, \neq 14$	4
≥ 5	$\geq 9, \not\equiv 0 \bmod 5$	5

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